WARING–GOLDBACH PROBLEM: ONE SQUARE, FOUR FIFTH POWERS AND FIFTEEN SIXTH POWERS

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Abstract. We prove that for every sufficiently large even integer n, the equation

$$n = p_1^2 + p_2^5 + p_3^5 + p_4^5 + p_5^5 + p_6^6 + \cdots + p_{20}^6$$

has a solution with all p_j prime.

Contents

1. Introduction

Let n be a sufficiently large positive integer that satisfies some necessary congruence conditions. Using the Hardy–Littlewood circle method, many mathematicians studied the number of representations of n as the sum of mixed powers of prime numbers. Examples of recent works on this topic include [1], [2], [3], [4], [5] and [10]. In this paper, we prove the following result.

Theorem 1.1. *For every sufficiently large even integer n, the equation*

$$n = p_1^2 + p_2^5 + p_3^5 + p_4^5 + p_5^5 + p_6^6 + \cdots + p_{20}^6 \tag{1}$$

has a solution with all p_j prime.

2. Preliminaries

Let n be a sufficiently large positive integer, $P_k = n^{\frac{1}{k}}$ and $L = \log n$. Let

$$\lambda_1 = 1, \quad \lambda_2 = \frac{27}{32}, \quad \lambda_3 = \frac{729}{1024}, \quad \lambda_4 = \frac{19683}{32768}, \quad \lambda_5 = \frac{531441}{1048576}, \quad \lambda_6 = \frac{14348907}{33554432}, \quad \lambda_7 = \frac{387420489}{1073741824},$$

$$\lambda_8 = \frac{10460353203}{34359738368}, \qquad \lambda_9 = \frac{9365374360779057135}{36385377104158523392}, \qquad \lambda_{10} = \frac{248229150174961425}{1137043034504953856},$$

$$\lambda_{11} = \frac{13208549498258175}{71065189656559616}, \qquad \lambda_{12} = \frac{1418162385496725}{8883148707069952}, \qquad \lambda_{13} = \frac{1234060169123925}{8883148707069952},$$

$$\lambda_{14} = \lambda_{15} = \frac{560936440510875}{4441574353534976},$$

and write

$$\Lambda = \sum_{j=1}^{15} \lambda_j = \frac{169503075438540046537}{283983431056847011184}, \qquad \Theta = \frac{3}{10} + \frac{\Lambda}{6} = \frac{1103100465143862543341}{851950293170541035520} \approx 1.29479.$$

We define

$$f_k(\alpha) = \sum_{\frac{1}{2}P_k < p \leqslant P_k} e(\alpha p^k) \log p$$

and, for $1 \leqslant j \leqslant 15$,

$$f_{k,j}(\alpha) = \sum_{\frac{1}{2}P_k^{\lambda_j} < p \leqslant P_k^{\lambda_j}} e(\alpha p^k) \log p.$$

We also define

$$F(\alpha) = f_2(\alpha) f_5(\alpha)^4 \prod_{j=1}^{15} f_{6,j}(\alpha).$$

Then by the orthogonality of additive characters, the integral

$$\int_0^1 F(\alpha) e(-\alpha n) d\alpha \tag{2}$$

counts the number of solutions of (1) with prime variables lying in certain ranges weighted by a log factor. Our goal is to show the lower bound

$$\int_0^1 F(\alpha) e(-\alpha n) d\alpha \gg n^{\Theta} \tag{3}$$

holds for all even n.

Fix $B \geqslant 1$. Put

$$\mathfrak{M}(q,a) = \left\{ \alpha \in [0,\ 1] : \left| \alpha - \frac{a}{q} \right| \leqslant \frac{L^B}{n} \right\}, \qquad \mathfrak{M} = \bigcup_{1 \leqslant q \leqslant L^B} \bigcup_{\substack{0 \leqslant a \leqslant q \\ (a,q)=1}} \mathfrak{M}(q,a), \qquad \mathfrak{m} = [0,\ 1] \backslash \mathfrak{M}.$$

Then (3) can be obtained if we get

$$\int_{\mathfrak{M}} F(\alpha) e(-\alpha n) d\alpha \gg n^{\Theta} \tag{4}$$

and

$$\int_{\mathfrak{m}} F(\alpha) e(-\alpha n) d\alpha \ll n^{\Theta} L^{-1}. \tag{5}$$

3. The major arcs

Define

$$S_k(q,a) = \sum_{\substack{x=1 \\ (x,q)=1}}^{q} e\left(\frac{a x^k}{q}\right), \qquad v_{k,r}(\beta) = \frac{1}{k} \sum_{2^{-k} n^r < m \leqslant n^r} m^{1/k-1} e(\beta m), \qquad v_k(\beta) = v_{k,1}(\beta),$$

$$U(q,a) = S_2(q,a) S_5(q,a)^4 S_6(q,a)^{15}, \qquad w(\beta) = v_2(\beta) v_5(\beta)^4 \prod_{j=1}^{15} v_{6,\lambda_j}(\beta).$$

Lemma 3.1. *Let q, a be integers such as $1 \leqslant q \leqslant L^B$, $0 \leqslant a \leqslant q$ and $(a,q) = 1$. Then for $\alpha \in \mathfrak{M}(q,a)$, we have*

$$\sum_{\frac{1}{2}P_k^r < p \leqslant P_k^r} e(\alpha p^k) \log p = \frac{1}{\varphi(q)} S_k(q,a) v_{k,r}(\alpha - a/q) + O\left(P_k^r \exp(-c\sqrt{L})\right)$$

for some constant $c > 0$.

Proof. This lemma follows from [[6], Lemma 6] and partial summation. □

We shall use this lemma to give an approximation of $F(\alpha)$ using functions $S_k(q,a)$ and $v_{k,r}$ defined above. By Lemma 3.1, we have

$$F(\alpha) = \frac{1}{\varphi(q)^{20}} U(q,a) w(\alpha - a/q) + O\left(n^{1+\Theta} L^{-4B}\right). \tag{6}$$

An integration over $\mathfrak{M}$ gives

$$\int_{\mathfrak{M}} F(\alpha) e(-\alpha n) d\alpha = \sum_{q \leqslant L^B} \frac{A_n(q)}{\varphi(q)^{20}} \int_{-L^B/n}^{L^B/n} w(\beta) e(-\beta n) d\beta + O\left(n^{\Theta} L^{-B}\right), \tag{7}$$

where

$$A_n(q)=\sum_{\substack{a=1\\(a,q)=1}}^{q}U(q,a)e(-an/q).$$

We first show that

$$\int_{-L^B/n}^{L^B/n}w(\beta)e(-\beta n)d\beta\gg n^{\Theta}. \tag{8}$$

Lemma 3.2. *([9], Lemma 6.2]). Suppose that $|\beta|\leqslant\frac{1}{2}$. Then,*

$$v_k(\beta)\ll n^{\frac{1}{k}}(1+n|\beta|)^{-1}.$$

By Lemma 3.2 we know that

$$w(\beta)\ll n^{1+\Theta}(1+n|\beta|)^{-2}. \tag{9}$$

By (9), we get

$$\begin{aligned}
&\int_{-L^B/n}^{L^B/n}w(\beta)e(-\beta n)d\beta\\
=&\int_{-1/2}^{1/2}w(\beta)e(-\beta n)d\beta+O\left(\int_{L^B/n}^{1/2}|w(\beta)|d\beta\right)\\
=&\int_{-1/2}^{1/2}w(\beta)e(-\beta n)d\beta+O\left(n^{\Theta}L^{-B}\right)\\
=&\frac{1}{2\cdot 5^4\cdot 6^{15}}\sum_{\substack{m_1+\cdots+m_{20}=n\\ 2^{-2}n<m_1\leqslant n\\ 2^{-5}n<m_2,m_3,m_4,m_5\leqslant n\\ 2^{-6}n^{\lambda_{j-5}}<m_j\leqslant n^{\lambda_{j-5}}\,(6\leqslant j\leqslant 20)}} m_1^{-\frac{1}{2}}(m_2m_3m_4m_5)^{-\frac{4}{5}}\prod_{j=6}^{20}m_j^{-\frac{5}{6}}+O\left(n^{\Theta}L^{-B}\right)\\
\gg&\sum_{\substack{m_1+\cdots+m_{20}=n\\ \frac{1}{4}n<m_1\leqslant\frac{1}{2}n\\ \frac{1}{32}n<m_2,m_3,m_4\leqslant\frac{3}{64}n\\ \frac{1}{64}n^{\lambda_{j-5}}<m_j\leqslant\frac{3}{128}n^{\lambda_{j-5}}\,(6\leqslant j\leqslant 20)}} m_1^{-\frac{1}{2}}(m_2m_3m_4m_5)^{-\frac{4}{5}}\prod_{j=6}^{20}m_j^{-\frac{5}{6}}\quad\gg n^{\Theta}.
\end{aligned} \tag{10}$$

Next, we want to deal with the sum

$$\sum_{q\leqslant L^B}\frac{A_n(q)}{\varphi(q)^{20}}. \tag{11}$$

Define the series

$$\mathfrak{S}(n)=\sum_{q=1}^{\infty}\frac{A_n(q)}{\varphi(q)^{20}}.$$

Lemma 3.3. *([6], Lemma 5]). For $k\geqslant 2$, we have*

$$S_k(q,a)\ll q^{\frac{1}{2}+\varepsilon}.$$

Lemma 3.4. *([9], Lemma 4.3]). For $(a,p)=1$, we have*

$$|S_k(p,a)|\leqslant((k,p-1)-1)\sqrt{p}+1.$$

Lemma 3.5. *([6], Lemma 4]). Let $p^n\mid k$, $p^{n+1}\nmid k$ and*

$$l=\begin{cases}n+2, & p=2,\\ n+1, & p\neq 2.\end{cases}$$

If $t>l$, then

$$S_k(p^t,a)=0.$$

By Lemma 3.3, we know that $U(q,a) \ll q^{10+\varepsilon}$ for $(a,q)=1$, and hence $A_n(q) \ll q^{11+\varepsilon}$ uniformly in n. Thus, $\mathfrak{S}(n)$ converges absolutely, and

$$\sum_{q \leqslant L^B} \frac{A_n(q)}{\varphi(q)^{20}} = \mathfrak{S}(n) + O\left(L^{-B}\right). \tag{12}$$

Since the complete exponential sums are multiplicative, for $(q_1,q_2)=1$ we have

$$A_n(q_1q_2) = A_n(q_1)A_n(q_2).$$

Moreover, By Lemma 3.5 and the fact that $A_n(4) = S_5(4,a) = 0$, we have $A_n(q) = 0$ unless q is square-free. Hence

$$\mathfrak{S}(n) = \prod_p \mathfrak{S}_p(n) = \prod_p \left(1 + \frac{A_n(p)}{(p-1)^{20}}\right). \tag{13}$$

By the orthogonality of additive characters modulo p, we have

$$\mathfrak{S}_p(n) = \frac{pM_n(p)}{(p-1)^{20}}, \tag{14}$$

where $M_n(p)$ denote the number of solutions $(x_1,\ldots,x_{20}) \in \left(\mathbb{F}_p^\times\right)^{20}$ of

$$x_1^2 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \sum_{j=6}^{20} x_j^6 \equiv n (\mathrm{mod} p). \tag{15}$$

When $p=2$, every unit is 1, and

$$x_1^2 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \sum_{j=6}^{20} x_j^6 \equiv 0 (\mathrm{mod} 2).$$

Hence $n \equiv 0 (\mathrm{mod} 2)$ is necessary.

When $p=3$, every square and sixth power is 1, and every fifth power is ± 1. It is easy to see that the sum of four fifth powers contains all residue classes modulo 3, and (15) has a solution.

When $p=5$, we have $x^5 = x$ and $x^6 = \pm 1$. Fix $x_1 = 1$. Since $\mathbb{F}_5^\times + \mathbb{F}_5^\times = \mathbb{F}_5$, the sum of two fifth powers contains all residue classes modulo 5, and (15) has a solution.

When $p=7$, we have $x^6=1$. Fix $x_1,x_2,x_3 = 1$. Since $\mathbb{F}_7^\times + \mathbb{F}_7^\times = \mathbb{F}_7$, the sum of two fifth powers contains all residue classes modulo 7, and (15) has a solution.

When $p \geqslant 11$, we define $\mathcal{A}_k = \mathcal{A}_k(p) = \{x^k : x \in \mathbb{F}_p\}$. It is known that $|\mathcal{A}_k(p)| = \frac{p-1}{(k,p-1)}$. Applying the Cauchy–Davenport theorem repeatedly, we have

$$\begin{aligned} |\mathcal{A}_2 + 4\mathcal{A}_5 + 15\mathcal{A}_6| &\geqslant \min(p, |\mathcal{A}_2| + 4|\mathcal{A}_5| + 15|\mathcal{A}_6| - 19) \\ &\geqslant \min\left(p, \frac{p-1}{2} + \frac{4(p-1)}{5} + \frac{15(p-1)}{6} - 19\right) \\ &\geqslant \min\left(p, \frac{19p-114}{5}\right). \end{aligned}$$

Since $p \geqslant 11$, we have $\frac{19p-114}{5} > p$, and thus $\mathcal{A}_2 + 4\mathcal{A}_5 + 15\mathcal{A}_6 = \mathbb{F}_p$. Now (15) has a solution for every prime $p \geqslant 11$.

Combining above cases, we know that $M_n(p) \geqslant 1$ if n is even, and also $\mathfrak{S}(n) > 0$ by (13). Now, by Lemma 3.4 and trivial estimates, we have

$$\begin{aligned} \left|\frac{A_n(p)}{(p-1)^{20}}\right| &\leqslant \frac{p(\sqrt{p}+1)(4\sqrt{p}+1)^4(5\sqrt{p}+1)^{15}}{(p-1)^{20}} \\ &= \frac{p^{11}}{(p-1)^{20}}\left(1+\frac{1}{\sqrt{p}}\right)\left(4+\frac{1}{\sqrt{p}}\right)^4\left(5+\frac{1}{\sqrt{p}}\right)^{15} \\ &\leqslant \frac{p^{11}}{(p/2)^{20}}\left(1+\frac{1}{\sqrt{p}}\right)\left(4+\frac{1}{\sqrt{p}}\right)^4\left(5+\frac{1}{\sqrt{p}}\right)^{15} \end{aligned}$$

$$\leqslant \frac{2^{20}\cdot 2\cdot 5^5\cdot 6^{15}}{p^9}$$

for $p > 10^{10}$. Thus, we get

$$\prod_{p>10^{10}}\left(1+\frac{A_n(p)}{(p-1)^{20}}\right) \geqslant \prod_{p>10^{10}}\left(1-\frac{2^{21}\cdot 5^5\cdot 6^{15}}{p^9}\right) > c \tag{16}$$

with some $c > 0$. By (13), (14), (16) and $M_n(p) \geqslant 1$, we have

$$\mathfrak{S}(n) > c \prod_{p\leqslant 10^{10}} \frac{pM_n(p)}{(p-1)^{20}} > c \prod_{p\leqslant 10^{10}} p^{-19} \gg 1. \tag{17}$$

Now, by (7), (10), (12) and (17), the proof of (4) is completed.

4. The minor arcs

We first recall some mean value estimates proved by Andrade [3] and Zhao [11]. Note that we can remove the logarithmic weights and the restriction to primes in $f_k(\alpha)$ and $f_{k,j}(\alpha)$ at the cost of a power of L, and deal with the classical Weyl sums.

Lemma 4.1. *We have*

$$\int_0^1 |f_2(\alpha)|^2\,|f_5(\alpha)|^6\,d\alpha \ll n^{\frac{6}{5}+\varepsilon}.$$

Proof. Taking $k_1 = k_2 = k_3 = 5$ in [[3], Lemma 4], we get Lemma 4.1. ☐

Lemma 4.2. *We have*

$$\int_0^1 \left|\prod_{j=1}^{15} f_{6,j}(\alpha)\right|^2 d\alpha \ll P_6^{\Lambda+\varepsilon}.$$

Proof. Taking $t = 1$ and $U = P_6$ in [[11], Lemma 6.1], we get Lemma 4.2. ☐

Define

$$\mathcal{E} = \{\alpha \in [0,\ 1] : |f_5(\alpha)| \leqslant P_5^{(\Lambda-4)/2-10\tau}\}.$$

Using Hölder's inequality, together with Lemma 4.1, Lemma 4.2 and the definition of $\mathcal{E}$, we get

$$\begin{aligned}\int_{\mathcal{E}} |F(\alpha)|d\alpha &\leqslant \left(\int_0^1 |f_2(\alpha)|^2\,|f_5(\alpha)|^6\,d\alpha\right)^{\frac{1}{2}}\left(\int_0^1 \left|\prod_{j=1}^{15} f_{6,j}(\alpha)\right|^2 d\alpha\right)^{\frac{1}{2}} \sup_{\alpha\in\mathcal{E}}|f_5(\alpha)| \\ &\ll n^{\frac{3}{5}+\varepsilon} P_6^{\Lambda/2+\varepsilon} P_5^{(\Lambda-4)/2-10\tau} \ll n^{\theta-\tau},\end{aligned} \tag{18}$$

where

$$\theta = \frac{441062789418431385223}{340780117268216414208} \approx 1.29427.$$

Note that $\Theta - \theta > 0.0005$.

5. Proof of Theorem 1.1

Lemma 5.1. *Suppose that $X^{\frac{1}{24}} \leqslant P \leqslant X^{\frac{9}{20}}$. Then either we have*

$$\sum_{X<p\leqslant 2X} e(\alpha p^5) \ll X^{\frac{47}{48}+\varepsilon},$$

or there exist integers q and a such that $1 \leqslant q \leqslant P$, $(a,q) = 1$ and $|q\alpha - a| \leqslant PX^{-5}$.

Proof. Taking $k = 5$ in [[8], Lemma 2.2], we get Lemma 5.1. ☐

Now suppose that $\alpha \in [0,\ 1]\backslash\mathcal{E}$. We have $|f_5(\alpha)| > P_5^{(\Lambda-4)/2-10\tau}$ in this case. Taking $X = \frac{1}{2}P_5$ and $P = \frac{1}{32}n^{\frac{1}{119}}$ in Lemma 5.1, we find that the first case will never occur since $\frac{\Lambda-4}{2} - \frac{47}{48} > 0.005$. Hence, we must have

$$[0,\ 1]\backslash\mathcal{E} \subset \mathfrak{R} = \bigcup_{1\leqslant q\leqslant \frac{1}{32}n^{\frac{1}{119}}} \bigcup_{\substack{0\leqslant a\leqslant q\\(a,q)=1}} \mathfrak{R}(q,a), \tag{19}$$

where

$$\mathfrak{R}(q,a) = \{\alpha \in [0,\ 1] : |q\alpha - a| \leqslant n^{-\frac{118}{119}}\}. \tag{20}$$

Define a function $\Upsilon : \mathfrak{R} \to [0,\ 1]$ by

$$\Upsilon(\alpha) = (q + n|q\alpha - a|)^{-1}.$$

Taking $k = 2, 5$ in [[7], Theorem 2] and by partial summation, we have

$$f_2(\alpha) \ll P_2 L^C \Upsilon(\alpha)^{\frac{1}{2}-\varepsilon} \tag{21}$$

and

$$f_5(\alpha) \ll P_5 L^C \Upsilon(\alpha)^{\frac{1}{2}-\varepsilon} \tag{22}$$

for some constant $C > 0$. By (21), (22) and trivial upper bounds for $f_{6,j}(\alpha)$ $(1 \leqslant j \leqslant 15)$, we have

$$F(\alpha) = f_2(\alpha) f_5(\alpha)^4 \prod_{j=1}^{15} f_{6,j}(\alpha) \ll n^{1+\Theta} L^{20C} \Upsilon(\alpha)^{\frac{49}{20}}. \tag{23}$$

Integrating over $\mathfrak{R}\backslash\mathfrak{M}$ gives

$$\int_{\mathfrak{R}\backslash\mathfrak{M}} |F(\alpha)| d\alpha \ll n^{1+\Theta} L^{20C} \int_{\mathfrak{R}\backslash\mathfrak{M}} |\Upsilon(\alpha)|^{\frac{49}{20}} d\alpha \ll n^{\Theta} L^{20C - \frac{9}{20}B + 1}. \tag{24}$$

Choosing $B = 200C + 100$ in (24) and by the fact that $\mathfrak{m} \subset \mathcal{E} \cup \mathfrak{R}\backslash\mathfrak{M}$, the proof of (5) is completed. From (4) and (5) we get (3), and the proof of Theorem 1.1 is completed.

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