The explicit expressions for the Buchstab function $\omega(u)$ $(3 \leqslant u < 4)$

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Let $\omega(u)$ denote the Buchstab function determined by the following differential-difference equation

$$\begin{cases} \omega(u) = \frac{1}{u}, & 1 \leqslant u \leqslant 2, \\ (u\omega(u))' = \omega(u-1), & u \geqslant 2. \end{cases}$$

Explicitly, we have

$$\omega(u) = \begin{cases} \frac{1}{u}, & 1 \leqslant u < 2, \\ \frac{1+\log(u-1)}{u}, & 2 \leqslant u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t} dt, & 3 \leqslant u < 4, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t} dt + \frac{1}{u}\int_3^{u-1} \frac{dt}{t} \int_2^{t-1} \frac{\log(s-1)}{s} ds, & 4 \leqslant u < 5. \end{cases}$$

In many applications of the sieve methods, we need to numerically calculate integrals involving $\omega(u)$. For $1 \leqslant u < 3$ the explicit expressions for $\omega(u)$ are simple and easy to compute, and for $u \geqslant 4$ we can use the known bound

$$0.5612 \leqslant \omega(u) \leqslant 0.5617$$

for $\omega(u)$. For $3 \leqslant u < 4$ we also have a bound

$$0.5607 \leqslant \omega(u) \leqslant 0.5644$$

for $\omega(u)$, but sometimes we decide to use another expression for $\omega(u)$:

$$\omega(u) = \frac{1+\log(u-1)}{u} + \frac{1}{u}\left(\frac{\pi^2}{12} + \log(u-2)\log(u-1) + \mathrm{Li}_2(2-u)\right) \quad \text{for} \quad 3 \leqslant u < 4,$$

where $\mathrm{Li}_2(z)$ is the dilogarithm function. This expression is very useful when performing calculations using Mathematica, and we shall prove that these two expressions are equivalent.

We only need to prove that

$$\int_2^{u-1} \frac{\log(t-1)}{t} dt = \frac{\pi^2}{12} + \log(u-2)\log(u-1) + \mathrm{Li}_2(2-u).$$

Let $x = t - 1$, then

$$\int_2^{u-1} \frac{\log(t-1)}{t} dt = \int_1^{u-2} \frac{\log x}{x+1} dx.$$

For $z \in \mathbb{C} \backslash [1, \infty)$, the dilogarithm function $\mathrm{Li}_2(z)$ satisfies

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-s)}{s} ds.$$

Thus,

$$\frac{d}{dz} \mathrm{Li}_2(z) = -\frac{\log(1-z)}{z}.$$

Let

$$v = \log x, \quad dw = \frac{1}{x+1} dx.$$

Then

$$dv = \frac{1}{x} dx, \quad w = \log(x+1).$$

Integration by parts gives

$$\int \frac{\log x}{x+1} dx = \log x \log(x+1) - \int \frac{\log(x+1)}{x} dx.$$

Since

$$\frac{d}{dx} \mathrm{Li}_2(-x) = -1 \cdot \left(-\frac{\log(x+1)}{-x} \right) = -\frac{\log(x+1)}{x},$$

we have

$$\int \frac{\log(x+1)}{x} dx = -\mathrm{Li}_2(-x) + C$$

and

$$\int \frac{\log x}{x+1} dx = \log x \log(x+1) + \mathrm{Li}_2(-x) + C.$$

Now,

$$\begin{aligned}
\int_2^{u-1} \frac{\log(t-1)}{t} dt &= \int_1^{u-2} \frac{\log x}{x+1} dx \\
&= \log(u-2)\log(u-1) + \mathrm{Li}_2(2-u) - \mathrm{Li}_2(-1) \\
&= \frac{\pi^2}{12} + \log(u-2)\log(u-1) + \mathrm{Li}_2(2-u),
\end{aligned}$$

since $\mathrm{Li}_2(-1) = -\frac{\pi^2}{12}$. The proof is completed.

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