The number of Piatetski-Shapiro primes in short intervals

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Abstract

We study the distribution of Piatetski-Shapiro primes, or primes of the form $[n^c]$, in short intervals of length x^θ. We give an asymptotic formula of the number of such primes in short intervals under certain restrictions on θ and c, which is better than a previous result of Guo and Guo (2026) when $\theta > \frac{11}{12}$. We present two proofs of our Theorem based on different sieve decompositions, using Buchstab's identity and the Heath-Brown–Linnik identity respectively.

MSC (2020): 11N05, 11N35, 11N36, 11B83
Keywords: Primes, Piatetski-Shapiro sequences, Sieve methods

1 Introduction

One of the famous topics in number theory is to find prime numbers in short intervals. Let x be a sufficiently large integer, $\pi(x)$ is the prime counting function and $0 < \theta \leqslant 1$. The Riemann Hypothesis implies that for all sufficiently large x, the asymptotic formula

$$\pi(x + x^\theta) - \pi(x) \sim \frac{x^\theta}{\log x} \tag{1}$$

holds for every $\theta > \frac{1}{2}$. In 2026, Guth and Maynard [3] proved this asymptotic formula holds for every $\theta > \frac{17}{30}$ unconditionally.

In 1953, Piatetski-Shapiro [5] has proposed to investigate the prime numbers of the form $[n^c]$, where $c > 1$ and $[n^c]$ denotes the integer part of n^c. Clearly $[n^c]$ can be regarded as "polynomials of degree c". Define

$$\pi_c(x) := \left|\{p \leqslant x : p = [n^c]\}\right|,$$

then he has shown that

$$\pi_c(x) \sim \frac{x^{1/c}}{\log x} \tag{2}$$

holds for any $1 < c < \frac{12}{11} \approx 1.0909$. This range has been improved by many authors, and the best record now is due to Guo, Guo and Lu [2], where they proved that (2) holds for any $1 < c < \frac{10318869}{8886224} \approx 1.1612$. We call this type of primes *Piatetski-Shapiro primes*.

In 2026, Guo and Guo [1] considered a mixed problem. They studied the distribution of Piatetski-Shapiro primes in short intervals and proved the following asymptotic formula

$$\pi_c(x+x^\theta)-\pi_c(x)=(1+o(1))\frac{x^{1/c+\theta-1}}{c\log x} \tag{3}$$

holds when $\frac{2}{3}<\theta<1$ and

$$\frac{1}{c}>\begin{cases}\frac{20-3\theta}{18}, & \frac{2}{3}<\theta\leqslant\frac{40}{51},\\ \max\left(\frac{1+\theta}{2},\frac{10-4\theta}{7},\frac{17-3\theta}{15}\right), & \frac{40}{51}<\theta\leqslant\frac{10}{11},\\ \max\left(\frac{15-6\theta}{10},\frac{17-3\theta}{15}\right), & \frac{10}{11}<\theta<1.\end{cases} \tag{4}$$

In this paper, we shall prove the following result.

Theorem 1.1. *We have (3) when*

$$\frac{5}{6}<\theta<1 \quad \textit{and} \quad \frac{1}{c}>\frac{15-6\theta}{10}.$$

Clearly our Theorem 1.1 is better than [[1], Theorem 1.1] when $\theta>\frac{11}{12}$. We will use sieve method to prove our Theorem 1.1, and we shall give two proofs based on different sieve decompositions: in the first proof we use Buchstab's identity directly, while in the second proof we apply the Heath-Brown–Linnik identity. The proof of these two identities can be found in [4]. In the proof of [[1], Theorem 1.1] another version of the Heath-Brown identity is used, and that is why they need the extra condition $\frac{1}{c}>\frac{17-3\theta}{15}$.

Let $\mathcal{C}$ denote a finite set of positive integers and put

$$P(z)=\prod_{p<z}p, \quad \mathcal{C}_d=\{n:nd\in\mathcal{C}\}, \quad S(\mathcal{C},z)=\sum_{\substack{n\in\mathcal{C}\\(n,P(z))=1}}1.$$

Let $\frac{5}{6}+10\varepsilon\leqslant\theta<1$, where ε is a sufficiently small positive number. Define

$$\mathcal{A}=\{a:a\in\mathbb{Z},\ x<a\leqslant x+x^\theta,\ a=[n^c]\}, \quad \mathcal{B}=\{b:b\in\mathbb{Z},\ x<b\leqslant x+x^\theta\}.$$

Since $\frac{17}{30}<\frac{5}{6}$, by Prime Number Theorem in short intervals [3], we have

$$S\left(\mathcal{B},(2x)^{\frac{1}{2}}\right)=\sum_{p\in\mathcal{B}}1\sim\frac{x^\theta}{\log x}. \tag{5}$$

In order to prove our Theorem 1.1, we only need to show that

$$S\left(\mathcal{A},(2x)^{\frac{1}{2}}\right)=(1+o(1))\frac{x^{1/c+\theta-1}}{c\log x}. \tag{6}$$

We also need the following important result.

Lemma 1.2. *For any $a_m \ll x^{\varepsilon}$, if we have*

$$\frac{3}{4} < \theta < 1, \quad \frac{9-4\theta}{6} < \frac{1}{c} < 1 \quad \textit{and} \quad 2-\theta-\frac{1}{c}+\varepsilon \leqslant 3\theta+\frac{5}{c}-7-\varepsilon,$$

then for $M \ll x^{3\theta+\frac{5}{c}-7-\varepsilon}$, we have

$$\sum_{m\sim M} a_m S\left(\mathcal{A}_m, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) = \frac{x^{1/c-1}}{c} \sum_{m\sim M} a_m S\left(\mathcal{B}_m, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) + O\left(x^{1/c+\theta-1-\varepsilon}\right).$$

Proof. This is a special case of [[1], Lemma 5.3(ii)]. □

Remark. When $c > 1$, $\frac{5}{6}+10\varepsilon \leqslant \theta < 1$ and $\frac{1}{c} \geqslant \frac{15-6\theta}{10}+\varepsilon$, we have

$$\frac{9-4\theta}{6} < \frac{15-6\theta}{10} < \frac{1}{c} < 1$$

and

$$2-\theta-\frac{1}{c}+\varepsilon < 2-\theta-\frac{15-6\theta}{10}+\varepsilon < 3\theta+5\cdot\frac{15-6\theta}{10}-7-\varepsilon < 3\theta+\frac{5}{c}-7-\varepsilon.$$

Hence all conditions in Lemma 1.2 are satisfied. Moreover, we have

$$3\theta+\frac{5}{c}-7-\varepsilon \geqslant \frac{1}{2}+4\varepsilon$$

and

$$1-\varepsilon = 4+6-9-\varepsilon > 4\theta+\frac{6}{c}-9-\varepsilon > 4\theta+6\cdot\frac{15-6\theta}{10}-9-\varepsilon > \frac{1}{3}+\varepsilon.$$

If

$$x^{4\theta+\frac{6}{c}-9-\varepsilon} \geqslant (2x)^{\frac{1}{2}},$$

then we have

$$S\left(\mathcal{A}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) = \sum_{p\in\mathcal{A}} 1 = S\left(\mathcal{A}, (2x)^{\frac{1}{2}}\right)$$

and

$$S\left(\mathcal{B}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) = \sum_{p\in\mathcal{B}} 1 = S\left(\mathcal{B}, (2x)^{\frac{1}{2}}\right),$$

and Theorem 1.1 follows immediately from Lemma 1.2 and (5). Thus, we assume that

$$x^{4\theta+\frac{6}{c}-9-\varepsilon} < (2x)^{\frac{1}{2}}.$$

2 The first proof: Buchstab's identity

Buchstab's identity is the equation

$$S(\mathcal{C}, z) = S(\mathcal{C}, w) - \sum_{w \leqslant p < z} S(\mathcal{C}_p, p),$$

where $2 \leqslant w < z$.

We apply Buchstab's identity twice to get

$$S\left(\mathcal{A}, (2x)^{\frac{1}{2}}\right) = S\left(\mathcal{A}, x^{4\theta + \frac{6}{c} - 9 - \varepsilon}\right) - \sum_{x^{4\theta + \frac{6}{c} - 9 - \varepsilon} \leqslant p_1 < (2x)^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1}, x^{4\theta + \frac{6}{c} - 9 - \varepsilon}\right) + \sum_{x^{4\theta + \frac{6}{c} - 9 - \varepsilon} \leqslant p_2 < p_1 < (2x)^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right). \tag{7}$$

Note that the last sum on the right-hand side of (7) counts numbers of the form $p_1 p_2 \beta \in \mathcal{A}$, where $x^{4\theta + \frac{6}{c} - 9 - \varepsilon} \leqslant p_2 < p_1 < (2x)^{\frac{1}{2}}$ and the least prime factor of β is larger than p_2. However, we have

$$p_1 p_2 \beta \geqslant p_1 p_2^2 > x^{3\left(4\theta + \frac{6}{c} - 9 - \varepsilon\right)} > x^{1+3\varepsilon},$$

making a contradiction with the fact that $p_1 p_2 \beta \in \mathcal{A}$ ($p_1 p_2 \beta < 2x$). Thus, we have

$$\sum_{x^{4\theta + \frac{6}{c} - 9 - \varepsilon} \leqslant p_2 < p_1 < (2x)^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) = 0, \tag{8}$$

and

$$S\left(\mathcal{A}, (2x)^{\frac{1}{2}}\right) = S\left(\mathcal{A}, x^{4\theta + \frac{6}{c} - 9 - \varepsilon}\right) - \sum_{x^{4\theta + \frac{6}{c} - 9 - \varepsilon} \leqslant p_1 < (2x)^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1}, x^{4\theta + \frac{6}{c} - 9 - \varepsilon}\right). \tag{9}$$

Similarly, we have

$$\sum_{x^{4\theta + \frac{6}{c} - 9 - \varepsilon} \leqslant p_2 < p_1 < (2x)^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1 p_2}, p_2\right) = 0, \tag{10}$$

and by Buchstab's identity twice,

$$\begin{aligned} S\left(\mathcal{B}, (2x)^{\frac{1}{2}}\right) = {} & S\left(\mathcal{B}, x^{4\theta + \frac{6}{c} - 9 - \varepsilon}\right) - \sum_{x^{4\theta + \frac{6}{c} - 9 - \varepsilon} \leqslant p_1 < (2x)^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1}, x^{4\theta + \frac{6}{c} - 9 - \varepsilon}\right) \\ & + \sum_{x^{4\theta + \frac{6}{c} - 9 - \varepsilon} \leqslant p_2 < p_1 < (2x)^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1 p_2}, p_2\right) \\ = {} & S\left(\mathcal{B}, x^{4\theta + \frac{6}{c} - 9 - \varepsilon}\right) - \sum_{x^{4\theta + \frac{6}{c} - 9 - \varepsilon} \leqslant p_1 < (2x)^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1}, x^{4\theta + \frac{6}{c} - 9 - \varepsilon}\right). \end{aligned} \tag{11}$$

Since $x^{3\theta+\frac{5}{c}-7-\varepsilon} \gg x^{\frac{1}{2}+4\varepsilon}$, by Lemma 1.2 we have

$$S\left(\mathcal{A}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) = \frac{x^{1/c-1}}{c} S\left(\mathcal{B}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) + O\left(x^{1/c+\theta-1-\varepsilon}\right) \tag{12}$$

and

$$\begin{aligned}&\sum_{x^{4\theta+\frac{6}{c}-9-\varepsilon} \leqslant p_1 < (2x)^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) \\ &= \frac{x^{1/c-1}}{c} \sum_{x^{4\theta+\frac{6}{c}-9-\varepsilon} \leqslant p_1 < (2x)^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) + O\left(x^{1/c+\theta-1-\varepsilon}\right). \end{aligned} \tag{13}$$

Now, by (5), (9) and (11)–(13), we have

$$\begin{aligned} S\left(\mathcal{A}, (2x)^{\frac{1}{2}}\right) &= S\left(\mathcal{A}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) - \sum_{x^{4\theta+\frac{6}{c}-9-\varepsilon} \leqslant p_1 < (2x)^{\frac{1}{2}}} S\left(\mathcal{A}_{p_1}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) \\ &= \frac{x^{1/c-1}}{c} \left(S\left(\mathcal{B}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) - \sum_{x^{4\theta+\frac{6}{c}-9-\varepsilon} \leqslant p_1 < (2x)^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) \right) \\ &\quad + O\left(x^{1/c+\theta-1-\varepsilon}\right) \\ &= \frac{x^{1/c-1}}{c} S\left(\mathcal{B}, (2x)^{\frac{1}{2}}\right) + O\left(x^{1/c+\theta-1-\varepsilon}\right) \\ &= (1+o(1)) \frac{x^{1/c+\theta-1}}{c \log x}, \end{aligned} \tag{14}$$

and the proof of Theorem 1.1 is completed.

3 The second proof: Heath-Brown–Linnik identity

Heath-Brown–Linnik identity is the equation

$$S\left(\mathcal{C}, (2x)^{\frac{1}{2}}\right) = \sum_{1 \leqslant j \leqslant k} \frac{(-1)^{j-1}}{j} S\left(\mathcal{C}^j, z\right) + O\left(x^{\frac{1}{2}}\right),$$

where $k \in \mathbb{Z}$, $z^k > 2x$ and $\mathcal{C}^j = \{n_1 \cdots n_j \in \mathcal{C}\}$.

Since $x^{4\theta+\frac{6}{c}-9-\varepsilon} > x^{\frac{1}{3}+\varepsilon}$, we can take $k=3$ and $z = x^{4\theta+\frac{6}{c}-9-\varepsilon}$. Now we have

$$\begin{aligned} S\left(\mathcal{A}, (2x)^{\frac{1}{2}}\right) &= \sum_{1 \leqslant j \leqslant 3} \frac{(-1)^{j-1}}{j} S\left(\mathcal{A}^j, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) + O\left(x^{\frac{1}{2}}\right) \\ &= S\left(\mathcal{A}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) - \frac{1}{2} S\left(\mathcal{A}^2, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{3}S\left(\mathcal{A}^3, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right)+O\left(x^{\frac{1}{2}}\right)\\
&=S\left(\mathcal{A}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right)-\frac{1}{2}\sum_{\substack{n_1n_2\in\mathcal{A}\\(n_1n_2,P(x^{4\theta+\frac{6}{c}-9-\varepsilon}))=1}}1\\
&+\frac{1}{3}\sum_{\substack{n_1n_2n_3\in\mathcal{A}\\(n_1n_2n_3,P(x^{4\theta+\frac{6}{c}-9-\varepsilon}))=1}}1+O\left(x^{\frac{1}{2}}\right). \qquad (15)
\end{aligned}
$$

Note that the last sum on the right-hand side of (15) counts numbers of the form $n_1n_2n_3\in\mathcal{A}$, where the least prime factors of n_1, n_2 and n_3 are all larger than $x^{3(4\theta+\frac{6}{c}-9-\varepsilon)}$. However, we have

$$
n_1n_2n_3 \geqslant x^{3(4\theta+\frac{6}{c}-9-\varepsilon)} > x^{1+3\varepsilon},
$$

making a contradiction with the fact that $n_1n_2n_3\in\mathcal{A}$ $(n_1n_2n_3<2x)$. Thus, we have

$$
\sum_{\substack{n_1n_2n_3\in\mathcal{A}\\(n_1n_2n_3,P(x^{4\theta+\frac{6}{c}-9-\varepsilon}))=1}}1=0, \qquad (16)
$$

and

$$
S\left(\mathcal{A},(2x)^{\frac{1}{2}}\right)=S\left(\mathcal{A}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right)-\frac{1}{2}\sum_{\substack{n_1n_2\in\mathcal{A}\\(n_1n_2,P(x^{4\theta+\frac{6}{c}-9-\varepsilon}))=1}}1+O\left(x^{\frac{1}{2}}\right). \qquad (17)
$$

Similarly, we have

$$
\sum_{\substack{n_1n_2n_3\in\mathcal{B}\\(n_1n_2n_3,P(x^{4\theta+\frac{6}{c}-9-\varepsilon}))=1}}1=0, \qquad (18)
$$

and by Heath-Brown–Linnik identity,

$$
S\left(\mathcal{B},(2x)^{\frac{1}{2}}\right)=S\left(\mathcal{B}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right)-\frac{1}{2}\sum_{\substack{n_1n_2\in\mathcal{B}\\(n_1n_2,P(x^{4\theta+\frac{6}{c}-9-\varepsilon}))=1}}1+O\left(x^{\frac{1}{2}}\right). \qquad (19)
$$

Since $x^{3\theta+\frac{5}{c}-7-\varepsilon}\gg x^{\frac{1}{2}+4\varepsilon}$ and $\min(n_1,n_2)<(2x)^{\frac{1}{2}}$, taking $m=\min(n_1,n_2)$ in Lemma 1.2, we have

$$
\sum_{\substack{n_1n_2\in\mathcal{A}\\(n_1n_2,P(x^{4\theta+\frac{6}{c}-9-\varepsilon}))=1}}1=\frac{x^{1/c-1}}{c}\sum_{\substack{n_1n_2\in\mathcal{B}\\(n_1n_2,P(x^{4\theta+\frac{6}{c}-9-\varepsilon}))=1}}1+O\left(x^{1/c+\theta-1-\varepsilon}\right). \qquad (20)
$$

Now, by (5), (12), (17) and (19)–(20), we have

$$
S\left(\mathcal{A},(2x)^{\frac{1}{2}}\right)=S\left(\mathcal{A}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right)-\frac{1}{2}\sum_{\substack{n_1n_2\in\mathcal{A}\\(n_1n_2,P(x^{4\theta+\frac{6}{c}-9-\varepsilon}))=1}}1+O\left(x^{\frac{1}{2}}\right)
$$

$$
\begin{aligned}
&= \frac{x^{1/c-1}}{c}\left(S\left(\mathcal{B}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) - \frac{1}{2}\sum_{\substack{n_1 n_2 \in \mathcal{B} \\ (n_1 n_2, P(x^{4\theta+\frac{6}{c}-9-\varepsilon}))=1}} 1\right) \\
&\quad + O\left(x^{1/c+\theta-1-\varepsilon}\right) + O\left(x^{\frac{1}{2}}\right) \\
&= \frac{x^{1/c-1}}{c}\left(S\left(\mathcal{B}, (2x)^{\frac{1}{2}}\right) - O\left(x^{\frac{1}{2}}\right)\right) + O\left(x^{1/c+\theta-1-\varepsilon}\right) + O\left(x^{\frac{1}{2}}\right) \\
&= (1+o(1))\frac{x^{1/c+\theta-1}}{c\log x}, \qquad (21)
\end{aligned}
$$

since we have $\theta > \frac{5}{6} > \frac{1}{2}$ and $\frac{1}{c} + \theta - 1 > \frac{9}{10} + \frac{5}{6} - 1 = \frac{11}{15} > \frac{1}{2}$. The proof of Theorem 1.1 is completed.

4 The connection

Let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity. Consider the sum

$$\sum_{\substack{n_1 n_2 \in \mathcal{B} \\ (n_1 n_2, P(x^{4\theta+\frac{6}{c}-9-\varepsilon}))=1}} 1 \qquad (22)$$

occurred in Section 3. If $\Omega(n_1 n_2) \geqslant 3$, then

$$n_1 n_2 \geqslant x^{3(4\theta+\frac{6}{c}-9-\varepsilon)} > x^{1+3\varepsilon},$$

making a contradiction with the fact that $n_1 n_2 \in \mathcal{B}$ $(n_1 n_2 < 2x)$. Thus, we have $\Omega(n_1 n_2) = 2$, which means that both n_1 and n_2 are prime. Now, the sum (22) can be rewritten as

$$\sum_{\substack{p_1 p_2 \in \mathcal{B} \\ p_1 \geqslant x^{4\theta+\frac{6}{c}-9-\varepsilon} \\ p_2 \geqslant x^{4\theta+\frac{6}{c}-9-\varepsilon}}} 1. \qquad (23)$$

Consider the sum

$$\sum_{x^{4\theta+\frac{6}{c}-9-\varepsilon} \leqslant p_1 < (2x)^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) \qquad (24)$$

occurred in Section 2. This sum counts numbers of the form $p_1\beta \in \mathcal{B}$, where $x^{4\theta+\frac{6}{c}-9-\varepsilon} \leqslant p_1 < (2x)^{\frac{1}{2}}$ and the least prime factor of β is larger than $x^{4\theta+\frac{6}{c}-9-\varepsilon}$. If $\Omega(\beta) \geqslant 2$, then

$$p_1\beta \geqslant x^{3(4\theta+\frac{6}{c}-9-\varepsilon)} > x^{1+3\varepsilon},$$

making a contradiction with the fact that $p_1\beta \in \mathcal{B}$ ($p_1\beta < 2x$). Thus, we have $\Omega(\beta) = 1$, which means that β is a prime. Now, the sum (24) can be rewritten as

$$\sum_{\substack{p_1p_2\in\mathcal{B}\\ x^{4\theta+\frac{6}{c}-9-\varepsilon}\leqslant p_1<(2x)^{\frac{1}{2}}\\ p_2\geqslant x^{4\theta+\frac{6}{c}-9-\varepsilon}}} 1. \tag{25}$$

It is obvious that the part of the sum (23) with $p_1 < (2x)^{\frac{1}{2}}$ is equal to the sum (25). The remaining part of the sum (23) is

$$\sum_{\substack{p_1p_2\in\mathcal{B}\\ p_1\geqslant(2x)^{\frac{1}{2}}\\ p_2\geqslant x^{4\theta+\frac{6}{c}-9-\varepsilon}}} 1. \tag{26}$$

Since $p_1p_2 < 2x$, we have $p_2 < (2x)^{\frac{1}{2}}$ if $p_1 \geqslant (2x)^{\frac{1}{2}}$. Swapping the roles of p_1 and p_2, we get

$$\sum_{\substack{p_1p_2\in\mathcal{B}\\ p_1\geqslant(2x)^{\frac{1}{2}}\\ p_2\geqslant x^{4\theta+\frac{6}{c}-9-\varepsilon}}} 1 = \sum_{\substack{p_1p_2\in\mathcal{B}\\ x^{4\theta+\frac{6}{c}-9-\varepsilon}\leqslant p_1<(2x)^{\frac{1}{2}}\\ p_2\geqslant(2x)^{\frac{1}{2}}}} 1. \tag{27}$$

Since $p_1p_2 > x$ and $p_1 < (2x)^{\frac{1}{2}}$ imply $p_2 > (\frac{x}{2})^{\frac{1}{2}}$, we know that the difference between the sum (25) and the sum (26) is

$$\sum_{\substack{p_1p_2\in\mathcal{B}\\ x^{4\theta+\frac{6}{c}-9-\varepsilon}\leqslant p_1<(2x)^{\frac{1}{2}}\\ (\frac{x}{2})^{\frac{1}{2}}<p_2<(2x)^{\frac{1}{2}}}} 1 \ll \frac{x^{\theta}}{(\log x)^2}. \tag{28}$$

Combining these two parts we get

$$\sum_{\substack{n_1n_2\in\mathcal{B}\\ (n_1n_2,P(x^{4\theta+\frac{6}{c}-9-\varepsilon}))=1}} 1 = 2\sum_{x^{4\theta+\frac{6}{c}-9-\varepsilon}\leqslant p_1<(2x)^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1}, x^{4\theta+\frac{6}{c}-9-\varepsilon}\right) - O\left(\frac{x^{\theta}}{(\log x)^2}\right). \tag{29}$$

Thus, the decompositions used in the two proofs are essentially same.

References

[1] L. Guo and V. Z. Guo. Piatetski-Shapiro primes in short intervals. *arXiv e-prints*, page arXiv:2606.01115v1, 2026.

[2] L. Guo, V. Z. Guo, and L. Lu. The Piatetski-Shapiro prime number theorem. *arXiv e-prints*, page arXiv:2505.10391v2, 2025.

[3] L. Guth and J. Maynard. New large value estimates for Dirichlet polynomials. *Ann. of Math.*, 203(2):623–675, 2026.

[4] G. Harman. *Prime-detecting Sieves*, volume 33 of *London Mathematical Society Monographs (New Series)*. Princeton University Press, Princeton, NJ, 2007.

[5] I. I. Piatetski-Shapiro. On the distribution of prime numbers in sequences of the form $[f(n)]$. *Mat. Sb.*, 33:559–566, 1953.

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