REMARKS ON ADDITIVE REPRESENTATIONS OF NATURAL NUMBERS

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Abstract. For two relatively prime square-free positive integers a and b, we study integers of the form $ap + bP_2$ and give a new lower bound for the number of such representations, where ap and bP_2 are both square-free, p denote a prime, and P_2 has at most two prime factors. We also consider some special cases where p is small, p and P_2 are within short intervals, p and P_2 are within arithmetical progressions and a Goldbach-type upper bound result. Our new results generalize and improve previous results.

Contents

1. Introduction

Let N_e denote a sufficiently large even integer, p and q, with or without subscript, denote prime numbers, and let P_r denote an integer with at most r prime factors counted with multiplicity. For each $N_e \geqslant 4$ and $r \geqslant 2$, we define

$$D_{1,r}(N_e) := |\{p : p \leqslant N_e, N_e - p = P_r\}| . \tag{1}$$

In 1966 Chen [8] announced his remarkable Chen's theorem: let N_e be a sufficiently large even integer, then

$$D_{1,2}(N_e) \geqslant 0.67 \frac{C(N_e) N_e}{(\log N_e)^2} \tag{2}$$

where

$$C(N_e) := \prod_{\substack{p \mid N_e \\ p>2}} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right). \tag{3}$$

and the detail was published in [9]. The original proof of Chen was simplified by Pan, Ding and Wang [29], Halberstam and Richert [16], Halberstam [15] and Ross [32]. As Halberstam and Richert indicated in [16], it would be interesting to know whether a more elaborate weighting procedure could be adapted to the purpose of (2). This might lead to numerical improvements and could be important. Chen's constant 0.67 was improved successively to

$$0.689,\ 0.7544,\ 0.81,\ 0.8285,\ 0.836,\ 0.867,\ 0.899$$

by Halberstam and Richert [16] [15], Chen [12] [10], Cai and Lu [7], Wu [39], Cai [2] and Wu [40] respectively.

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In 1990, Wu [36] generalized Chen's theorem and showed that

$$D_{1,r}(N_e) \geqslant 0.67\frac{C(N_e)N_e}{(\log N_e)^2}(\log\log N_e)^{r-2}. \tag{4}$$

Kan [17] also proved the similar result in 1991:

$$D_{1,r}(N_e) \geqslant \frac{0.77}{(r-2)!}\frac{C(N_e)N_e}{(\log N_e)^2}(\log\log N_e)^{r-2}, \tag{5}$$

which is better than Wu's result when $r = 3$. Kan [19] proved the more generalized theorem in 1992:

$$D_{s,r}(N_e) \geqslant \frac{0.77}{(s-1)!(r-2)!}\frac{C(N_e)N_e}{(\log N_e)^2}(\log\log N_e)^{s+r-3}, \tag{6}$$

where $s \geqslant 1$,

$$D_{s,r}(N_e) := \left|\{P_s : P_s \leqslant N_e, N_e - P_s = P_r\}\right|. \tag{7}$$

Furthermore, for two relatively prime square-free positive integers a and b, let N denote a sufficiently large integer that is relatively prime to both a and b, $a, b < N^{\varepsilon}$ and let N be even if a and b are both odd. Let $R_{a,b}(N)$ denote the number of primes p such that ap and $N - ap$ are both square-free, $b \mid (N - ap)$, and $\frac{N-ap}{b} = P_2$. In 1976, Ross [[33], Chapter 3] got a similar result without the square-free restrictions on ap and $N - ap$. In 2023, Li [26] established that

$$R_{a,b}(N) \geqslant 0.68\frac{C(abN)N}{ab(\log N)^2}. \tag{8}$$

In this paper, we improve the result by using a delicate sieve process similar to that of [2] and prove that

Theorem 1.1.

$$R_{a,b}(N) \geqslant 0.8671\frac{C(abN)N}{ab(\log N)^2}.$$

It is easy to see that when we take $a = 1$ and $b = 1$, Theorem 1.1 implies Cai's result on Chen's theorem [[2], Theorem 1]; when we take $a = 1$ and $b = 2$, Theorem 1.1 improves Li's result related to the Lemoine's conjecture [[25], Theorem 1]. When we take $a = q_1q_2\cdots q_s$ and $b = q_1'q_2'\cdots q_r'$ where q, q' denote prime numbers satisfy

$$s, r \geqslant 1, \quad q_i, q_j' < N^{\varepsilon}, \quad (q_i, N) = (q_j', N) = 1 \text{ for every } 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant r,$$

Theorem 1.1 generalizes and improves the previous results of Kan [[17], Theorem 2] [[19], Theorem 2] and Wu [[36], Theorems 1 and 2]. Clearly one can modify our proof of Theorem 1.1 to get a similar lower bound on the twin prime version. For this, we refer the interested readers to Ross's PhD thesis [33] and [[14], Sect. 25.6], as well as [18], [20] and [22] for some interesting applications.

Chen's theorem with small primes was first studied by Cai [1]. For $0 < \theta \leqslant 1$, we define

$$D_{1,r}^{\theta}(N_e) := \left|\left\{p : p \leqslant N_e^{\theta}, N_e - p = P_r\right\}\right|. \tag{9}$$

Then it is proved in [1] that for $0.95 \leqslant \theta \leqslant 1$, we have

$$D_{1,2}^{\theta}(N_e) \gg \frac{C(N_e)N_e^{\theta}}{(\log N_e)^2}. \tag{10}$$

Cai's range $0.95 \leqslant \theta \leqslant 1$ was extended successively to $0.945 \leqslant \theta \leqslant 1$ in [4] and to $0.941 \leqslant \theta \leqslant 1$ in [3].

In this paper, we generalize their results to integers of the form $ap + bP_2$. Let $R_{a,b}^{\theta}(N)$ denote the number of primes $p \leqslant N^{\theta}$ such that ap and $N - ap$ are both square-free, $b \mid (N - ap)$, and $\frac{N-ap}{b} = P_2$. In 1976, Ross [[33], Chapter 5] got a similar result without the square-free restrictions on ap and $N - ap$ and showed that $0.959 \leqslant \theta \leqslant 1$ is admissible. Now by using a delicate sieve process similar to that of [3], we prove that

Theorem 1.2. *For $0.9409 \leqslant \theta \leqslant 1$ we have*

$$R_{a,b}^{\theta}(N) \gg \frac{C(abN)N^{\theta}}{ab(\log N)^2}.$$

For similar results on the twin prime version with small primes, we refer the interested readers to [27], [41], [13] and [28].

Chen's theorem in short intervals was first studied by Ross [34]. For $0 < \kappa \leqslant 1$, we define

$$D_{1,r}(N_e, \kappa) := \left|\{p : N_e/2 - N_e^{\kappa} \leqslant p, P_r \leqslant N_e/2 + N_e^{\kappa}, N_e = p + P_r\}\right|. \tag{11}$$

Then it is proved in [34] that for $0.98 \leqslant \kappa \leqslant 1$, we have

$$D_{1,2}(N_e, \kappa) \gg \frac{C(N_e)N_e^{\kappa}}{(\log N_e)^2}. \tag{12}$$

The constant 0.98 was improved successively to

$$0.974,\ 0.973,\ 0.9729,\ 0.972,\ 0.971,\ 0.97$$

by Wu [37] [38], Salerno and Vitolo [35], Cai and Lu [6], Wu [39] and Cai [2] respectively.

In this paper, we generalize their results to integers of the form $ap + bP_2$. Let $R_{a,b}(N, \kappa)$ denote the number of primes $N/2 - N^{\kappa} \leqslant p \leqslant N/2 + N^{\kappa}$ such that ap and $N - ap$ are both square-free, $b \mid (N - ap)$ and $\frac{N-ap}{b} = P_2$. In [34], Ross mentioned that his method can be used to prove similar results of $R_{a,b}(N,\kappa)$ with $0.98 \leqslant \kappa \leqslant 1$ and a detailed proof was given in [[33], Chapter 5]. Now by using a delicate sieve process similar to that of [2], we prove that

Theorem 1.3. *For* $0.97 \leqslant \kappa \leqslant 1$ *we have*

$$R_{a,b}(N,\kappa) \gg \frac{C(abN)N^{\kappa}}{ab(\log N)^2}.$$

From our Theorems 1.1–1.3, it can be seen that the first aim of this paper is to improve the old results on the natural numbers of the form $ap+bP_2$ to be consistent with or better than the results on the even numbers of the form $p+P_2$. Before our work, all results on this topic are weaker than those of binary Goldbach problem. For Theorem 1.1, the constants 0.608 in [33] and 0.68 in [26] are smaller than 0.867 in [2]. For Theorems 1.2–1.3, Ross's exponent 0.959 and 0.98 are again weaker than those in [3] and [2].

Chen's theorem in arithmetical progressions was first studied by Kan and Shan [23]. If we define

$$D_{1,r}(N_e,c,d) := \left|\{p : p \leqslant N_e, p \equiv d(\bmod c), (c,d)=1, (N_e-p,c)=1, N_e-p=P_r\}\right|, \tag{13}$$

then it is proved in [23] that for $c \leqslant (\log N_e)^C$ where C is a positive constant, we have

$$D_{1,r}(N_e,c,d) \geqslant \frac{0.77}{(r-2)!} \prod_{\substack{p\mid c\\ p\nmid N_e\\ p>2}} \left(\frac{p-1}{p-2}\right) \frac{C(N_e)N_e}{\varphi(c)(\log N_e)^2} (\log\log N_e)^{r-2}, \tag{14}$$

where φ denote the Euler's totient function. Clearly their result (14) generalized the previous results (2) and (5). They also got the similar results on the twin prime version (or even the "safe prime" version, see [21]) and Lewulis [24] considered the similar problem. However, their results are only valid when c is "small". In 1999, Cai and Lu [5] considered this problem with "large" c and proved that for $c \leqslant N_e^{\frac{1}{37}}$, except for $O\left(N_e^{\frac{1}{37}}(\log N_e)^{-A}\right)$ exceptional values, we have

$$D_{1,2}(N_e,c,d) \gg \prod_{\substack{p\mid c\\ p\nmid N_e\\ p>2}} \left(\frac{p-1}{p-2}\right) \frac{C(N_e)N_e}{\varphi(c)(\log N_e)^2} \tag{15}$$

and they mentioned that the exponent $\frac{1}{37}$ can be improved to 0.028. In this paper, we further generalize their results to integers of the form $ap+bP_2$. Let $R_{a,b}(N,c,d)$ denote the number of primes $p \equiv d(\bmod c)$ such that ap and $N-ap$ are both square-free, $b \mid (N-ap)$, and $\frac{N-ap}{b}=P_2$. Then by using a delicate sieve process similar to that of [2], we prove that

Theorem 1.4. *For* $c \leqslant (\log N)^C$*, we have*

$$R_{a,b}(N,c,d) \geqslant 0.8671 \prod_{\substack{p\mid c\\ p\nmid N\\ p>2}} \left(\frac{p-1}{p-2}\right) \frac{C(abN)N}{\varphi(c)ab(\log N)^2}.$$

Theorem 1.5. *For* $c \leqslant N^{0.028}$*, except for* $O\left(N^{0.028}(\log N)^{-A}\right)$ *exceptional values, we have*

$$R_{a,b}(N,c,d) \gg \prod_{\substack{p\mid c\\ p\nmid N\\ p>2}} \left(\frac{p-1}{p-2}\right) \frac{C(abN)N}{\varphi(c)ab(\log N)^2}.$$

Now we combine Theorem 1.4 with Theorems 1.2–1.3. Let $R^{\theta}_{a,b}(N,c,d)$ denote the number of primes $p \equiv d(\bmod c)$ such that $p \leqslant N^{\theta}$, ap and $N-ap$ are both square-free, $b \mid (N-ap)$, and $\frac{N-ap}{b}=P_2$. And let $R_{a,b}(N,c,d,\kappa)$ denote the number of primes $p \equiv d(\bmod c)$ such that $N/2-N^{\kappa} \leqslant p \leqslant N/2+N^{\kappa}$, ap and $N-ap$ are both square-free, $b \mid (N-ap)$, $\frac{N-ap}{b}=P_2$, and $N/2-N^{\kappa} \leqslant \frac{N-ap}{b} \leqslant N/2+N^{\kappa}$. Then by using a delicate sieve process similar to that of [2] and [3], we prove that

Theorem 1.6. *For* $c \leqslant (\log N)^C$ *and* $0.9409 \leqslant \theta \leqslant 1$*, we have*

$$R^{\theta}_{a,b}(N,c,d) \gg \prod_{\substack{p\mid c\\ p\nmid N\\ p>2}} \left(\frac{p-1}{p-2}\right) \frac{C(abN)N^{\theta}}{\varphi(c)ab(\log N)^2}.$$

Theorem 1.7. *For* $c \leqslant (\log N)^C$ *and* $0.97 \leqslant \kappa \leqslant 1$*, we have*

$$R_{a,b}(N,c,d,\kappa) \gg \prod_{\substack{p\mid c\\ p\nmid N\\ p>2}} \left(\frac{p-1}{p-2}\right) \frac{C(abN)N^{\kappa}}{\varphi(c)ab(\log N)^2}.$$

Clearly our Theorems 1.6–1.7 focus on the case when c is "small". For "large" c, we need to control the size of both θ (or κ) and c, and it seems hard to say what is "optimal". For example, we can show that for some $0 < \delta_1 < 0.028$, $0.9409 < \delta_2 < 1$ and $c \leqslant N^{\delta_1}$, except for $O\left(N^{\delta_1}(\log N)^{-A}\right)$ exceptional values, we have

$$R_{a,b}^{\delta_2}(N,c,d) \gg \prod_{\substack{p \mid c\\ p \nmid N\\ p>2}} \left(\frac{p-1}{p-2}\right) \frac{C(abN)N^{\delta_2}}{\varphi(c)ab(\log N)^2}, \tag{16}$$

but we cannot say what choice of δ_1 and δ_2 are the optimal values.

From our Theorems 1.4–1.7, it can be seen that the second aim of this paper is to construct some new results on the natural numbers of the form $ap+bP_2$ that generalize the results on the even numbers of the form $p+P_2$, $p+P_r$ and P_s+P_r.

The last theorem in this paper is a Goldbach-type upper bound result. Similar to [[26], Theorem 1. (2)], we also improve the upper bound of the number of primes p such that ap and $N-ap$ are both square-free, $b \mid (N-ap)$, and $\frac{N-ap}{b}$ is also a prime number. By using a delicate sieve process similar to that of [[30], Chap. 9.2], we prove that

Theorem 1.8.

$$\sum_{\substack{ap_1+bp_2=N\\ p_1 \text{ and } p_2 \text{ are primes}}} 1 \leqslant 7.928\frac{C(abN)N}{ab(\log N)^2}.$$

In fact, Lemmas 5.1–5.6 are also valid for the sets $\mathcal{A}_3$–$\mathcal{A}_6$ in section 2 if we make some suitable modifications. Since the detail of the proof of Theorems 1.3–1.8 is similar to those of [6], [23], [5], [30] and Theorems 1.1–1.2 so we omit them in this paper.

In this paper, we do not focus on Chen's double sieve technique. Maybe this can be used to improve our Theorems 1.1–1.8. For this, we refer the interested readers to [11], [39], [40] and Quarel's thesis [31].

It is worth to mention that if we relax the number of prime factors of $\frac{N-ap}{b}$ from two to three, we can extend the range of θ in Theorems 1.2 and 1.6 and κ in Theorems 1.3 and 1.7 to $0.838 \leqslant \theta \leqslant 1$ and $0.919 \leqslant \kappa \leqslant 1$ respectively. This improvement partially relies on the cancellation of the use of Wu's mean value theorem (see [37], this is because we don't need Chen's switching principle to prove such results that involve integers of the form $ap+bP_3$).

2. The sets we want to sieve

We first list the sets that we will work with later. Let $\theta = 0.9409$ in the following sections. Put

$$\begin{aligned}
\mathcal{A}_1 = \Bigg\{ &\frac{N-ap}{b} : p \leqslant \frac{N}{a},\ (p,abN)=1,\\
&p \equiv Na_{b^2}^{-1} + kb \left(\bmod b^2\right),\ 0 \leqslant k \leqslant b-1,\ (k,b)=1 \Bigg\},\\
\mathcal{A}_2 = \Bigg\{ &\frac{N-ap}{b} : p \leqslant \frac{N^\theta}{a},\ (p,abN)=1,\\
&p \equiv Na_{b^2}^{-1} + kb \left(\bmod b^2\right),\ 0 \leqslant k \leqslant b-1,\ (k,b)=1 \Bigg\},\\
\mathcal{A}_3 = \Bigg\{ &\frac{N-ap}{b} : \frac{N/2-N^{0.97}}{a} \leqslant p \leqslant \frac{N/2+N^{0.97}}{a},\ (p,abN)=1,\\
&p \equiv Na_{b^2}^{-1} + kb \left(\bmod b^2\right),\ 0 \leqslant k \leqslant b-1,\ (k,b)=1 \Bigg\},\\
\mathcal{A}_4 = \Bigg\{ &\frac{N-ap}{b} : p \leqslant \frac{N}{a},\ (p,abN)=1,\ p \equiv d(\bmod c), (c,d)=1,\\
&\left(\frac{N-ad}{b}, c\right)=1,\ p \equiv Na_{b^2}^{-1} + kb \left(\bmod b^2\right),\ 0 \leqslant k \leqslant b-1,\ (k,b)=1 \Bigg\},\\
\mathcal{A}_5 = \Bigg\{ &\frac{N-ap}{b} : p \leqslant \frac{N^\theta}{a},\ (p,abN)=1,\ p \equiv d(\bmod c),\ (c,d)=1,\\
&\left(\frac{N-ad}{b}, c\right)=1,\ p \equiv Na_{b^2}^{-1} + kb \left(\bmod b^2\right),\ 0 \leqslant k \leqslant b-1,\ (k,b)=1 \Bigg\},\\
\mathcal{A}_6 = \Bigg\{ &\frac{N-ap}{b} : \frac{N/2-N^{0.97}}{a} \leqslant p \leqslant \frac{N/2+N^{0.97}}{a},\ (p,abN)=1,\ p \equiv d(\bmod c),\\
&(c,d)=1,\ \left(\frac{N-ad}{b}, c\right)=1,\ p \equiv Na_{b^2}^{-1} + kb \left(\bmod b^2\right),\ 0 \leqslant k \leqslant b-1,\ (k,b)=1 \Bigg\},\\
\mathcal{B}_1 = \Bigg\{ &\frac{N-bp_1p_2p_3}{a} : (p_1p_2p_3,abN)=1,\ p_3 \leqslant \frac{N}{bp_1p_2},\\
&\left(\frac{N}{b}\right)^{\frac{1}{13.2}} \leqslant p_1 < \left(\frac{N}{b}\right)^{\frac{1}{3}} \leqslant p_2 < \left(\frac{N}{bp_1}\right)^{\frac{1}{2}},\\
&p_3 \equiv N\left(bp_1p_2\right)_{a^2}^{-1} + ja \left(\bmod a^2\right),\ 0 \leqslant j \leqslant a-1,\ (j,a)=1 \Bigg\},
\end{aligned}$$

$$\mathcal{B}_2=\Bigg\{\frac{N-bp_1p_2p_3}{a}:(p_1p_2p_3,abN)=1,\ \frac{N-N^\theta}{bp_1p_2}\leqslant p_3\leqslant\frac{N}{bp_1p_2},$$
$$\left(\frac{N}{b}\right)^{\frac{1}{14}}\leqslant p_1<\left(\frac{N}{b}\right)^{\frac{1}{3.1}}\leqslant p_2<\left(\frac{N}{bp_1}\right)^{\frac{1}{2}},$$
$$p_3\equiv N\left(bp_1p_2\right)_{a^2}^{-1}+ja\left(\bmod a^2\right),\ 0\leqslant j\leqslant a-1,\ (j,a)=1\Bigg\},$$

$$\mathcal{C}_1=\Bigg\{\frac{N-bmp_1p_2p_3p_4}{a}:(p_1p_2p_4,\ abN)=1,\ \left(\frac{N}{b}\right)^{\frac{1}{13.2}}\leqslant p_1<p_4<p_2<\left(\frac{N}{b}\right)^{\frac{1}{8.4}},$$
$$1\leqslant m\leqslant\frac{N}{bp_1p_2^2p_4},\ \left(m,p_1^{-1}abNP\left(p_4\right)\right)=1,$$
$$p_3\equiv N\left(bmp_1p_2p_4\right)_{a^2}^{-1}+ja\left(\bmod a^2\right),\ 0\leqslant j\leqslant a-1,\ (j,a)=1,$$
$$p_2<p_3<\min\left(\left(\frac{N}{b}\right)^{\frac{1}{8.4}},\frac{N}{bmp_1p_2p_4}\right)\Bigg\},$$

$$\mathcal{C}_2=\Bigg\{\frac{N-bmp_1p_2p_3p_4}{a}:(p_1p_2p_3p_4,abN)=1,\ \left(\frac{N}{b}\right)^{\frac{1}{14}}\leqslant p_1<p_2<p_3<p_4<\left(\frac{N}{b}\right)^{\frac{1}{8.8}},$$
$$mp_1p_2p_3p_4\equiv Nb_{a^2}^{-1}+ja\left(\bmod a^2\right),\ 0\leqslant j\leqslant a-1,\ (j,a)=1,$$
$$\frac{N-N^\theta}{bp_1p_2p_3p_4}\leqslant m\leqslant\frac{N}{bp_1p_2p_3p_4},\ \left(m,p_1^{-1}abNP\left(p_2\right)\right)=1\Bigg\},$$

$$\mathcal{C}_3=\Bigg\{\frac{N-bmp_1p_2p_3p_4}{a}:(p_1p_2p_3p_4,abN)=1,\ \left(\frac{N}{b}\right)^{\frac{1}{14}}\leqslant p_1<p_2<p_3<\left(\frac{N}{b}\right)^{\frac{1}{8.8}}\leqslant p_4<\left(\frac{N}{b}\right)^{\frac{4.5863}{14}}p_3^{-1},$$
$$mp_1p_2p_3p_4\equiv Nb_{a^2}^{-1}+ja\left(\bmod a^2\right),\ 0\leqslant j\leqslant a-1,\ (j,a)=1,$$
$$\frac{N-N^\theta}{bp_1p_2p_3p_4}\leqslant m\leqslant\frac{N}{bp_1p_2p_3p_4},\ \left(m,p_1^{-1}abNP\left(p_2\right)\right)=1\Bigg\},$$

$$\mathcal{E}_1=\left\{p_1p_2:(p_1p_2,abN)=1,\ \left(\frac{N}{b}\right)^{\frac{1}{13.2}}\leqslant p_1<\left(\frac{N}{b}\right)^{\frac{1}{3}}\leqslant p_2<\left(\frac{N}{bp_1}\right)^{\frac{1}{2}}\right\},$$

$$\mathcal{E}_2=\left\{p_1p_2:(p_1p_2,abN)=1,\ \left(\frac{N}{b}\right)^{\frac{1}{14}}\leqslant p_1<\left(\frac{N}{b}\right)^{\frac{1}{3.1}}\leqslant p_2<\left(\frac{N}{bp_1}\right)^{\frac{1}{2}}\right\},$$

$$\mathcal{F}_1=\Bigg\{mp_1p_2p_4:(p_1p_2p_4,abN)=1,\ \left(\frac{N}{b}\right)^{\frac{1}{13.2}}\leqslant p_1<p_4<p_2<\left(\frac{N}{b}\right)^{\frac{1}{8.4}},$$
$$1\leqslant m\leqslant\frac{N}{bp_1p_2^2p_4},\ \left(m,p_1^{-1}abNP\left(p_4\right)\right)=1\Bigg\},$$

$$\mathcal{F}_2=\Bigg\{mp_1p_2p_3p_4:(p_1p_2p_3p_4,abN)=1,\ \left(\frac{N}{b}\right)^{\frac{1}{14}}\leqslant p_1<p_2<p_3<p_4<\left(\frac{N}{b}\right)^{\frac{1}{8.8}},$$
$$\frac{N-N^\theta}{bp_1p_2p_3p_4}\leqslant m\leqslant\frac{N}{bp_1p_2p_3p_4},\ \left(m,p_1^{-1}abNP\left(p_2\right)\right)=1\Bigg\},$$

$$\mathcal{F}_3=\Bigg\{mp_1p_2p_3p_4:(p_1p_2p_3p_4,abN)=1,\ \left(\frac{N}{b}\right)^{\frac{1}{14}}\leqslant p_1<p_2<p_3<\left(\frac{N}{b}\right)^{\frac{1}{8.8}}\leqslant p_4<\left(\frac{N}{b}\right)^{\frac{4.5863}{14}}p_3^{-1},$$
$$\frac{N-N^\theta}{bp_1p_2p_3p_4}\leqslant m\leqslant\frac{N}{bp_1p_2p_3p_4},\ \left(m,p_1^{-1}abNP\left(p_2\right)\right)=1\Bigg\},$$

where $a_{b^2}^{-1}$ is the multiplicative inverse of a mod b^2, which exists by our assumption $(a,b)=1$.

3. Preliminary lemmas

Let $\mathcal{A}$ denote a finite set of positive integers, $\mathcal{P}$ denote an infinite set of primes and $z\geqslant 2$. Suppose that $|\mathcal{A}|\sim X_{\mathcal{A}}$ and for square-free d, put

$$\mathcal{P}=\{p:(p,N)=1\},\quad \mathcal{P}(r)=\{p:p\in\mathcal{P},(p,r)=1\},$$
$$P(z)=\prod_{\substack{p\in\mathcal{P}\\p<z}}p,\quad \mathcal{A}_d=\{a:ad\in\mathcal{A}\},\quad S(\mathcal{A};\mathcal{P},z)=\sum_{\substack{a\in\mathcal{A}\\(a,P(z))=1}}1.$$

Lemma 3.1. *([19], Lemma 1]). If*

$$\sum_{z_1\leqslant p<z_2}\frac{\omega(p)}{p}=\log\frac{\log z_2}{\log z_1}+O\left(\frac{1}{\log z_1}\right),\quad z_2>z_1\geqslant 2,$$

where $\omega(d)$ is a multiplicative function, $0 \leqslant \omega(p) < p, X > 1$ is independent of d. Then

$$S(\mathcal{A};\mathcal{P},z) \geqslant X_{\mathcal{A}} W(z)\left\{f\left(\frac{\log D}{\log z}\right) + O\left(\frac{1}{\log^{\frac{1}{3}} D}\right)\right\} - \sum_{\substack{n \leqslant D \\ n \mid P(z)}} |\eta(X_{\mathcal{A}}, n)|$$

$$S(\mathcal{A};\mathcal{P},z) \leqslant X_{\mathcal{A}} W(z)\left\{F\left(\frac{\log D}{\log z}\right) + O\left(\frac{1}{\log^{\frac{1}{3}} D}\right)\right\} + \sum_{\substack{n \leqslant D \\ n \mid P(z)}} |\eta(X_{\mathcal{A}}, n)|$$

where

$$W(z) = \prod_{\substack{p<z \\ (p,N)=1}} \left(1 - \frac{\omega(p)}{p}\right), \quad \eta(X_{\mathcal{A}}, n) = |\mathcal{A}_n| - \frac{\omega(n)}{n} X_{\mathcal{A}} = \sum_{\substack{a \in \mathcal{A} \\ a \equiv 0 (\bmod n)}} 1 - \frac{\omega(n)}{n} X_{\mathcal{A}},$$

γ denote the Euler's constant, $f(s)$ and $F(s)$ are determined by the following differential-difference equation

$$\begin{cases} F(s) = \frac{2e^{\gamma}}{s}, \quad f(s) = 0, & 0 < s \leqslant 2, \\ (sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), & s \geqslant 2. \end{cases}$$

Lemma 3.2. *([2], Lemma 2).*

$$\begin{aligned}
F(s) =& \frac{2e^{\gamma}}{s}, \quad 0 < s \leqslant 3; \\
F(s) =& \frac{2e^{\gamma}}{s}\left(1 + \int_2^{s-1} \frac{\log(t-1)}{t} dt\right), \quad 3 \leqslant s \leqslant 5; \\
F(s) =& \frac{2e^{\gamma}}{s}\left(1 + \int_2^{s-1} \frac{\log(t-1)}{t} dt + \int_2^{s-3} \frac{\log(t-1)}{t} dt \int_{t+2}^{s-1} \frac{1}{u} \log\frac{u-1}{t+1} du\right), \quad 5 \leqslant s \leqslant 7; \\
f(s) =& \frac{2e^{\gamma}\log(s-1)}{s}, \quad 2 \leqslant s \leqslant 4; \\
f(s) =& \frac{2e^{\gamma}}{s}\left(\log(s-1) + \int_3^{s-1} \frac{dt}{t} \int_2^{t-1} \frac{\log(u-1)}{u} du\right), \quad 4 \leqslant s \leqslant 6; \\
f(s) =& \frac{2e^{\gamma}}{s}\left(\log(s-1) + \int_3^{s-1} \frac{dt}{t} \int_2^{t-1} \frac{\log(u-1)}{u} du \right. \\
&\left. + \int_2^{s-4} \frac{\log(t-1)}{t} dt \int_{t+2}^{s-2} \frac{1}{u} \log\frac{u-1}{t+1} \log\frac{s}{u+2} du\right), \quad 6 \leqslant s \leqslant 8.
\end{aligned}$$

Lemma 3.3. *([2], Lemma 4). Let*

$$x > 1, \quad z = x^{\frac{1}{u}}, \quad Q(z) = \prod_{p<z} p.$$

Then for $u \geqslant 1$, we have

$$\sum_{\substack{n \leqslant x \\ (n,Q(z))=1}} 1 = w(u)\frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right),$$

where $w(u)$ is determined by the following differential-difference equation

$$\begin{cases} w(u) = \frac{1}{u}, & 1 \leqslant u \leqslant 2, \\ (uw(u))' = w(u-1), & u \geqslant 2. \end{cases}$$

Moreover, we have

$$\begin{cases} w(u) \leqslant \frac{1}{1.763}, & u \geqslant 2, \\ w(u) < 0.5644, & u \geqslant 3, \\ w(u) < 0.5617, & u \geqslant 4. \end{cases}$$

Lemma 3.4. *([4], Lemma 2.6], [6], Lemma 4]). Let*

$$x > 1, \quad x^{\frac{19}{24}+\varepsilon} \leqslant y_1 \leqslant \frac{x}{\log x}, \quad x^{\frac{3}{5}} \leqslant y_2 < x, \quad z = x^{\frac{1}{u}}, \quad Q(z) = \prod_{p<z} p.$$

Then for $u > 1$, we have

$$\sum_{\substack{x-y_1 \leqslant n \leqslant x \\ (n,Q(z))=1}} 1 = w(u)\frac{y_1}{\log z} + O\left(\frac{y_1}{\log^2 z}\right),$$

$$\sum_{\substack{x \leqslant n < x+y_2 \\ (n,Q(z))=1}} 1 = w(u)\frac{y_2}{\log z} + O\left(\frac{y_2}{\log^2 z}\right),$$

where $w(u)$ is defined in Lemma 3.3.

Lemma 3.5. *If we define the function ω as $\omega(p)=0$ for primes $p \mid abN$ and $\omega(p)=\frac{p}{p-1}$ for other primes and $N^{\frac{1}{\alpha}-\varepsilon}<z\leqslant N^{\frac{1}{\alpha}}$, then we have*

$$W(z)=\frac{2\alpha e^{-\gamma}C(abN)(1+o(1))}{\log N}.$$

Proof. By [[26], Lemma 2] we have

$$W(z)=\frac{N}{\varphi(N)}\prod_{(p,N)=1}\left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-1}\frac{e^{-\gamma}}{\log z}\left(1+O\left(\frac{1}{\log z}\right)\right).$$

Since $2 \mid abN$, we have

$$\begin{aligned}
W(z)&=\frac{N}{\varphi(N)}\prod_{(p,N)=1}\left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-1}\frac{e^{-\gamma}}{\log z}\left(1+O\left(\frac{1}{\log z}\right)\right)\\
&=\prod_{p\mid N}\frac{p}{p-1}\prod_{(p,N)=1}\left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-1}\frac{\alpha e^{-\gamma}(1+o(1))}{\log N}\\
&=\prod_{p\mid N}\frac{p}{p-1}\prod_{p\mid ab}\left(1-\frac{1}{p}\right)^{-1}\prod_{(p,abN)=1}\left(1-\frac{1}{p-1}\right)\left(1-\frac{1}{p}\right)^{-1}\frac{\alpha e^{-\gamma}(1+o(1))}{\log N}\\
&=\prod_{\substack{p\mid N\\ p>2}}\frac{p}{p-1}\prod_{\substack{p\mid ab\\ p>2}}\frac{p}{p-1}\frac{\prod_{p>2}\frac{p(p-2)}{(p-1)^2}}{\prod_{\substack{p\mid abN\\ p>2}}\frac{p(p-2)}{(p-1)^2}}\frac{2\alpha e^{-\gamma}(1+o(1))}{\log N}\\
&=\frac{2\alpha e^{-\gamma}C(abN)(1+o(1))}{\log N}.
\end{aligned}$$

□

4. Mean value theorems

Now we provide some mean value theorems which will be used in bounding various sieve error terms later. The first two lemmas come from Pan and Pan's book [30] and they were first proven by Pan, Ding and Wang.

Lemma 4.1. *([30], Corollary 8.2]). Let*

$$\pi(x;k,d,l)=\sum_{\substack{kp\leq x\\ kp\equiv l(\bmod d)}}1$$

and let $g(k)$ be a real function, $g(k)\ll 1$. Then, for any given constant $A>0$, there exists a constant $B=B(A)>0$ such that

$$\sum_{d\leqslant x^{1/2}(\log x)^{-B}}\max_{y\leqslant x}\max_{(l,d)=1}\left|\sum_{\substack{k\leqslant E(x)\\ (k,d)=1}}g(k)H(y;k,d,l)\right|\ll\frac{x}{\log^A x},$$

where

$$H(y;k,d,l)=\pi(y;k,d,l)-\frac{1}{\varphi(d)}\pi(y;k,1,1)=\sum_{\substack{kp\leqslant y\\ kp\equiv l(\bmod d)}}1-\frac{1}{\varphi(d)}\sum_{kp\leqslant y}1,$$

$$\frac{1}{2}\leqslant E(x)\ll x^{1-\alpha},\quad 0<\alpha\leqslant 1,\quad B(A)=\frac{3}{2}A+17.$$

Lemma 4.2. *([30], Corollary 8.3 and 8.4]). Let $r_1(y)$ be a positive function depending on x and satisfying $r_1(y)\ll x^\alpha$ for $y\leqslant x$. Then under the conditions in Lemma 4.1, we have*

$$\sum_{d\leqslant x^{1/2}(\log x)^{-B}}\max_{y\leqslant x}\max_{(l,d)=1}\left|\sum_{\substack{k\leqslant E(x)\\ (k,d)=1}}g(k)H\left(kr_1(y);k,d,l\right)\right|\ll\frac{x}{\log^A x}.$$

Let $r_2(k)$ be a positive function depending on x and y such that $kr_2(k)\ll x$ for $k\leqslant E(x)$, $y\leqslant x$. Then under the conditions in Lemma 4.1, we have

$$\sum_{d\leqslant x^{1/2}(\log x)^{-B}}\max_{y\leqslant x}\max_{(l,d)=1}\left|\sum_{\substack{k\leqslant E(x)\\ (k,d)=1}}g(k)H\left(kr_2(k);k,d,l\right)\right|\ll\frac{x}{\log^A x}.$$

The next two lemmas were first proven by Wu [37], and they are the "short interval" version of Lemmas 4.1–4.2. These will help us deal with the sieve error terms involved in evaluation of S_4' and S_7'.

Lemma 4.3. *([37], Theorem 2]). Let $g(k)$ be a real function such that*

$$\sum_{k\leqslant x}\frac{g^2(k)}{k}\ll\log^C x$$

for some $C>0$. Then, for any given constant $A>0$, there exists a constant $B=B(A,C)>0$ such that

$$\sum_{d\leqslant x^{t-1/2}(\log x)^{-B}}\max_{x/2\leqslant y\leqslant x}\max_{(l,d)=1}\max_{h\leqslant x^t}\left|\sum_{\substack{k\leqslant x^\beta\\(k,d)=1}}g(k)H_1(y,h,k,d,l)\right|\ll\frac{x^t}{\log^A x},$$

where

$$\begin{aligned}H_1(y,h,k,d,l)&=(\pi(y+h;k,d,l)-\pi(y;k,d,l))\\&\quad-\frac{1}{\varphi(d)}\left(\pi(y+h;k,1,1)-(\pi(y;k,1,1)\right)\\&=\sum_{\substack{y<kp\leqslant y+h\\kp\equiv l(\bmod d)}}1-\frac{1}{\varphi(d)}\sum_{y<kp\leqslant y+h}1,\\ \frac{3}{5}<t\leqslant 1,&\quad 0\leqslant\beta<\frac{5t-3}{2},\quad B(A,C)=3A+C+34.\end{aligned}$$

Lemma 4.4. *([3], Lemma 7], [6], Remark]). Let $g(k)$ be a real function such that*

$$\sum_{k\leqslant x}\frac{g^2(k)}{k}\ll\log^C x$$

for some $C>0$. Let $r_1(k,h)$ and $r_2(k,h)$ be positive function such that

$$y\leqslant kr_1(k,h),kr_2(k,h)\leqslant y+h.$$

Then, for any given constant $A>0$, there exists a constant $B=B(A,C)>0$ such that

$$\sum_{d\leqslant x^{t-1/2}(\log x)^{-B}}\max_{x/2\leqslant y\leqslant x}\max_{(l,d)=1}\max_{h\leqslant x^t}\left|\sum_{\substack{k\leqslant x^\beta\\(k,d)=1}}g(k)H_2(y,h,k,d,l)\right|\ll\frac{x^t}{\log^A x},$$

where

$$\begin{aligned}H_2(y,h,k,d,l)&=(\pi(kr_2(k,h);k,d,l)-\pi(kr_1(k,h);k,d,l))\\&\quad-\frac{1}{\varphi(d)}\left(\pi(kr_2(k,h);k,1,1)-(\pi(kr_1(k,h);k,1,1)\right)\\&=\sum_{\substack{kr_1(k,h)<kp\leqslant kr_2(k,h)\\kp\equiv l(\bmod d)}}1-\frac{1}{\varphi(d)}\sum_{kr_1(k,h)<kp\leqslant kr_2(k,h)}1,\\ \frac{3}{5}<t\leqslant 1,&\quad 0\leqslant\beta<\frac{5t-3}{2},\quad B(A,C)=3A+C+34.\end{aligned}$$

In [3], Cai said that we faced the difficulty that cannot be overcome by our Lemmas 4.3–4.4 which are not sufficient to deal with some of the sieve error terms involved. Actually, the function $g(k)$ cannot be well defined to control the sieve error terms occurred in the evaluation of S_6'. (i.e. $\frac{5\theta-3}{2}<\frac{13}{14}$). So we need a new mean value theorem to overcome that. The next lemma is a new mean value theorem for products of large primes over short intervals and it was first proven by Cai [3]. This lemma will help us deal with the sieve error terms involved in evaluation of S_6'.

Lemma 4.5. *For $j=2,3$ and any given constant $A>0$, there exists a constant $B=B(A)>0$ such that*

$$\sum_{d\leqslant x^{\theta-1/2}(\log x)^{-B}}\max_{(l,d)=1}\left|\sum_{\substack{mp_1p_2p_3p_4\in\mathcal{F}_j\\mp_1p_2p_3p_4\equiv l(\bmod d)}}1-\frac{1}{\varphi(d)}\sum_{\substack{mp_1p_2p_3p_4\in\mathcal{F}_j\\(mp_1p_2p_3p_4,d)=1}}1\right|\ll\frac{x^\theta}{\log^A x}.$$

Proof. This result can be proved in the same way as [[3], Lemma 8] by showing that for $j=2,3$ and $5\leqslant r\leqslant 14$, the bounds

$$\sum_{d\leqslant x^{\theta-1/2}(\log x)^{-B}}\max_{(l,d)=1}\left|\sum_{\substack{p_1p_2\cdots p_r\in\mathcal{F}_j\\p_1p_2\cdots p_r\equiv l(\bmod d)}}1-\frac{1}{\varphi(d)}\sum_{\substack{p_1p_2\cdots p_r\in\mathcal{F}_j\\(p_1p_2\cdots p_r,d)=1}}1\right|\ll\frac{x^\theta}{\log^A x}$$

hold. ∎

The following lemmas are the "arithmetical progression with almost all large c" versions of the above lemmas, and they will help us prove Theorem 1.5. We can also obtain variants of Theorems 1.6–1.7 with "large" c by using the following lemmas.

Lemma 4.6. *([5], Lemma 4]). For any given constant $A>0$, under the conditions in Lemmas 4.1–4.2, there exists a constant $B=B(A)>0$ such that for $c\leqslant x^{0.028}$, except for $O\left(x^{0.028}(\log x)^{-A}\right)$ exceptional values, we have*

$$R_1=\sum_{d\leqslant\left(x^{1/2}(\log x)^{-B}\right)/c}\max_{y\leqslant x}\max_{(l,dc)=1}\left|\sum_{\substack{k\leqslant E(x)\\(k,d)=1}}g(k)H(y;k,dc,l)\right|\ll\frac{x^{1-0.028}}{\log^A x},$$

$$R_2=\sum_{d\leqslant\left(x^{1/2}(\log x)^{-B}\right)/c}\max_{y\leqslant x}\max_{(l,dc)=1}\left|\sum_{\substack{k\leqslant E(x)\\(k,d)=1}}g(k)H(kr_1(y);k,dc,l)\right|\ll\frac{x^{1-0.028}}{\log^A x},$$

$$R_3=\sum_{d\leqslant\left(x^{1/2}(\log x)^{-B}\right)/c}\max_{y\leqslant x}\max_{(l,dc)=1}\left|\sum_{\substack{k\leqslant E(x)\\(k,d)=1}}g(k)H(kr_2(k);k,dc,l)\right|\ll\frac{x^{1-0.028}}{\log^A x}.$$

Proof. We prove Lemma 4.6 in the case R_1 only, the same argument can be applied to the cases R_2 and R_3. Let $\tau(d)$ denote the divisor function, By Lemma 4.1 and similar arguments as in [[28], Lemma 3], we have

$$\begin{aligned}\sum_{c\leqslant N^{0.028}}R_1&=\sum_{c\leqslant N^{0.028}}\sum_{d\leqslant\left(x^{1/2}(\log x)^{-B}\right)/c}\max_{y\leqslant x}\max_{(l,dc)=1}\left|\sum_{\substack{k\leqslant E(x)\\(k,d)=1}}g(k)H(y;k,dc,l)\right|\\&\leqslant\sum_{d\leqslant x^{1/2}(\log x)^{-B}}\tau(d)\max_{y\leqslant x}\max_{(l,d)=1}\left|\sum_{\substack{k\leqslant E(x)\\(k,d)=1}}g(k)H(y;k,d,l)\right|\ll\frac{x}{\log^{2A}x},\end{aligned}$$

$$\sum_{\substack{c\leqslant N^{0.028}\\R_1>\frac{x^{1-0.028}}{\log^A x}}}1\ll\frac{\log^A x}{x^{1-0.028}}\sum_{c\leqslant N^{0.028}}R_1\ll\frac{x^{0.028}}{\log^A x}.$$

Now the proof of Lemma 4.6 is completed. □

Lemma 4.7. *For any given constant $A>0$, under the conditions in Lemmas 4.3–4.4, there exists a constant $B=B(A,C)>0$ such that for $c\leqslant x^{0.028}$, except for $O\left(x^{0.028}(\log x)^{-A}\right)$ exceptional values, we have*

$$R_4=\sum_{d\leqslant\left(x^{t-1/2}(\log x)^{-B}\right)/c}\max_{x/2\leqslant y\leqslant x}\max_{(l,dc)=1}\max_{h\leqslant x^t}\left|\sum_{\substack{k\leqslant x^\beta\\(k,d)=1}}g(k)H_1(y,h,k,dc,l)\right|\ll\frac{x^{t-0.028}}{\log^A x},$$

$$R_5=\sum_{d\leqslant\left(x^{t-1/2}(\log x)^{-B}\right)/c}\max_{x/2\leqslant y\leqslant x}\max_{(l,dc)=1}\max_{h\leqslant x^t}\left|\sum_{\substack{k\leqslant x^\beta\\(k,d)=1}}g(k)H_2(y,h,k,dc,l)\right|\ll\frac{x^{t-0.028}}{\log^A x}.$$

Proof. We prove Lemma 4.7 in the case R_4 only, the same argument can be applied to the case R_5. By Lemma 4.3 and similar arguments as in [[28], Lemma 3], we have

$$\begin{aligned}\sum_{c\leqslant N^{0.028}}R_4&=\sum_{c\leqslant N^{0.028}}\sum_{d\leqslant\left(x^{t-1/2}(\log x)^{-B}\right)/c}\max_{x/2\leqslant y\leqslant x}\max_{(l,dc)=1}\max_{h\leqslant x^t}\left|\sum_{\substack{k\leqslant x^\beta\\(k,d)=1}}g(k)H_1(y,h,k,dc,l)\right|\\&\leqslant\sum_{d\leqslant x^{t-1/2}(\log x)^{-B}}\tau(d)\max_{x/2\leqslant y\leqslant x}\max_{(l,d)=1}\max_{h\leqslant x^t}\left|\sum_{\substack{k\leqslant x^\beta\\(k,d)=1}}g(k)H_1(y,h,k,d,l)\right|\ll\frac{x^t}{\log^{2A}x},\end{aligned}$$

$$\sum_{\substack{c\leqslant N^{0.028}\\ R_4>\frac{x^{t-0.028}}{\log^A x}}} 1 \ll \frac{\log^A x}{x^{t-0.028}} \sum_{c\leqslant N^{0.028}} R_4 \ll \frac{x^{0.028}}{\log^A x}.$$

Now the proof of Lemma 4.7 is completed. □

Lemma 4.8. *For $j=2,3$, let*

$$\mathcal{F}_j^{\prime}=\left\{m p_1 p_2 p_3 p_4: m p_1 p_2 p_3 p_4 \in \mathcal{F}_j,\left(p_1 p_2 p_3 p_4, c\right)=1\right\},$$

then for any given constant $A>0$, there exists a constant $B=B(A)>0$ such that for $c\leqslant x^{0.028}$, except for $O\left(x^{0.028}(\log x)^{-A}\right)$ exceptional values, we have

$$R_j^{\prime}=\sum_{d\leqslant\left(x^{\theta-1/2}(\log x)^{-B}\right)/c} \max_{(l,dc)=1}\left|\sum_{\substack{mp_1p_2p_3p_4\in\mathcal{F}_j^{\prime}\\ mp_1p_2p_3p_4\equiv l(\bmod dc)}} 1-\frac{1}{\varphi(dc)} \sum_{\substack{mp_1p_2p_3p_4\in\mathcal{F}_j^{\prime}\\ (mp_1p_2p_3p_4,dc)=1}} 1\right| \ll \frac{x^{\theta-0.028}}{\log^A x}.$$

Proof. We prove Lemma 4.8 in the case $R_2^{\prime}$ only, the same argument can be applied to the case $R_3^{\prime}$. By Lemma 4.5 and similar arguments as in [[28], Lemma 3], we have

$$\begin{aligned}
\sum_{c\leqslant N^{0.028}} R_2^{\prime}= & \sum_{c\leqslant N^{0.028}} \sum_{d\leqslant\left(x^{\theta-1/2}(\log x)^{-B}\right)/c} \max_{(l,dc)=1}\left|\sum_{\substack{mp_1p_2p_3p_4\in\mathcal{F}_j^{\prime}\\ mp_1p_2p_3p_4\equiv l(\bmod dc)}} 1-\frac{1}{\varphi(dc)} \sum_{\substack{mp_1p_2p_3p_4\in\mathcal{F}_j^{\prime}\\ (mp_1p_2p_3p_4,dc)=1}} 1\right| \\
\leqslant & \sum_{d\leqslant x^{\theta-1/2}(\log x)^{-B}} \tau(d) \max_{(l,d)=1}\left|\sum_{\substack{mp_1p_2p_3p_4\in\mathcal{F}_j\\ mp_1p_2p_3p_4\equiv l(\bmod d)}} 1-\frac{1}{\varphi(d)} \sum_{\substack{mp_1p_2p_3p_4\in\mathcal{F}_j\\ (mp_1p_2p_3p_4,d)=1}} 1\right| \ll \frac{x^\theta}{\log^{2A} x},
\end{aligned}$$

$$\sum_{\substack{c\leqslant N^{0.028}\\ R_2^{\prime}>\frac{x^{\theta-0.028}}{\log^A x}}} 1 \ll \frac{\log^A x}{x^{\theta-0.028}} \sum_{c\leqslant N^{0.028}} R_2^{\prime} \ll \frac{x^{0.028}}{\log^A x}.$$

Now the proof of Lemma 4.8 is completed. □

5. Weighted sieve method

Now we provide the delicate weighted sieves in order to prove our Theorems 1.1–1.8.

Lemma 5.1. *Let $\mathcal{A}=\mathcal{A}_1$ in section 2 and $0<\alpha<\beta\leqslant\frac{1}{3}$. Then we have*

$$\begin{aligned}
2R_{a,b}(N)\geqslant & 2S\left(\mathcal{A};\mathcal{P},\left(\frac{N}{b}\right)^\alpha\right)-\sum_{\substack{(\frac{N}{b})^\alpha\leqslant p<(\frac{N}{b})^\beta\\ (p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P},\left(\frac{N}{b}\right)^\alpha\right) \\
& -\sum_{\substack{(\frac{N}{b})^\alpha\leqslant p_1<(\frac{N}{b})^\beta\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)-2\sum_{\substack{(\frac{N}{b})^\beta\leqslant p_1<p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right) \\
& +\sum_{\substack{(\frac{N}{b})^\alpha\leqslant p_1<p_2<p_3<(\frac{N}{b})^\beta\\ (p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(p_1),p_2\right)+O\left(N^{1-\alpha}\right).
\end{aligned}$$

Proof. It is similar to that of [[2], Lemma 5]. By the trivial inequality

$$R_{a,b}(N)\geqslant S\left(\mathcal{A};\mathcal{P},\left(\frac{N}{b}\right)^\beta\right)-\sum_{\substack{(\frac{N}{b})^\beta\leqslant p_1<p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)$$

and Buchstab's identity we have

$$R_{a,b}(N)\geqslant S\left(\mathcal{A};\mathcal{P},\left(\frac{N}{b}\right)^\beta\right)-\sum_{\substack{(\frac{N}{b})^\beta\leqslant p_1<p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)$$

$$
\begin{aligned}
=&S\left(\mathcal{A};\mathcal{P},\left(\frac{N}{b}\right)^{\alpha}\right)-\sum_{\substack{(\frac{N}{b})^{\alpha}\leqslant p<(\frac{N}{b})^{\beta}\\(p,N)=1}}S\left(\mathcal{A}_p;\mathcal{P},\left(\frac{N}{b}\right)^{\alpha}\right)\\
&+\sum_{\substack{(\frac{N}{b})^{\alpha}\leqslant p_1<p_2<(\frac{N}{b})^{\beta}\\(p_1p_2,N)=1}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P},p_1\right)-\sum_{\substack{(\frac{N}{b})^{\beta}\leqslant p_1<p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right).
\end{aligned}\tag{17}
$$

On the other hand, we have the trivial inequality

$$
\begin{aligned}
R_{a,b}(N)\geqslant&S\left(\mathcal{A};\mathcal{P},\left(\frac{N}{b}\right)^{\alpha}\right)-\sum_{\substack{(\frac{N}{b})^{\alpha}\leqslant p_1<p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)\\
=&S\left(\mathcal{A};\mathcal{P},\left(\frac{N}{b}\right)^{\alpha}\right)-\sum_{\substack{(\frac{N}{b})^{\alpha}\leqslant p_1<p_2<(\frac{N}{b})^{\beta}\\(p_1p_2,N)=1}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\alpha}\leqslant p_1<(\frac{N}{b})^{\beta}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\beta}\leqslant p_1<p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right).
\end{aligned}\tag{18}
$$

Now by Buchstab's identity we have

$$
\begin{aligned}
&\sum_{\substack{(\frac{N}{b})^{\alpha}\leqslant p_1<p_2<(\frac{N}{b})^{\beta}\\(p_1p_2,N)=1}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P},p_1\right)-\sum_{\substack{(\frac{N}{b})^{\alpha}\leqslant p_1<p_2<(\frac{N}{b})^{\beta}\\(p_1p_2,N)=1}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)\\
=&\sum_{\substack{(\frac{N}{b})^{\alpha}\leqslant p_1<p_2<p_3<(\frac{N}{b})^{\beta}\\(p_1p_2p_3,N)=1}}S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(p_1),p_2\right)+O\left(N^{1-\alpha}\right),
\end{aligned}\tag{19}
$$

where the trivial bound

$$
\sum_{\substack{(\frac{N}{b})^{\alpha}\leqslant p_1<p_2<(\frac{N}{b})^{\beta}\\(p_1p_2,N)=1}}S\left(\mathcal{A}_{p_1^2p_2};\mathcal{P},p_1\right)\ll N^{1-\alpha}\tag{20}
$$

is used. Now we add (17) and (18) and by (19), Lemma 5.1 follows. □

Lemma 5.2. *Let $\mathcal{A}=\mathcal{A}_2$ in section 2 and $0<\alpha<\beta\leqslant\frac{1}{3}$. Then we have*

$$
\begin{aligned}
2R_{a,b}^{\theta}(N)\geqslant&2S\left(\mathcal{A};\mathcal{P},\left(\frac{N}{b}\right)^{\alpha}\right)-\sum_{\substack{(\frac{N}{b})^{\alpha}\leqslant p<(\frac{N}{b})^{\beta}\\(p,N)=1}}S\left(\mathcal{A}_p;\mathcal{P},\left(\frac{N}{b}\right)^{\alpha}\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\alpha}\leqslant p_1<(\frac{N}{b})^{\beta}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)-2\sum_{\substack{(\frac{N}{b})^{\beta}\leqslant p_1<p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)\\
&+\sum_{\substack{(\frac{N}{b})^{\alpha}\leqslant p_1<p_2<p_3<(\frac{N}{b})^{\beta}\\(p_1p_2p_3,N)=1}}S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(p_1),p_2\right)+O\left(N^{1-\alpha}\right).
\end{aligned}
$$

Proof. It is similar to that of Lemma 5.1 so we omit it here. □

Lemma 5.3. *Let $\mathcal{A}=\mathcal{A}_1$ in section 2, then we have*

$$
\begin{aligned}
4R_{a,b}(N)\geqslant&3S\left(\mathcal{A};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right)+S\left(\mathcal{A};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{8.4}}\right)\\
&+\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<p_2<(\frac{N}{b})^{\frac{1}{8.4}}\\(p_1p_2,N)=1}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{8.4}}\leqslant p_2<(\frac{N}{b})^{\frac{4.6}{13.2}}p_1^{-1}\\(p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p<(\frac{N}{b})^{\frac{4.1001}{13.2}}\\(p,N)=1}} S\left(\mathcal{A}_{p};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p<(\frac{N}{b})^{\frac{3.6}{13.2}}\\(p,N)=1}} S\left(\mathcal{A}_{p};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3.604}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),\left(\frac{N}{bp_1p_2}\right)^{\frac{1}{2}}\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{4.1001}{13.2}}\leqslant p<(\frac{N}{b})^{\frac{1}{3}}\\(p,N)=1}} S\left(\mathcal{A}_{p};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{3.6}{13.2}}\leqslant p<(\frac{N}{b})^{\frac{1}{3.604}}\\(p,N)=1}} S\left(\mathcal{A}_{p};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{8.4}}\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<p_2<p_3<p_4<(\frac{N}{b})^{\frac{1}{8.4}}\\(p_1p_2p_3p_4,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<p_2<p_3<(\frac{N}{b})^{\frac{1}{8.4}}\leqslant p_4<(\frac{N}{b})^{\frac{4.6}{13.2}}p_3^{-1}\\(p_1p_2p_3p_4,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right)\\
&-2\sum_{\substack{(\frac{N}{b})^{\frac{1}{3.604}}\leqslant p_1<p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)+O\left(N^{\frac{12.2}{13.2}}\right)\\
=&(3S_{11}+S_{12})+(S_{21}+S_{22})-(S_{31}+S_{32})-(S_{41}+S_{42})\\
&-(S_{51}+S_{52})-(S_{61}+S_{62})-2S_7+O\left(N^{\frac{12.2}{13.2}}\right)\\
=&S_1+S_2-S_3-S_4-S_5-S_6-2S_7+O\left(N^{\frac{12.2}{13.2}}\right).
\end{aligned}
$$

Proof. It is similar to that of [[2], Lemma 6]. By Buchstab's identity, we have

$$
\begin{aligned}
S\left(\mathcal{A};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{8.4}}\right)=&S\left(\mathcal{A};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p<(\frac{N}{b})^{\frac{1}{8.4}}\\(p,N)=1}} S\left(\mathcal{A}_{p};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right)\\
&+\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<p_2<(\frac{N}{b})^{\frac{1}{8.4}}\\(p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<p_2<p_3<(\frac{N}{b})^{\frac{1}{8.4}}\\(p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P},p_1\right),
\end{aligned}
\tag{21}
$$

$$\sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p < (\frac{N}{b})^{\frac{3.6}{13.2}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}, \left(\frac{N}{b}\right)^{\frac{1}{8.4}}\right)$$

$$\leqslant \sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p < (\frac{N}{b})^{\frac{3.6}{13.2}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}, \left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right)$$

$$- \sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}} \leqslant p_1 < (\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_2 < (\frac{N}{b})^{\frac{4.6}{13.2}} p_1^{-1} \\ (p_1 p_2,N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}, \left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right)$$

$$+ \sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}} \leqslant p_1 < p_2 < (\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_3 < (\frac{N}{b})^{\frac{4.6}{13.2}} p_2^{-1} \\ (p_1 p_2 p_3,N)=1}} S\left(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}, p_1\right), \tag{22}$$

$$\sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_1 < (\frac{N}{b})^{\frac{1}{3.604}} \leqslant p_2 < (\frac{N}{bp_1})^{\frac{1}{2}} \\ (p_1 p_2,N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2\right)$$

$$= \sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_1 < (\frac{N}{b})^{\frac{1}{3.604}} \leqslant p_2 < (\frac{N}{bp_1})^{\frac{1}{3}} \\ (p_1 p_2,N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2\right)$$

$$+ \sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_1 < (\frac{N}{b})^{\frac{1}{3.604}},\ (\frac{N}{bp_1})^{\frac{1}{3}} \leqslant p_2 < (\frac{N}{bp_1})^{\frac{1}{2}} \\ (p_1 p_2,N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2\right). \tag{23}$$

If $p_2 \leqslant \left(\frac{N}{bp_1}\right)^{\frac{1}{3}}$, then $p_2 \leqslant \left(\frac{N}{bp_1 p_2}\right)^{\frac{1}{2}}$ and by Buchstab's identity we have

$$\sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_1 < (\frac{N}{b})^{\frac{1}{3.604}} \leqslant p_2 < (\frac{N}{bp_1})^{\frac{1}{3}} \\ (p_1 p_2,N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2\right)$$

$$= \sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_1 < (\frac{N}{b})^{\frac{1}{3.604}} \leqslant p_2 < (\frac{N}{bp_1})^{\frac{1}{3}} \\ (p_1 p_2,N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), \left(\frac{N}{bp_1 p_2}\right)^{\frac{1}{2}}\right)$$

$$+ \sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_1 < (\frac{N}{b})^{\frac{1}{3.604}} \leqslant p_2 \leqslant p_3 < (\frac{N}{bp_1 p_2})^{\frac{1}{2}} \\ (p_1 p_2 p_3,N)=1}} S\left(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(p_1 p_2), p_3\right). \tag{24}$$

On the other hand, if $p_2 \geqslant \left(\frac{N}{bp_1}\right)^{\frac{1}{3}}$, then $p_2 \geqslant \left(\frac{N}{bp_1 p_2}\right)^{\frac{1}{2}}$ and we have

$$\sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_1 < (\frac{N}{b})^{\frac{1}{3.604}},\ (\frac{N}{bp_1})^{\frac{1}{3}} \leqslant p_2 < (\frac{N}{bp_1})^{\frac{1}{2}} \\ (p_1 p_2,N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2\right)$$

$$\leqslant \sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_1 < (\frac{N}{b})^{\frac{1}{3.604}},\ (\frac{N}{bp_1})^{\frac{1}{3}} \leqslant p_2 < (\frac{N}{bp_1})^{\frac{1}{2}} \\ (p_1 p_2,N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), \left(\frac{N}{bp_1 p_2}\right)^{\frac{1}{2}}\right). \tag{25}$$

By (23)–(25) we get

$$\sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_1 < (\frac{N}{b})^{\frac{1}{3.604}} \leqslant p_2 < (\frac{N}{bp_1})^{\frac{1}{2}} \\ (p_1 p_2,N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2\right)$$

$$\leqslant \sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_1 < (\frac{N}{b})^{\frac{1}{3.604}} \leqslant p_2 < (\frac{N}{bp_1})^{\frac{1}{2}} \\ (p_1 p_2,N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), \left(\frac{N}{bp_1 p_2}\right)^{\frac{1}{2}}\right)$$

$$+\sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3.604}}\leqslant p_2\leqslant p_3<(\frac{N}{bp_1p_2})^{\frac{1}{2}}\\ (p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(p_1p_2),p_3\right). \tag{26}$$

By Buchstab's identity we have

$$\begin{aligned}
&\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<p_2<p_3<(\frac{N}{b})^{\frac{1}{3}}\\ (p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<p_2<p_3<(\frac{N}{b})^{\frac{1}{8.4}}\\ (p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P},p_1\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<p_2<(\frac{N}{b})^{\frac{1}{8.4}}\leqslant p_3<(\frac{N}{b})^{\frac{4.6}{13.2}}p_2^{-1}\\ (p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P},p_1\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{8.4}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3.604}}\leqslant p_2\leqslant p_3<(\frac{N}{bp_1p_2})^{\frac{1}{2}}\\ (p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(p_1p_2),p_3\right)\\
\geqslant&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<p_2<p_3<p_4<(\frac{N}{b})^{\frac{1}{8.4}}\\ (p_1p_2p_3p_4,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<p_2<p_3<(\frac{N}{b})^{\frac{1}{8.4}}\leqslant p_4<(\frac{N}{b})^{\frac{4.6}{13.2}}p_3^{-1}\\ (p_1p_2p_3p_4,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right)+O\left(N^{\frac{12.2}{13.2}}\right),
\end{aligned} \tag{27}$$

where an argument similar to (20) is used. By Lemma 5.1 with $(\alpha,\beta)=\left(\frac{1}{13.2},\frac{1}{3}\right)$ and $(\alpha,\beta)=\left(\frac{1}{8.4},\frac{1}{3.604}\right)$ and (21)–(22), (26)–(27) we complete the proof of Lemma 5.3. □

Lemma 5.4. *Let $\mathcal{A}=\mathcal{A}_2$ in section 2, then we have*

$$\begin{aligned}
4R_{a,b}^{\theta}(N)\geqslant&3S\left(\mathcal{A};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{14}}\right)+S\left(\mathcal{A};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{8.8}}\right)\\
&+\sum_{\substack{(\frac{N}{b})^{\frac{1}{14}}\leqslant p_1<p_2<(\frac{N}{b})^{\frac{1}{8.8}}\\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{14}}\right)\\
&+\sum_{\substack{(\frac{N}{b})^{\frac{1}{14}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{8.8}}\leqslant p_2<(\frac{N}{b})^{\frac{4.5863}{14}}p_1^{-1}\\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{14}}\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{14}}\leqslant p<(\frac{N}{b})^{\frac{4.08631}{14}}\\ (p,N)=1}} S\left(\mathcal{A}_{p};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{14}}\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{14}}\leqslant p<(\frac{N}{b})^{\frac{3.5863}{14}}\\ (p,N)=1}} S\left(\mathcal{A}_{p};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{14}}\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{14}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3.1}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{1}{8.8}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3.7}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),\left(\frac{N}{bp_1p_2}\right)^{\frac{1}{2}}\right)\\
&-\sum_{\substack{(\frac{N}{b})^{\frac{4.08631}{14}}\leqslant p<(\frac{N}{b})^{\frac{1}{3.1}}\\ (p,N)=1}} S\left(\mathcal{A}_{p};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{14}}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{(\frac{N}{b})^{\frac{3.5863}{14}} \leqslant p<(\frac{N}{b})^{\frac{1}{3.7}} \\ (p,N)=1}} S\left(\mathcal{A}_{p};\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{8.8}}\right) \\
&- \sum_{\substack{(\frac{N}{b})^{\frac{1}{14}} \leqslant p_1<p_2<p_3<p_4<(\frac{N}{b})^{\frac{1}{8.8}} \\ (p_1p_2p_3p_4,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right) \\
&- \sum_{\substack{(\frac{N}{b})^{\frac{1}{14}} \leqslant p_1<p_2<p_3<(\frac{N}{b})^{\frac{1}{8.8}} \leqslant p_4<(\frac{N}{b})^{\frac{4.5863}{14}} p_3^{-1} \\ (p_1p_2p_3p_4,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right) \\
&- 2 \sum_{\substack{(\frac{N}{b})^{\frac{1}{3.1}} \leqslant p_1<p_2<(\frac{N}{bp_1})^{\frac{1}{2}} \\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right) \\
&- 2 \sum_{\substack{(\frac{N}{b})^{\frac{1}{3.7}} \leqslant p_1<p_2<(\frac{N}{bp_1})^{\frac{1}{2}} \\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right) + O\left(N^{\frac{13}{14}}\right) \\
=& \left(3S_{11}'+S_{12}'\right)+\left(S_{21}'+S_{22}'\right)-\left(S_{31}'+S_{32}'\right)-\left(S_{41}'+S_{42}'\right) \\
&-\left(S_{51}'+S_{52}'\right)-\left(S_{61}'+S_{62}'\right)-2\left(S_{71}'+S_{72}'\right)+O\left(N^{\frac{13}{14}}\right) \\
=& S_1'+S_2'-S_3'-S_4'-S_5'-S_6'-2S_7'+O\left(N^{\frac{13}{14}}\right).
\end{aligned}
$$

Proof. It is similar to that of Lemma 5.3 and [[3], Lemma 9] so we omit it here. □

Lemma 5.5. *See* [2]. *Let* $\mathcal{A}=\mathcal{A}_1$ *in section 2,* $D_1=\left(\frac{N}{b}\right)^{1/2}\left(\log\left(\frac{N}{b}\right)\right)^{-B}$ *with* $B=B(A)>0$ *in Lemma 4.1, and* $\underline{p}=\frac{D_1}{p}$. *Then we have*

$$
\begin{aligned}
&\sum_{\substack{(\frac{N}{b})^{\frac{4.1001}{13.2}} \leqslant p<(\frac{N}{b})^{\frac{1}{3}} \\ (p,N)=1}} S\left(\mathcal{A}_{p};\mathcal{P},\underline{p}^{\frac{1}{2.5}}\right) \\
\leqslant & \sum_{\substack{(\frac{N}{b})^{\frac{4.1001}{13.2}} \leqslant p<(\frac{N}{b})^{\frac{1}{3}} \\ (p,N)=1}} S\left(\mathcal{A}_{p};\mathcal{P},\underline{p}^{\frac{1}{3.675}}\right) \\
&-\frac{1}{2} \sum_{\substack{(\frac{N}{b})^{\frac{4.1001}{13.2}} \leqslant p<(\frac{N}{b})^{\frac{1}{3}} \\ (p,N)=1}} \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1<\underline{p}^{\frac{1}{2.5}} \\ (p_1,N)=1}} S\left(\mathcal{A}_{pp_1};\mathcal{P},\underline{p}^{\frac{1}{3.675}}\right) \\
&+\frac{1}{2} \sum_{\substack{(\frac{N}{b})^{\frac{4.1001}{13.2}} \leqslant p<(\frac{N}{b})^{\frac{1}{3}} \\ (p,N)=1}} \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1<p_2<p_3<\underline{p}^{\frac{1}{2.5}} \\ (p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{pp_1p_2p_3};\mathcal{P}(p_1),p_2\right) + O\left(N^{\frac{19}{20}}\right).
\end{aligned}
$$

Proof. It is similar to that of [[2], Lemma 7]. By Buchstab's identity, we have

$$
\begin{aligned}
S\left(\mathcal{A}_{p};\mathcal{P},\underline{p}^{\frac{1}{2.5}}\right) =& S\left(\mathcal{A}_{p};\mathcal{P},\underline{p}^{\frac{1}{3.675}}\right) - \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1<\underline{p}^{\frac{1}{2.5}} \\ (p_1,N)=1}} S\left(\mathcal{A}_{pp_1};\mathcal{P},\underline{p}^{\frac{1}{3.675}}\right) \\
&+ \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1<p_2<\underline{p}^{\frac{1}{2.5}} \\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{pp_1p_2};\mathcal{P},p_1\right), \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
S\left(\mathcal{A}_{p};\mathcal{P},\underline{p}^{\frac{1}{2.5}}\right) =& S\left(\mathcal{A}_{p};\mathcal{P},\underline{p}^{\frac{1}{3.675}}\right) - \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1<\underline{p}^{\frac{1}{2.5}} \\ (p_1,N)=1}} S\left(\mathcal{A}_{pp_1};\mathcal{P},\underline{p}^{\frac{1}{2.5}}\right) \\
&- \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1<p_2<\underline{p}^{\frac{1}{2.5}} \\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{pp_1p_2};\mathcal{P}(p_1),p_2\right), \qquad (29)
\end{aligned}
$$

$$\sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1 < p_2 < \underline{p}^{\frac{1}{2.5}} \\ (p_1 p_2, N)=1}} S\left(\mathcal{A}_{pp_1p_2}; \mathcal{P}, p_1\right) - \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1 < p_2 < \underline{p}^{\frac{1}{2.5}} \\ (p_1 p_2, N)=1}} S\left(\mathcal{A}_{pp_1p_2}; \mathcal{P}(p_1), p_2\right)$$
$$= \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1 < p_2 < p_3 < \underline{p}^{\frac{1}{2.5}} \\ (p_1 p_2 p_3, N)=1}} S\left(\mathcal{A}_{pp_1p_2p_3}; \mathcal{P}(p_1), p_2\right) + \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1 < p_2 < \underline{p}^{\frac{1}{2.5}} \\ (p_1 p_2, N)=1}} S\left(\mathcal{A}_{pp_1^2p_2}; \mathcal{P}, p_1\right). \tag{30}$$

Now we add (28) and (29), sum over p in the interval $\left[\left(\frac{N}{b}\right)^{\frac{4.1001}{13.2}}, \left(\frac{N}{b}\right)^{\frac{1}{3}}\right)$ and by (30), we get Lemma 5.5, where the trivial inequality

$$\sum_{\substack{\left(\frac{N}{b}\right)^{\frac{4.1001}{13.2}} \leqslant p < \left(\frac{N}{b}\right)^{\frac{1}{3}} \\ (p,N)=1}} \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1 < p_2 < \underline{p}^{\frac{1}{2.5}} \\ (p_1 p_2, N)=1}} S\left(\mathcal{A}_{pp_1^2p_2}; \mathcal{P}, p_1\right) \ll N^{\frac{19}{20}}$$

is used. □

Lemma 5.6. *See* [4]. *Let $\mathcal{A} = \mathcal{A}_2$ in section 2, $D_2 = \left(\frac{N}{b}\right)^{\theta/2}\left(\log\left(\frac{N}{b}\right)\right)^{-B}$ with $B = B(A) > 0$ in Lemma 4.1, and $\underline{p}' = \frac{D_2}{p}$. Then we have*

$$\sum_{\substack{\left(\frac{N}{b}\right)^{\frac{4.08631}{14}} \leqslant p < \left(\frac{N}{b}\right)^{\frac{1}{3.1}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}, \underline{p}'^{\frac{1}{2.5}}\right)$$
$$\leqslant \sum_{\substack{\left(\frac{N}{b}\right)^{\frac{4.08631}{14}} \leqslant p < \left(\frac{N}{b}\right)^{\frac{1}{3.1}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}, \underline{p}'^{\frac{1}{3.675}}\right)$$
$$- \frac{1}{2} \sum_{\substack{\left(\frac{N}{b}\right)^{\frac{4.08631}{14}} \leqslant p < \left(\frac{N}{b}\right)^{\frac{1}{3.1}} \\ (p,N)=1}} \sum_{\substack{\underline{p}'^{\frac{1}{3.675}} \leqslant p_1 < \underline{p}'^{\frac{1}{2.5}} \\ (p_1, N)=1}} S\left(\mathcal{A}_{pp_1}; \mathcal{P}, \underline{p}'^{\frac{1}{3.675}}\right)$$
$$+ \frac{1}{2} \sum_{\substack{\left(\frac{N}{b}\right)^{\frac{4.08631}{14}} \leqslant p < \left(\frac{N}{b}\right)^{\frac{1}{3.1}} \\ (p,N)=1}} \sum_{\substack{\underline{p}'^{\frac{1}{3.675}} \leqslant p_1 < p_2 < p_3 < \underline{p}'^{\frac{1}{2.5}} \\ (p_1 p_2 p_3, N)=1}} S\left(\mathcal{A}_{pp_1p_2p_3}; \mathcal{P}(p_1), p_2\right) + O\left(N^{\theta - \frac{1}{20}}\right).$$

Proof. It is similar to that of Lemma 5.5 and [[3], Lemma 10] so we omit it here. □

Lemma 5.7. *See* [30]. *Let $\mathcal{A} = \mathcal{A}_1$ in section 2, then we have*

$$\sum_{\substack{ap_1 + bp_2 = N \\ p_1 \text{ and } p_2 \text{ are primes}}} 1 \leqslant S\left(\mathcal{A}; \mathcal{P}, \left(\frac{N}{b}\right)^{\frac{1}{5}}\right) + O\left(N^{\frac{1}{5}}\right)$$
$$\leqslant S\left(\mathcal{A}; \mathcal{P}, \left(\frac{N}{b}\right)^{\frac{1}{7}}\right)$$
$$- \frac{1}{2} \sum_{\substack{\left(\frac{N}{b}\right)^{\frac{1}{7}} \leqslant p < \left(\frac{N}{b}\right)^{\frac{1}{5}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}, \left(\frac{N}{b}\right)^{\frac{1}{7}}\right)$$
$$+ \frac{1}{2} \sum_{\substack{\left(\frac{N}{b}\right)^{\frac{1}{7}} \leqslant p_1 < p_2 < p_3 < \left(\frac{N}{b}\right)^{\frac{1}{5}} \\ (p_1 p_2 p_3, N)=1}} S\left(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(p_1), p_2\right) + O\left(N^{\frac{6}{7}}\right)$$
$$= \Upsilon_1 - \frac{1}{2}\Upsilon_2 + \frac{1}{2}\Upsilon_3 + O\left(N^{\frac{6}{7}}\right).$$

Proof. It is similar to that of Lemma 5.5 and [[30], p. 211, Lemma 5] so we omit it here. □

6. Proof of Theorem 1.1

In this section, sets $\mathcal{A}_1$, $\mathcal{B}_1$, $\mathcal{C}_1$, $\mathcal{E}_1$ and $\mathcal{F}_1$ are defined respectively. We define the function ω as $\omega(p) = 0$ for primes $p \mid abN$ and $\omega(p) = \frac{p}{p-1}$ for other primes.

6.1. **Evaluation of S_1, S_2, S_3.** Let $D_{\mathcal{A}_1} = \left(\frac{N}{b}\right)^{1/2}\left(\log\left(\frac{N}{b}\right)\right)^{-B}$ for some positive constant B. We can take

$$X_{\mathcal{A}_1} = \sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \pi\left(\frac{N}{a}; b^2, Na_{b^2}^{-1}+kb\right) \sim \frac{\varphi(b)N}{a\varphi\left(b^2\right)\log N} \sim \frac{N}{ab\log N} \tag{31}$$

so that $|\mathcal{A}_1| \sim X_{\mathcal{A}_1}$. By Lemma 3.5 for $z_{\mathcal{A}_1} = \left(\frac{N}{b}\right)^{\frac{1}{\alpha}}$ we have

$$W(z_{\mathcal{A}_1}) = \frac{2\alpha e^{-\gamma}C(abN)(1+o(1))}{\log N}. \tag{32}$$

To deal with the error terms, any $\frac{N-ap}{b}$ in $\mathcal{A}_1$ is relatively prime to b, so $\eta\left(X_{\mathcal{A}_1}, n\right) = 0$ for any integer n that shares a common prime divisor with b. If n and a share a common prime divisor r, say $n = rn'$ and $a = ra'$, then $\frac{N-ap}{bn} = \frac{N-ra'p}{brn'} \in \mathbb{Z}$ implies $r \mid N$, which is a contradiction to $(a, N) = 1$. Similarly, we have $\eta\left(X_{\mathcal{A}_1}, n\right) = 0$ if $(n, N) > 1$. We conclude that $\eta\left(X_{\mathcal{A}_1}, n\right) = 0$ if $(n, abN) > 1$. For a square-free integer $n \leqslant D_{\mathcal{A}_1}$ such that $(n, abN) = 1$, to make $n \mid \frac{N-ap}{b}$ for some $\frac{N-ap}{b} \in \mathcal{A}_1$, we need $ap \equiv N(\bmod bn)$, which implies $ap \equiv N + kbn \left(\bmod b^2n\right)$ for some $0 \leqslant k \leqslant b-1$. Since $\left(\frac{N-ap}{bn}, b\right) = 1$, we can further require $(k, b) = 1$. When k runs through the reduced residues modulo b, we know $ka_{b^2n}^{-1}$ also runs through the reduced residues modulo b. Therefore, we have $p \equiv Na_{b^2n}^{-1} + kbn \left(\bmod b^2n\right)$ for some $0 \leqslant k \leqslant b-1$ such that $(k, b) = 1$. Conversely, if $p = Na_{b^2n}^{-1} + kbn + mb^2n$ for some integer m and some $0 \leqslant k \leqslant b-1$ such that $(k, b) = 1$, then $\left(\frac{N-ap}{bn}, b\right) = \left(\frac{N-aa_{b^2n}^{-1}N-akbn-amb^2n}{bn}, b\right) = (-ak, b) = 1$. Therefore, for square-free integers n such that $(n, abN) = 1$, we have

$$\begin{aligned}
\left|\eta\left(X_{\mathcal{A}_1}, n\right)\right| &= \left|\sum_{\substack{a\in\mathcal{A}_1\\a\equiv 0(\bmod n)}} 1 - \frac{\omega(n)}{n}X_{\mathcal{A}_1}\right| \\
&= \left|\sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \pi\left(\frac{N}{a}; b^2n, Na_{b^2n}^{-1}+kbn\right) - \frac{X_{\mathcal{A}_1}}{\varphi(n)}\right| \\
&= \left|\sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \left(\pi\left(\frac{N}{a}; b^2n, Na_{b^2n}^{-1}+kbn\right) - \frac{\pi\left(\frac{N}{a}; b^2, Na_{b^2}^{-1}+kb\right)}{\varphi(n)}\right)\right| \\
&\ll \left|\sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \left(\pi\left(\frac{N}{a}; b^2n, Na_{b^2n}^{-1}+kbn\right) - \frac{\pi\left(\frac{N}{a}; 1, 1\right)}{\varphi\left(b^2n\right)}\right)\right| \\
&\quad + \left|\sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \left(\frac{\pi\left(\frac{N}{a}; b^2, Na_{b^2}^{-1}+kb\right)}{\varphi(n)} - \frac{\pi\left(\frac{N}{a}; 1, 1\right)}{\varphi\left(b^2n\right)}\right)\right| \\
&\ll \sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \left|\pi\left(\frac{N}{a}; b^2n, Na_{b^2n}^{-1}+kbn\right) - \frac{\pi\left(\frac{N}{a}; 1, 1\right)}{\varphi\left(b^2n\right)}\right| \\
&\quad + \frac{1}{\varphi(n)}\sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \left|\pi\left(\frac{N}{a}; b^2, Na_{b^2}^{-1}+kb\right) - \frac{\pi\left(\frac{N}{a}; 1, 1\right)}{\varphi\left(b^2\right)}\right|.
\end{aligned} \tag{33}$$

By Lemma 4.1 with $g(k) = 1$ for $k = 1$ and $g(k) = 0$ for $k > 1$, we have

$$\sum_{\substack{n\leqslant D_{\mathcal{A}_1}\\n\mid P(z_{\mathcal{A}_1})}} \left|\eta\left(X_{\mathcal{A}_1}, n\right)\right| \ll N(\log N)^{-5} \tag{34}$$

and

$$\sum_{p}\sum_{\substack{n\leqslant \frac{D_{\mathcal{A}_1}}{p}\\n\mid P(z_{\mathcal{A}_1})}} \left|\eta\left(X_{\mathcal{A}_1}, pn\right)\right| \ll N(\log N)^{-5}. \tag{35}$$

Then by (31)–(35), Lemma 3.1, Lemma 3.2 and some routine arguments we have

$$\begin{aligned}
S_{11} \geqslant & X_{\mathcal{A}_1} W\left(z_{\mathcal{A}_1}\right)\left\{f\left(\frac{1/2}{1/13.2}\right)+O\left(\frac{1}{\log^{\frac{1}{3}} D_{\mathcal{A}_1}}\right)\right\}-\sum_{\substack{n<D_{\mathcal{A}_1} \\ n \mid P(z_{\mathcal{A}_1})}}\left|\eta\left(X_{\mathcal{A}_1}, n\right)\right| \\
\geqslant & \frac{N}{ab\log N} \frac{2\times 13.2 e^{-\gamma} C(abN)(1+o(1))}{\log N}\left(\frac{2e^{\gamma}}{\frac{13.2}{2}}\left(\log 5.6+\int_{2}^{4.6} \frac{\log(s-1)}{s} \log\frac{5.6}{s+1} ds\right)\right) \\
\geqslant & (1+o(1)) \frac{8C(abN)N}{ab(\log N)^2}\left(\log 5.6+\int_{2}^{4.6} \frac{\log(s-1)}{s} \log\frac{5.6}{s+1} ds\right) \\
\geqslant & 14.82216 \frac{C(abN)N}{ab(\log N)^2}, \\
S_{12} \geqslant & X_{\mathcal{A}_1} W\left(z_{\mathcal{A}_1}\right)\left\{f\left(\frac{1/2}{1/8.4}\right)+O\left(\frac{1}{\log^{\frac{1}{3}} D_{\mathcal{A}_1}}\right)\right\}-\sum_{\substack{n<D_{\mathcal{A}_1} \\ n \mid P(z_{\mathcal{A}_1})}}\left|\eta\left(X_{\mathcal{A}_1}, n\right)\right| \\
\geqslant & \frac{N}{ab\log N} \frac{2\times 8.4 e^{-\gamma} C(abN)(1+o(1))}{\log N}\left(\frac{2e^{\gamma}}{\frac{8.4}{2}}\left(\log 3.2+\int_{2}^{2.2} \frac{\log(s-1)}{s} \log\frac{3.2}{s+1} ds\right)\right) \\
\geqslant & (1+o(1)) \frac{8C(abN)N}{ab(\log N)^2}\left(\log 3.2+\int_{2}^{2.2} \frac{\log(s-1)}{s} \log\frac{3.2}{s+1} ds\right) \\
\geqslant & 9.30664 \frac{C(abN)N}{ab(\log N)^2}, \\
S_1 = & 3S_{11}+S_{12} \geqslant 53.77312 \frac{C(abN)N}{ab(\log N)^2}.
\end{aligned} \tag{36}$$

Similarly, we have

$$\begin{aligned}
S_{21} \geqslant & \frac{N}{ab\log N} \frac{2\times 13.2 e^{-\gamma} C(abN)(1+o(1))}{\log N} \times \\
& \sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}} \leqslant p_1<p_2<(\frac{N}{b})^{\frac{1}{8.4}} \\ (p_1p_2, N)=1}} \frac{1}{p_1p_2} f\left(13.2\left(\frac{1}{2}-\frac{\log p_1p_2}{\log\frac{N}{b}}\right)\right) \\
\geqslant & \frac{N}{ab\log N} \frac{2\times 13.2 e^{-\gamma} C(abN)(1+o(1))}{\log N} \times \\
& \sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}} \leqslant p_1<p_2<(\frac{N}{b})^{\frac{1}{8.4}} \\ (p_1p_2, N)=1}} \frac{1}{p_1p_2} \frac{2e^{\gamma}\log\left(13.2\left(\frac{1}{2}-\frac{\log p_1p_2}{\log\frac{N}{b}}\right)-1\right)}{13.2\left(\frac{1}{2}-\frac{\log p_1p_2}{\log\frac{N}{b}}\right)} \\
\geqslant & (1+o(1)) \frac{4C(abN)N}{ab(\log N)^2}\left(\int_{\frac{1}{13.2}}^{\frac{1}{8.4}} \int_{t_1}^{\frac{1}{8.4}} \frac{\log\left(5.6-13.2\left(t_1+t_2\right)\right)}{t_1t_2\left(\frac{1}{2}-\left(t_1+t_2\right)\right)} dt_1 dt_2\right) \\
\geqslant & (1+o(1)) \frac{8C(abN)N}{ab(\log N)^2}\left(\int_{\frac{1}{13.2}}^{\frac{1}{8.4}} \int_{t_1}^{\frac{1}{8.4}} \frac{\log\left(5.6-13.2\left(t_1+t_2\right)\right)}{t_1t_2\left(1-2\left(t_1+t_2\right)\right)} dt_1 dt_2\right), \\
S_{22} \geqslant & \frac{N}{ab\log N} \frac{2\times 13.2 e^{-\gamma} C(abN)(1+o(1))}{\log N} \times \\
& \sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}} \leqslant p_1<(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_2<(\frac{N}{b})^{\frac{4.6}{13.2}} p_1^{-1} \\ (p_1p_2, N)=1}} \frac{1}{p_1p_2} f\left(13.2\left(\frac{1}{2}-\frac{\log p_1p_2}{\log\frac{N}{b}}\right)\right) \\
\geqslant & \frac{N}{ab\log N} \frac{2\times 13.2 e^{-\gamma} C(abN)(1+o(1))}{\log N} \times \\
& \sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}} \leqslant p_1<(\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_2<(\frac{N}{b})^{\frac{4.6}{13.2}} p_1^{-1} \\ (p_1p_2, N)=1}} \frac{1}{p_1p_2} \frac{2e^{\gamma}\log\left(13.2\left(\frac{1}{2}-\frac{\log p_1p_2}{\log\frac{N}{b}}\right)-1\right)}{13.2\left(\frac{1}{2}-\frac{\log p_1p_2}{\log\frac{N}{b}}\right)} \\
\geqslant & (1+o(1)) \frac{4C(abN)N}{ab(\log N)^2}\left(\int_{\frac{1}{13.2}}^{\frac{1}{8.4}} \int_{\frac{1}{8.4}}^{\frac{4.6}{13.2}-t_1} \frac{\log\left(5.6-13.2\left(t_1+t_2\right)\right)}{t_1t_2\left(\frac{1}{2}-\left(t_1+t_2\right)\right)} dt_1 dt_2\right)
\end{aligned}$$

$$
\begin{aligned}
&\geqslant(1+o(1))\frac{8C(abN)N}{ab(\log N)^2}\left(\int_{\frac{1}{13.2}}^{\frac{1}{8.4}}\int_{\frac{1}{8.4}}^{\frac{4.6}{13.2}-t_1}\frac{\log\left(5.6-13.2\left(t_1+t_2\right)\right)}{t_1t_2\left(1-2\left(t_1+t_2\right)\right)}dt_1dt_2\right),\\
S_2=&S_{21}+S_{22}\\
&\geqslant(1+o(1))\frac{8C(abN)N}{ab(\log N)^2}\left(\int_{\frac{1}{13.2}}^{\frac{1}{8.4}}\int_{t_1}^{\frac{4.6}{13.2}-t_1}\frac{\log\left(5.6-13.2\left(t_1+t_2\right)\right)}{t_1t_2\left(1-2\left(t_1+t_2\right)\right)}dt_1dt_2\right)\\
&\geqslant 5.201296\frac{C(abN)N}{ab(\log N)^2},
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
S_{31}\leqslant&\frac{N}{ab\log N}\frac{2\times 13.2e^{-\gamma}C(abN)(1+o(1))}{\log N}\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p<(\frac{N}{b})^{\frac{4.1001}{13.2}}\\(p,N)=1}}\frac{1}{p}F\left(13.2\left(\frac{1}{2}-\frac{\log p}{\log\frac{N}{b}}\right)\right)\\
\leqslant&\frac{N}{ab\log N}\frac{2\times 13.2e^{-\gamma}C(abN)(1+o(1))}{\log N}\int_{(\frac{N}{b})^{\frac{1}{13.2}}}^{(\frac{N}{b})^{\frac{4.1001}{13.2}}}\frac{1}{u\log u}F\left(13.2\left(\frac{1}{2}-\frac{\log u}{\log\frac{N}{b}}\right)\right)du\\
\leqslant&(1+o(1))\frac{8C(abN)N}{ab(\log N)^2}\left(\log\frac{4.1001(13.2-2)}{13.2-8.2002}+\int_2^{4.6}\frac{\log(s-1)}{s}\log\frac{5.6(5.6-s)}{s+1}ds\right.\\
&\left.+\int_2^{2.6}\frac{\log(s-1)}{s}ds\int_{s+2}^{4.6}\frac{1}{t}\log\frac{t-1}{s+1}\log\frac{5.6(5.6-t)}{t+1}dt\right)\leqslant 21.9016\frac{C(abN)N}{ab(\log N)^2},\\
S_{32}\leqslant&\frac{N}{ab\log N}\frac{2\times 13.2e^{-\gamma}C(abN)(1+o(1))}{\log N}\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p<(\frac{N}{b})^{\frac{3.6}{13.2}}\\(p,N)=1}}\frac{1}{p}F\left(13.2\left(\frac{1}{2}-\frac{\log p}{\log\frac{N}{b}}\right)\right)\\
\leqslant&\frac{N}{ab\log N}\frac{2\times 13.2e^{-\gamma}C(abN)(1+o(1))}{\log N}\int_{(\frac{N}{b})^{\frac{1}{13.2}}}^{(\frac{N}{b})^{\frac{3.6}{13.2}}}\frac{1}{u\log u}F\left(13.2\left(\frac{1}{2}-\frac{\log u}{\log\frac{N}{b}}\right)\right)du\\
\leqslant&(1+o(1))\frac{8C(abN)N}{ab(\log N)^2}\left(\log\frac{3.6(13.2-2)}{13.2-7.2}+\int_2^{4.6}\frac{\log(s-1)}{s}\log\frac{5.6(5.6-s)}{s+1}ds\right.\\
&\left.+\int_2^{2.6}\frac{\log(s-1)}{s}ds\int_{s+2}^{4.6}\frac{1}{t}\log\frac{t-1}{s+1}\log\frac{5.6(5.6-t)}{t+1}dt\right)\leqslant 19.40136\frac{C(abN)N}{ab(\log N)^2},\\
S_3=&S_{31}+S_{32}\leqslant 41.30296\frac{C(abN)N}{ab(\log N)^2}.
\end{aligned}
\tag{38}
$$

6.2. **Evaluation of S_4, S_7.** Let $D_{\mathcal{B}_1}=N^{1/2}(\log N)^{-B}$. By Chen's switching principle and similar arguments as in [7], we know that

$$
|\mathcal{E}_1|<\left(\frac{N}{b}\right)^{\frac{2}{3}},\quad \left(\frac{N}{b}\right)^{\frac{1}{3}}<e\leqslant\left(\frac{N}{b}\right)^{\frac{2}{3}}\ \text{ for }\ e\in\mathcal{E}_1,\quad S_{41}\leqslant S\left(\mathcal{B}_1;\mathcal{P},D_{\mathcal{B}_1}^{\frac{1}{2}}\right)+O\left(N^{\frac{2}{3}}\right).
\tag{39}
$$

Then we can take

$$
X_{\mathcal{B}_1}=\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\0\leqslant j\leqslant a-1,(j,a)=1}}\pi\left(\frac{N}{bp_1p_2};a^2,N\left(bp_1p_2\right)_{a^2}^{-1}+ja\right)
\tag{40}
$$

so that $|\mathcal{B}_1|\sim X_{\mathcal{B}_1}$. By Lemma 3.5 for $z_{\mathcal{B}_1}=D_{\mathcal{B}_1}^{\frac{1}{2}}=N^{\frac{1}{4}}(\log N)^{-B/2}$ we have

$$
W(z_{\mathcal{B}_1})=\frac{8e^{-\gamma}C(abN)(1+o(1))}{\log N},\quad F(2)=e^{\gamma}.
\tag{41}
$$

By the prime number theorem and integration by parts we get that

$$
\begin{aligned}
X_{\mathcal{B}_1}&=(1+o(1))\sum_{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}}}\frac{\varphi(a)\frac{N}{bp_1p_2}}{\varphi\left(a^2\right)\log\left(\frac{N}{bp_1p_2}\right)}\\
&=(1+o(1))\frac{N}{ab}\sum_{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}}}\frac{1}{p_1p_2\log\left(\frac{N}{p_1p_2}\right)}\\
&=(1+o(1))\frac{N}{ab}\int_{(\frac{N}{b})^{\frac{1}{13.2}}}^{(\frac{N}{b})^{\frac{1}{3}}}\frac{dt}{t\log t}\int_{(\frac{N}{b})^{\frac{1}{3}}}^{(\frac{N}{bt})^{\frac{1}{2}}}\frac{du}{u\log u\log\left(\frac{N}{ut}\right)}\\
&=(1+o(1))\frac{N}{ab\log N}\int_2^{12.2}\frac{\log\left(2-\frac{3}{s+1}\right)}{s}ds.
\end{aligned}
\tag{42}
$$

To deal with the error terms, for an integer n such that $(n,abN)>1$, similarly to the discussion for $\eta\left(X_{\mathcal{A}_1},n\right)$, we have $\eta\left(X_{\mathcal{B}_1},n\right)=0$. For a square-free integer n such that $(n,abN)=1$, if $n\mid\frac{N-bp_1p_2p_3}{a}$, then $(p_1,n)=1$ and $(p_2,n)=1$. Moreover, if $\left(\frac{N-bp_1p_2p_3}{an},a\right)=1$, then we have $bp_1p_2p_3\equiv N+jan\left(\bmod a^2n\right)$ for some j such that $0\leqslant j\leqslant a-1$ and $(j,a)=1$. Conversely, if $bp_1p_2p_3=N+jan+sa^2n$ for some integer j such that $0\leqslant j\leqslant a$ and $(j,a)=1$, some integer n relatively prime to p_1p_2 such that $an\mid(N-bp_1p_2p_3)$, and some integer s, then $\left(\frac{N-bp_1p_2p_3}{an},a\right)=(-j,a)=1$. Since jbp_1p_2 runs through the reduced residues modulo a when j runs through the reduced residues modulo a and $\pi(x;k,1,1)=\pi\left(\frac{x}{k};1,1\right)$, for square-free integers n such that $(n,abN)=1$, we have

$$
\begin{aligned}
\left|\eta\left(X_{\mathcal{B}_1},n\right)\right| &= \left|\sum_{\substack{a\in\mathcal{B}_1\\ a\equiv0(\bmod n)}}1-\frac{\omega(n)}{n}X_{\mathcal{B}_1}\right| = \left|\sum_{\substack{a\in\mathcal{B}_1\\ a\equiv0(\bmod n)}}1-\frac{X_{\mathcal{B}_1}}{\varphi(n)}\right| \\
&= \left|\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}},(p_1p_2,N)=1\\ (p_1p_2,n)=1,0\leqslant j\leqslant a-1,(j,a)=1}}\pi\left(N;bp_1p_2,a^2n,N+jan\right)\right.\\
&\quad\left.-\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\ (p_1p_2,n)=1,0\leqslant j\leqslant a-1,(j,a)=1}}\frac{\pi\left(\frac{N}{bp_1p_2};a^2,N\left(bp_1p_2\right)_{a^2}^{-1}+ja\right)}{\varphi(n)}\right|\\
&\ll\left|\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}},(p_1p_2,N)=1\\ (p_1p_2,n)=1,0\leqslant j\leqslant a-1,(j,a)=1}}\left(\pi\left(N;bp_1p_2,a^2n,N+jan\right)\right.\right.\\
&\quad\left.\left.-\frac{\pi\left(\frac{N}{bp_1p_2};a^2,N\left(bp_1p_2\right)_{a^2}^{-1}+ja\right)}{\varphi(n)}\right)\right|\\
&\quad+\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}}\\ (p_1p_2,nN)>1,0\leqslant j\leqslant a-1,(j,a)=1}}\frac{\pi\left(\frac{N}{bp_1p_2};a^2,N\left(bp_1p_2\right)_{a^2}^{-1}+ja\right)}{\varphi(n)}\\
&\ll\left|\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}},(p_1p_2,N)=1\\ (p_1p_2,n)=1,0\leqslant j\leqslant a-1,(j,a)=1}}\left(\pi\left(N;bp_1p_2,a^2n,N+jan\right)-\frac{\pi\left(N;bp_1p_2,1,1\right)}{\varphi\left(a^2n\right)}\right)\right|\\
&\quad+\left|\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}},(p_1p_2,N)=1\\ (p_1p_2,n)=1,0\leqslant j\leqslant a-1,(j,a)=1}}\left(\frac{\pi\left(\frac{N}{bp_1p_2};a^2,N\left(bp_1p_2\right)_{a^2}^{-1}+ja\right)}{\varphi(n)}-\frac{\pi\left(\frac{N}{bp_1p_2};1,1\right)}{\varphi\left(a^2n\right)}\right)\right|\\
&\quad+N^{\frac{12.2}{13.2}}(\log N)^2\\
&\ll\left|\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}},(p_1p_2,N)=1\\ (p_1p_2,n)=1,0\leqslant j\leqslant a-1,(j,a)=1}}\left(\pi\left(N;bp_1p_2,a^2n,N+jan\right)-\frac{\pi\left(N;bp_1p_2,1,1\right)}{\varphi\left(a^2n\right)}\right)\right|\\
&\quad+\frac{1}{\varphi(n)}\left|\sum_{\substack{(\frac{N}{b})^{\frac{1}{13.2}}\leqslant p_1<(\frac{N}{b})^{\frac{1}{3}}\leqslant p_2<(\frac{N}{bp_1})^{\frac{1}{2}},(p_1p_2,N)=1\\ (p_1p_2,n)=1,0\leqslant j\leqslant a-1,(j,a)=1}}\left(\pi\left(\frac{N}{bp_1p_2};a^2,N\left(bp_1p_2\right)_{a^2}^{-1}+ja\right)-\frac{\pi\left(\frac{N}{bp_1p_2};1,1\right)}{\varphi\left(a^2\right)}\right)\right|
\end{aligned}
$$

$$+ N^{\frac{12.2}{13.2}}(\log N)^2. \tag{43}$$

By Lemma 4.1 with

$$g(k) = \begin{cases} 1, & \text{if } k \in \mathcal{E}_1 \\ 0, & \text{otherwise} \end{cases},$$

we have

$$\sum_{\substack{n \leqslant D_{\mathcal{B}_1} \\ n \mid P(z_{\mathcal{B}_1})}} \left|\eta\left(X_{\mathcal{B}_1}, n\right)\right| \ll N(\log N)^{-5}. \tag{44}$$

Then by (39)–(44) and some routine arguments we have

$$S_{41} \leqslant (1+o(1))\frac{8C(abN)N}{ab(\log N)^2}\int_2^{12.2}\frac{\log\left(2-\frac{3}{s+1}\right)}{s}ds.$$

Similarly, we have

$$S_{42} \leqslant (1+o(1))\frac{8C(abN)N}{ab(\log N)^2}\int_{2.604}^{7.4}\frac{\log\left(2.604-\frac{3.604}{s+1}\right)}{s}ds,$$

$$S_4 = S_{41}+S_{42} \leqslant (1+o(1))\frac{8C(abN)N}{ab(\log N)^2}\left(\int_2^{12.2}\frac{\log\left(2-\frac{3}{s+1}\right)}{s}ds + \int_{2.604}^{7.4}\frac{\log\left(2.604-\frac{3.604}{s+1}\right)}{s}ds\right)$$

$$\leqslant 10.69152\frac{C(abN)N}{ab(\log N)^2}, \tag{45}$$

$$S_7 \leqslant (1+o(1))\frac{8C(abN)N}{ab(\log N)^2}\int_2^{2.604}\frac{\log(s-1)}{s}ds \leqslant 0.5160672\frac{C(abN)N}{ab(\log N)^2}. \tag{46}$$

6.3. **Evaluation of S_6.** Let $D_{\mathcal{C}_1} = N^{1/2}(\log N)^{-B}$. By Chen's switching principle and similar arguments as in [2], we know that

$$|\mathcal{F}_1| < \left(\frac{N}{b}\right)^{\frac{12.2}{13.2}}, \quad \left(\frac{N}{b}\right)^{\frac{1}{4.4}} < e < \left(\frac{N}{b}\right)^{\frac{12.2}{13.2}} \text{ for } e \in \mathcal{F}_1, \quad S_{61} \leqslant S\left(\mathcal{C}_1; \mathcal{P}, D_{\mathcal{C}_1}^{\frac{1}{2}}\right) + O\left(N^{\frac{12.2}{13.2}}\right). \tag{47}$$

By Lemma 3.5 for $z_{\mathcal{C}_1} = D_{\mathcal{C}_1}^{\frac{1}{2}} = N^{\frac{1}{4}}(\log N)^{-B/2}$ we have

$$W(z_{\mathcal{C}_1}) = \frac{8e^{-\gamma}C(abN)(1+o(1))}{\log N}, \quad F(2) = e^{\gamma}. \tag{48}$$

By Lemma 3.3 we have

$$|\mathcal{C}_1| = \sum_{mp_1p_2p_4 \in \mathcal{F}_1} \sum_{\substack{p_2 < p_3 < \min\left(\left(\frac{N}{b}\right)^{\frac{1}{8.4}}, \left(\frac{N}{bmp_1p_2p_4}\right)\right) \\ p_3 \equiv N(bmp_1p_2p_4)^{-1}_{a^2} + ja(\bmod a^2) \\ 0 \leqslant j \leqslant a-1, (j,a)=1}} 1$$

$$= \sum_{\substack{\left(\frac{N}{b}\right)^{\frac{1}{13.2}} \leqslant p_1 < p_4 < p_2 < p_3 < \left(\frac{N}{b}\right)^{\frac{1}{8.4}} \\ (p_1p_2p_3p_4, N)=1}} \sum_{\substack{1 \leqslant m \leqslant \frac{N}{bp_1p_2p_3p_4} \\ \left(m, p_1^{-1}abNP(p_4)\right)=1}} \frac{\varphi(a)}{\varphi(a^2)} + O\left(N^{\frac{12.2}{13.2}}\right)$$

$$< (1+o(1))\frac{N}{ab} \sum_{\left(\frac{N}{b}\right)^{\frac{1}{13.2}} \leqslant p_1 < p_4 < p_2 < p_3 < \left(\frac{N}{b}\right)^{\frac{1}{8.4}}} \frac{0.5617}{p_1p_2p_3p_4\log p_4} + O\left(N^{\frac{12.2}{13.2}}\right)$$

$$= (1+o(1))\frac{0.5617N}{ab\log N}\int_{\frac{1}{13.2}}^{\frac{1}{8.4}}\frac{dt_1}{t_1}\int_{t_1}^{\frac{1}{8.4}}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\frac{1}{8.4t_2}dt_2. \tag{49}$$

To deal with the error terms, for an integer n such that $(n, abN) > 1$, similarly to the discussion for $\eta\left(X_{\mathcal{B}_1}, n\right)$, we have $\eta\left(|\mathcal{C}_1|, n\right) = 0$. For a square-free integer n that is relatively prime to abN, if $n \mid \frac{N-bmp_1p_2p_3p_4}{a}$, then $(p_1, n) = 1, (p_2, n) = 1$ and $(p_4, n) = 1$. Moreover, if $\left(\frac{N-bmp_1p_2p_3p_4}{an}, a\right) = 1$, then we have $bmp_1p_2p_3p_4 \equiv N + jan \left(\bmod a^2n\right)$ for some j such that $0 \leqslant j \leqslant a-1$ and $(j, a) = 1$. Conversely, if $bmp_1p_2p_3p_4 = N + jan + sa^2n$ for some integer j such that $0 \leqslant j \leqslant a$ and $(j, a) = 1$, some integer n relatively prime to $p_1p_2p_4$ such that $an \mid (N - bmp_1p_2p_3p_4)$, and some integer s, then $\left(\frac{N-bmp_1p_2p_3p_4}{an}, a\right) = (-j, a) = 1$. Since $jbmp_1p_2p_4$ runs through the reduced residues modulo a when j runs through the reduced residues modulo a and $\pi(x; k, 1, 1) = \pi\left(\frac{x}{k}; 1, 1\right)$, for a square-free integer n relatively prime to abN, we have

$$\left|\eta\left(|\mathcal{C}_1|, n\right)\right| = \left|\sum_{\substack{a \in \mathcal{C}_1 \\ a \equiv 0(\bmod n)}} 1 - \frac{\omega(n)}{n}|\mathcal{C}_1|\right| = \left|\sum_{\substack{a \in \mathcal{C}_1 \\ a \equiv 0(\bmod n)}} 1 - \frac{|\mathcal{C}_1|}{\varphi(n)}\right|$$

$$
\begin{aligned}
&= \left| \sum_{\substack{e\in\mathcal{F}_1\\(e,n)=1}} \left(\sum_{\substack{p_2<p_3<\min((\frac{N}{b})^{\frac{1}{8.4}},(\frac{N}{be}))\\ bep_3\equiv N+jan(\bmod a^2n)\\ 0\leqslant j\leqslant a-1,(j,a)=1}} 1 - \frac{1}{\varphi(n)} \sum_{\substack{p_2<p_3<\min((\frac{N}{b})^{\frac{1}{8.4}},(\frac{N}{be}))\\ p_3\equiv N(bmp_1p_2p_4)^{-1}_{a^2}+ja(\bmod a^2)\\ 0\leqslant j\leqslant a-1,(j,a)=1}} 1 \right) \right| \\
&\quad + \frac{1}{\varphi(n)} \sum_{\substack{e\in\mathcal{F}_1\\(e,n)>1}} \sum_{\substack{p_2<p_3<\min((\frac{N}{b})^{\frac{1}{8.4}},(\frac{N}{be}))\\ p_3\equiv N(bmp_1p_2p_4)^{-1}_{a^2}+ja(\bmod a^2)\\ 0\leqslant j\leqslant a-1,(j,a)=1}} 1. \qquad (50)
\end{aligned}
$$

Let

$$
g(k) = \sum_{\substack{e=k\\ e\in\mathcal{F}_1\\ 0\leqslant j\leqslant a-1,(j,a)=1}} 1,
$$

then

$$
\begin{aligned}
|\eta(|\mathcal{C}_1|,n)| &\ll \left| \sum_{\substack{(\frac{N}{b})^{\frac{1}{4.4}}<k<(\frac{N}{b})^{\frac{12.2}{13.2}}\\(k,n)=1}} g(k) \left(\sum_{\substack{p_2<p_3<\min((\frac{N}{b})^{\frac{1}{8.4}},(\frac{N}{bk}))\\ bkp_3\equiv N+jan(\bmod a^2n)}} 1 - \frac{1}{\varphi(n)} \sum_{\substack{p_2<p_3<\min((\frac{N}{b})^{\frac{1}{8.4}},(\frac{N}{bk}))\\ p_3\equiv N(bmp_1p_2p_4)^{-1}_{a^2}+ja(\bmod a^2)}} 1 \right) \right| \\
&\quad + \frac{1}{\varphi(n)} \sum_{\substack{(\frac{N}{b})^{\frac{1}{4.4}}<k<(\frac{N}{b})^{\frac{12.2}{13.2}}\\(k,n)\geqslant(\frac{N}{b})^{\frac{1}{13.2}}}} \sum_{\substack{p_2<p_3<\min((\frac{N}{b})^{\frac{1}{8.4}},(\frac{N}{bk}))\\ p_3\equiv N(bmp_1p_2p_4)^{-1}_{a^2}+ja(\bmod a^2)}} 1 \\
&\ll \left| \sum_{\substack{(\frac{N}{b})^{\frac{1}{4.4}}<k<(\frac{N}{b})^{\frac{7.4}{8.4}}\\(k,n)=1}} g(k) \left(\pi\left(bk\left(\frac{N}{b}\right)^{\frac{1}{8.4}}; bk, a^2n, N+jan\right) - \frac{\pi\left(\left(\frac{N}{b}\right)^{\frac{1}{8.4}}; a^2, N(bmp_1p_2p_4)^{-1}_{a^2}+ja\right)}{\varphi(n)} \right) \right| \\
&\quad + \left| \sum_{\substack{(\frac{N}{b})^{\frac{7.4}{8.4}}<k<(\frac{N}{b})^{\frac{12.2}{13.2}}\\(k,n)=1}} g(k) \left(\pi\left(N; bk, a^2n, N+jan\right) - \frac{\pi\left(\frac{N}{bk}; a^2, N(bmp_1p_2p_4)^{-1}_{a^2}+ja\right)}{\varphi(n)} \right) \right| \\
&\quad + \left| \sum_{\substack{(\frac{N}{b})^{\frac{1}{4.4}}<k<(\frac{N}{b})^{\frac{12.2}{13.2}}\\(k,n)=1}} g(k) \left(\pi\left(bkp_2; bk, a^2n, N+jan\right) - \frac{\pi\left(p_2; a^2, N(bmp_1p_2p_4)^{-1}_{a^2}+ja\right)}{\varphi(n)} \right) \right| \\
&\quad + N^{\frac{12.2}{13.2}}(\log N)^2 \\
&\ll \left| \sum_{\substack{(\frac{N}{b})^{\frac{1}{4.4}}<k<(\frac{N}{b})^{\frac{7.4}{8.4}}\\(k,n)=1}} g(k) \left(\pi\left(bk\left(\frac{N}{b}\right)^{\frac{1}{8.4}}; bk, a^2n, N+jan\right) - \frac{\pi\left(bk\left(\frac{N}{b}\right)^{\frac{1}{8.4}}; bk, 1, 1\right)}{\varphi(a^2n)} \right) \right| \\
&\quad + \left| \sum_{\substack{(\frac{N}{b})^{\frac{1}{4.4}}<k<(\frac{N}{b})^{\frac{7.4}{8.4}}\\(k,n)=1}} g(k) \left(\frac{\pi\left(\left(\frac{N}{b}\right)^{\frac{1}{8.4}}; a^2, N(bmp_1p_2p_4)^{-1}_{a^2}+ja\right)}{\varphi(n)} - \frac{\pi\left(\left(\frac{N}{b}\right)^{\frac{1}{8.4}}; 1, 1\right)}{\varphi(a^2n)} \right) \right| \\
&\quad + \left| \sum_{\substack{(\frac{N}{b})^{\frac{7.4}{8.4}}<k<(\frac{N}{b})^{\frac{12.2}{13.2}}\\(k,n)=1}} g(k) \left(\pi\left(N; bk, a^2n, N+jan\right) - \frac{\pi(N; bk, 1, 1)}{\varphi(a^2n)} \right) \right|
\end{aligned}
$$

$$+\left|\sum_{\substack{(\frac{N}{b})^{\frac{7.4}{8.4}}<k<(\frac{N}{b})^{\frac{12.2}{13.2}}\\(k,n)=1}} g(k)\left(\frac{\pi\left(\frac{N}{bk};a^2,N(bmp_1p_2p_4)^{-1}_{a^2}+ja\right)}{\varphi(n)}-\frac{\pi\left(\frac{N}{bk};1,1\right)}{\varphi(a^2n)}\right)\right|$$

$$+\left|\sum_{\substack{(\frac{N}{b})^{\frac{1}{4.4}}<k<(\frac{N}{b})^{\frac{12.2}{13.2}}\\(k,n)=1}} g(k)\left(\pi\left(bkp_2;bk,a^2n,N+jan\right)-\frac{\pi\left(bkp_2;bk,1,1\right)}{\varphi(a^2n)}\right)\right|$$

$$+\left|\sum_{\substack{(\frac{N}{b})^{\frac{1}{4.4}}<k<(\frac{N}{b})^{\frac{12.2}{13.2}}\\(k,n)=1}} g(k)\left(\frac{\pi\left(p_2;a^2,N(bmp_1p_2p_4)^{-1}_{a^2}+ja\right)}{\varphi(n)}-\frac{\pi\left(p_2;1,1\right)}{\varphi(a^2n)}\right)\right|$$

$$+N^{\frac{12.2}{13.2}}(\log N)^2$$

$$\ll\left|\sum_{\substack{(\frac{N}{b})^{\frac{1}{4.4}}<k<(\frac{N}{b})^{\frac{7.4}{8.4}}\\(k,n)=1}} g(k)\left(\pi\left(bk\left(\frac{N}{b}\right)^{\frac{1}{8.4}};bk,a^2n,N+jan\right)-\frac{\pi\left(bk\left(\frac{N}{b}\right)^{\frac{1}{8.4}};bk,1,1\right)}{\varphi(a^2n)}\right)\right|$$

$$+\frac{1}{\varphi(n)}\left|\sum_{\substack{(\frac{N}{b})^{\frac{1}{4.4}}<k<(\frac{N}{b})^{\frac{7.4}{8.4}}\\(k,n)=1}} g(k)\left(\pi\left(\left(\frac{N}{b}\right)^{\frac{1}{8.4}};a^2,N(bmp_1p_2p_4)^{-1}_{a^2}+ja\right)-\frac{\pi\left(\left(\frac{N}{b}\right)^{\frac{1}{8.4}};1,1\right)}{\varphi(a^2)}\right)\right|$$

$$+\left|\sum_{\substack{(\frac{N}{b})^{\frac{7.4}{8.4}}<k<(\frac{N}{b})^{\frac{12.2}{13.2}}\\(k,n)=1}} g(k)\left(\pi\left(N;bk,a^2n,N+jan\right)-\frac{\pi\left(N;bk,1,1\right)}{\varphi(a^2n)}\right)\right|$$

$$+\frac{1}{\varphi(n)}\left|\sum_{\substack{(\frac{N}{b})^{\frac{7.4}{8.4}}<k<(\frac{N}{b})^{\frac{12.2}{13.2}}\\(k,n)=1}} g(k)\left(\pi\left(\frac{N}{bk};a^2,N(bmp_1p_2p_4)^{-1}_{a^2}+ja\right)-\frac{\pi\left(\frac{N}{bk};1,1\right)}{\varphi(a^2)}\right)\right|$$

$$+\left|\sum_{\substack{(\frac{N}{b})^{\frac{1}{4.4}}<k<(\frac{N}{b})^{\frac{12.2}{13.2}}\\(k,n)=1}} g(k)\left(\pi\left(bkp_2;bk,a^2n,N+jan\right)-\frac{\pi\left(bkp_2;bk,1,1\right)}{\varphi(a^2n)}\right)\right|$$

$$+\frac{1}{\varphi(n)}\left|\sum_{\substack{(\frac{N}{b})^{\frac{1}{4.4}}<k<(\frac{N}{b})^{\frac{12.2}{13.2}}\\(k,n)=1}} g(k)\left(\pi\left(p_2;a^2,N(bmp_1p_2p_4)^{-1}_{a^2}+ja\right)-\frac{\pi\left(p_2;1,1\right)}{\varphi(a^2)}\right)\right|$$

$$+N^{\frac{12.2}{13.2}}(\log N)^2. \tag{51}$$

By Lemmas 4.1–4.2, we have

$$\sum_{\substack{n\leqslant D_{\mathcal{C}_1}\\n\mid P(z_{\mathcal{C}_1})}}\left|\eta\left(|\mathcal{C}_1|,n\right)\right|\ll N(\log N)^{-5}. \tag{52}$$

By (47)–(52) we have

$$S_{61}\leqslant(1+o(1))\frac{0.5617\times 8C(abN)N}{ab(\log N)^2}\int_{\frac{1}{13.2}}^{\frac{1}{8.4}}\frac{dt_1}{t_1}\int_{t_1}^{\frac{1}{8.4}}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\frac{1}{8.4t_2}dt_2$$

$$\leqslant 0.0864362\frac{C(abN)N}{ab(\log N)^2}. \tag{53}$$

Similarly, we have

$$\begin{aligned}
S_{62} = & \sum_{(\frac{N}{b})^{\frac{1}{13.2}} \leqslant p_1 < p_2 < p_3 < (\frac{N}{b})^{\frac{1}{8.4}} \leqslant p_4 < (\frac{N}{b})^{\frac{1.4}{8.4}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(p_1), p_2\right) \\
& + \sum_{(\frac{N}{b})^{\frac{1}{13.2}} \leqslant p_1 < p_2 < p_3 < (\frac{N}{b})^{\frac{1}{8.4}} < (\frac{N}{b})^{\frac{1.4}{8.4}} \leqslant p_4 < (\frac{N}{b})^{\frac{4.6}{13.2}} p_3^{-1}} S\left(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(p_1), p_2\right) \\
\leqslant & (1+o(1))\frac{0.5617 \times 8C(abN)N}{ab(\log N)^2}\left(21.6\log\frac{13.2}{8.4} - 9.6\right)\log 1.4 \\
& + (1+o(1))\frac{0.5644 \times 8C(abN)N}{ab(\log N)^2}\int_{\frac{1}{13.2}}^{\frac{1}{8.4}}\frac{dt_1}{t_1}\int_{t_1}^{\frac{1}{8.4}}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\left(\frac{8.4}{1.4}\left(\frac{4.6}{13.2}-t_2\right)\right)dt_2 \\
\leqslant & 0.5208761\frac{C(abN)N}{ab(\log N)^2}.
\end{aligned} \tag{54}$$

By (53) and (54) we have

$$\begin{aligned}
S_6 = S_{61} + S_{62} &\leqslant 0.0864362\frac{C(abN)N}{ab(\log N)^2} + 0.5208761\frac{C(abN)N}{ab(\log N)^2} \\
&\leqslant 0.6073123\frac{C(abN)N}{ab(\log N)^2}.
\end{aligned} \tag{55}$$

6.4. **Evaluation of S_5.** For $p \geqslant \left(\frac{N}{b}\right)^{\frac{4.1001}{13.2}}$ we have

$$\underline{p}^{\frac{1}{2.5}} \leqslant \left(\frac{N}{b}\right)^{\frac{1}{13.2}}, \quad S\left(\mathcal{A}_p; \mathcal{P}, \left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right) \leqslant S\left(\mathcal{A}_p; \mathcal{P}, \underline{p}^{\frac{1}{2.5}}\right).$$

By Lemma 5.5 we have

$$\begin{aligned}
S_{51} = & \sum_{\substack{(\frac{N}{b})^{\frac{4.1001}{13.2}} \leqslant p < (\frac{N}{b})^{\frac{1}{3}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}, \left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right) \\
\leqslant & \sum_{\substack{(\frac{N}{b})^{\frac{4.1001}{13.2}} \leqslant p < (\frac{N}{b})^{\frac{1}{3}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}, \underline{p}^{\frac{1}{2.5}}\right) \leqslant \Gamma_1 - \frac{1}{2}\Gamma_2 + \frac{1}{2}\Gamma_3 + O\left(N^{\frac{19}{20}}\right).
\end{aligned} \tag{56}$$

By Lemmas 3.1, 3.2, 3.5, 4.1 and some routine arguments we get

$$\begin{aligned}
\Gamma_1 = & \sum_{\substack{(\frac{N}{b})^{\frac{4.1001}{13.2}} \leqslant p < (\frac{N}{b})^{\frac{1}{3}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}, \underline{p}^{\frac{1}{3.675}}\right) \\
\leqslant & (1+o(1))\frac{8C(abN)N}{ab(\log N)^2}\left(\int_{\frac{4.1001}{13.2}}^{\frac{1}{3}}\frac{dt}{t(1-2t)}\right)\left(1+\int_2^{2.675}\frac{\log(t-1)}{t}dt\right),
\end{aligned} \tag{57}$$

$$\begin{aligned}
\Gamma_2 = & \sum_{\substack{(\frac{N}{b})^{\frac{4.1001}{13.2}} \leqslant p < (\frac{N}{b})^{\frac{1}{3}} \\ (p,N)=1}} \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1 < \underline{p}^{\frac{1}{2.5}} \\ (p_1,N)=1}} S\left(\mathcal{A}_{pp_1}; \mathcal{P}, \underline{p}^{\frac{1}{3.675}}\right) \\
\geqslant & (1+o(1))\frac{8C(abN)N}{ab(\log N)^2}\left(\int_{\frac{4.1001}{13.2}}^{\frac{1}{3}}\frac{dt}{t(1-2t)}\right)\left(\int_{1.5}^{2.675}\frac{\log\left(2.675-\frac{3.675}{t+1}\right)}{t}dt\right).
\end{aligned} \tag{58}$$

By arguments similar to the evaluation of S_{61} we get that

$$\begin{aligned}
\Gamma_3 = & \sum_{\substack{(\frac{N}{b})^{\frac{4.1001}{13.2}} \leqslant p < (\frac{N}{b})^{\frac{1}{3}} \\ (p,N)=1}} \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1 < p_2 < p_3 < \underline{p}^{\frac{1}{2.5}} \\ (p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{pp_1p_2p_3}; \mathcal{P}(p_1), p_2\right) \\
\leqslant & (1+o(1))\frac{8C(abN)}{\log N}\sum_{\substack{(\frac{N}{b})^{\frac{4.1001}{13.2}} \leqslant p < (\frac{N}{b})^{\frac{1}{3}} \\ (p,N)=1}} \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \leqslant p_1 < p_2 < p_3 < \underline{p}^{\frac{1}{2.5}} \\ (p_1p_2p_3,N)=1}} \sum_{\substack{m \leqslant \frac{N}{bpp_1p_2p_3} \\ \left(m, p_1^{-1}abNP(p_2)\right)=1}} \frac{\varphi(a)}{\varphi(a^2)}
\end{aligned}$$

$$\leqslant (1+o(1))\frac{8C(abN)N}{1.763ab\log N}\sum_{\substack{(\frac{N}{b})^{\frac{4.1001}{13.2}}\leqslant p<(\frac{N}{b})^{\frac{1}{3}}\\(p,N)=1}}\frac{1}{p\log p}\int_{\frac{1}{3.675}}^{\frac{1}{2.5}}\int_{t_1}^{\frac{1}{2.5}}\int_{t_2}^{\frac{1}{2.5}}\frac{dt_1dt_2dt_3}{t_1t_2^2t_3}$$

$$\leqslant (1+o(1))\frac{16C(abN)N}{1.763ab(\log N)^2}\left(\int_{\frac{4.1001}{13.2}}^{\frac{1}{3}}\frac{dt}{t(1-2t)}\right)\left(6.175\log\frac{3.675}{2.5}-2.35\right). \tag{59}$$

By (56)–(59) we have

$$S_{51}=\sum_{\substack{(\frac{N}{b})^{\frac{4.1001}{13.2}}\leqslant p<(\frac{N}{b})^{\frac{1}{3}}\\(p,N)=1}}S\left(\mathcal{A}_p;\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{13.2}}\right)$$

$$\leqslant (1+o(1))\frac{8C(abN)N}{ab(\log N)^2}\left(\int_{\frac{4.1001}{13.2}}^{\frac{1}{3}}\frac{dt}{t(1-2t)}\right)\times$$

$$\left(1+\int_2^{2.675}\frac{\log(t-1)}{t}dt-\frac{1}{2}\int_{1.5}^{2.675}\frac{\log\left(2.675-\frac{3.675}{t+1}\right)}{t}dt+\frac{1}{1.763}\left(6.175\log\frac{3.675}{2.5}-2.35\right)\right).$$

Similarly, we have

$$S_{52}=\sum_{\substack{(\frac{N}{b})^{\frac{3.6}{13.2}}\leqslant p<(\frac{N}{b})^{\frac{1}{3.604}}\\(p,N)=1}}S\left(\mathcal{A}_p;\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{8.4}}\right)$$

$$\leqslant (1+o(1))\frac{8C(abN)N}{ab(\log N)^2}\left(\int_{\frac{3.6}{13.2}}^{\frac{1}{3.604}}\frac{dt}{t(1-2t)}\right)\times$$

$$\left(1+\int_2^{2.675}\frac{\log(t-1)}{t}dt-\frac{1}{2}\int_{1.5}^{2.675}\frac{\log\left(2.675-\frac{3.675}{t+1}\right)}{t}dt+\frac{1}{1.763}\left(6.175\log\frac{3.675}{2.5}-2.35\right)\right)$$

$$S_5=S_{51}+S_{52}$$

$$\leqslant (1+o(1))\frac{8C(abN)N}{ab(\log N)^2}\left(\int_{\frac{4.1001}{13.2}}^{\frac{1}{3}}\frac{dt}{t(1-2t)}+\int_{\frac{3.6}{13.2}}^{\frac{1}{3.604}}\frac{dt}{t(1-2t)}\right)\times$$

$$\left(1+\int_2^{2.675}\frac{\log(t-1)}{t}dt-\frac{1}{2}\int_{1.5}^{2.675}\frac{\log\left(2.675-\frac{3.675}{t+1}\right)}{t}dt+\frac{1}{1.763}\left(6.175\log\frac{3.675}{2.5}-2.35\right)\right)$$

$$\leqslant 1.87206\frac{C(abN)N}{ab(\log N)^2}. \tag{60}$$

6.5. **Proof of theorem 1.1.** By Lemma 5.3, (36)–(38), (45)–(46), (55) and (60) we get

$$S_1+S_2\geqslant 58.974416\frac{C(abN)N}{ab(\log N)^2},$$

$$S_3+S_4+S_5+S_6+2S_7\leqslant 55.505987\frac{C(abN)N}{ab(\log N)^2},$$

$$4R_{a,b}(N)\geqslant(S_1+S_2)-(S_3+S_4+S_5+S_6+2S_7)\geqslant 3.468429\frac{C(abN)N}{ab(\log N)^2},$$

$$R_{a,b}(N)\geqslant 0.8671\frac{C(abN)N}{ab(\log N)^2}.$$

Theorem 1.1 is proved.

7. Proof of Theorem 1.2

In this section, sets $\mathcal{A}_2$, $\mathcal{B}_2$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{E}_2$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are defined respectively. We define the function ω as $\omega(p)=0$ for primes $p\mid abN$ and $\omega(p)=\frac{p}{p-1}$ for other primes.

7.1. **Evaluation of S_1', S_2', S_3'.** Let $D_{\mathcal{A}_2} = \left(\frac{N}{b}\right)^{\theta/2}\left(\log\left(\frac{N}{b}\right)\right)^{-B}$ for some positive constant B. We can take

$$X_{\mathcal{A}_2} = \sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \pi\left(\frac{N^\theta}{a}; b^2, Na_{b^2}^{-1}+kb\right) \sim \frac{\varphi(b)N^\theta}{a\varphi(b^2)\log N^\theta} \sim \frac{N^\theta}{ab\theta\log N}. \tag{61}$$

so that $|\mathcal{A}_2| \sim X_{\mathcal{A}_2}$. By Lemma 3.5 for $z_{\mathcal{A}_2} = \left(\frac{N}{b}\right)^{\frac{1}{\alpha}}$ we have

$$W(z_{\mathcal{A}_2}) = \frac{2\alpha e^{-\gamma}C(abN)(1+o(1))}{\log N}. \tag{62}$$

To deal with the error terms, for an integer n such that $(n, abN) > 1$, similarly to the discussion for $\eta\left(X_{\mathcal{A}_1}, n\right)$, we have $\eta\left(X_{\mathcal{A}_2}, n\right) = 0$. For a square-free integer $n \leqslant D_{\mathcal{A}_2}$ such that $(n, abN) = 1$, to make $n \mid \frac{N-ap}{b}$ for some $\frac{N-ap}{b} \in \mathcal{A}_2$, we need $ap \equiv N(\bmod bn)$, which implies $ap \equiv N + kbn \left(\bmod b^2 n\right)$ for some $0 \leqslant k \leqslant b-1$. Since $\left(\frac{N-ap}{bn}, b\right) = 1$, we can further require $(k, b) = 1$. When k runs through the reduced residues modulo b, we know $ka_{b^2n}^{-1}$ also runs through the reduced residues modulo b. Therefore, we have $p \equiv Na_{b^2n}^{-1} + kbn \left(\bmod b^2 n\right)$ for some $0 \leqslant k \leqslant b-1$ such that $(k, b) = 1$. Conversely, if $p = Na_{b^2n}^{-1} + kbn + mb^2 n$ for some integer m and some $0 \leqslant k \leqslant b-1$ such that $(k, b) = 1$, then $\left(\frac{N-ap}{bn}, b\right) = \left(\frac{N-aa_{b^2n}^{-1}N-akbn-amb^2n}{bn}, b\right) = (-ak, b) = 1$. Therefore, for square-free integers n such that $(n, abN) = 1$, we have

$$\begin{aligned}
\left|\eta\left(X_{\mathcal{A}_2}, n\right)\right| &= \left|\sum_{\substack{a\in\mathcal{A}_2\\a\equiv 0(\bmod n)}} 1 - \frac{\omega(n)}{n}X_{\mathcal{A}_2}\right| \\
&= \left|\sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \pi\left(\frac{N^\theta}{a}; b^2 n, Na_{b^2n}^{-1}+kbn\right) - \frac{X_{\mathcal{A}_2}}{\varphi(n)}\right| \\
&= \left|\sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \left(\pi\left(\frac{N^\theta}{a}; b^2 n, Na_{b^2n}^{-1}+kbn\right) - \frac{\pi\left(\frac{N^\theta}{a}; b^2, Na_{b^2}^{-1}+kb\right)}{\varphi(n)}\right)\right| \\
&\ll \left|\sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \left(\pi\left(\frac{N^\theta}{a}; b^2 n, Na_{b^2n}^{-1}+kbn\right) - \frac{\pi\left(\frac{N^\theta}{a}; 1, 1\right)}{\varphi\left(b^2 n\right)}\right)\right| \\
&\quad + \left|\sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \left(\frac{\pi\left(\frac{N^\theta}{a}; b^2, Na_{b^2}^{-1}+kb\right)}{\varphi(n)} - \frac{\pi\left(\frac{N^\theta}{a}; 1, 1\right)}{\varphi\left(b^2 n\right)}\right)\right| \\
&\ll \sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \left|\pi\left(\frac{N^\theta}{a}; b^2 n, Na_{b^2n}^{-1}+kbn\right) - \frac{\pi\left(\frac{N^\theta}{a}; 1, 1\right)}{\varphi\left(b^2 n\right)}\right| \\
&\quad + \frac{1}{\varphi(n)} \sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}} \left|\pi\left(\frac{N^\theta}{a}; b^2, Na_{b^2}^{-1}+kb\right) - \frac{\pi\left(\frac{N^\theta}{a}; 1, 1\right)}{\varphi\left(b^2\right)}\right|.
\end{aligned} \tag{63}$$

By Lemma 4.1 with $g(k) = 1$ for $k = 1$ and $g(k) = 0$ for $k > 1$, we have

$$\sum_{\substack{n\leqslant D_{\mathcal{A}_2}\\n\mid P(z_{\mathcal{A}_2})}} \left|\eta\left(X_{\mathcal{A}_2}, n\right)\right| \ll N^\theta(\log N)^{-5} \tag{64}$$

and

$$\sum_p \sum_{\substack{n\leqslant \frac{D_{\mathcal{A}_2}}{p}\\n\mid P(z_{\mathcal{A}_2})}} \left|\eta\left(X_{\mathcal{A}_2}, pn\right)\right| \ll N^\theta(\log N)^{-5}. \tag{65}$$

Then by (61)–(65), Lemma 3.1, Lemma 3.2 and some routine arguments we have

$$
\begin{aligned}
S'_{11} \geqslant & X_{\mathcal{A}_2} W\left(z_{\mathcal{A}_2}\right)\left\{f\left(\frac{\theta/2}{1/14}\right)+O\left(\frac{1}{\log^{\frac{1}{3}} D_{\mathcal{A}_2}}\right)\right\}-\sum_{\substack{n<D_{\mathcal{A}_2}\\ n \mid P(z_{\mathcal{A}_2})}}\left|\eta\left(X_{\mathcal{A}_2}, n\right)\right| \\
\geqslant & \frac{N^\theta}{ab\theta \log N} \frac{2 \times 14 e^{-\gamma} C(abN)(1+o(1))}{\log N} \times \\
& \left(\frac{2e^\gamma}{\frac{14\theta}{2}}\left(\log(7\theta-1)+\int_2^{7\theta-2} \frac{\log(s-1)}{s} \log \frac{7\theta-1}{s+1} ds\right)\right) \\
\geqslant & (1+o(1)) \frac{8C(abN)N^\theta}{ab\theta^2(\log N)^2}\left(\log(7\theta-1)+\int_2^{7\theta-2} \frac{\log(s-1)}{s} \log \frac{7\theta-1}{s+1} ds\right) \\
\geqslant & 16.70802 \frac{C(abN)N^\theta}{ab(\log N)^2}, \\
S'_{12} \geqslant & X_{\mathcal{A}_2} W\left(z_{\mathcal{A}_2}\right)\left\{f\left(\frac{\theta/2}{1/8.8}\right)+O\left(\frac{1}{\log^{\frac{1}{3}} D_{\mathcal{A}_2}}\right)\right\}-\sum_{\substack{n<D_{\mathcal{A}_2}\\ n \mid P(z_{\mathcal{A}_2})}}\left|\eta\left(X_{\mathcal{A}_2}, n\right)\right| \\
\geqslant & \frac{N^\theta}{ab\theta \log N} \frac{2 \times 8.8 e^{-\gamma} C(abN)(1+o(1))}{\log N} \times \\
& \left(\frac{2e^\gamma}{\frac{8.8\theta}{2}}\left(\log(4.4\theta-1)+\int_2^{4.4\theta-2} \frac{\log(s-1)}{s} \log \frac{4.4\theta-1}{s+1} ds\right)\right) \\
\geqslant & (1+o(1)) \frac{8C(abN)N^\theta}{ab\theta^2(\log N)^2}\left(\log(4.4\theta-1)+\int_2^{4.4\theta-2} \frac{\log(s-1)}{s} \log \frac{4.4\theta-1}{s+1} ds\right) \\
\geqslant & 10.340342 \frac{C(abN)N^\theta}{ab(\log N)^2}, \\
S'_1 = & 3S'_{11}+S'_{12} \geqslant 60.464402 \frac{C(abN)N^\theta}{ab(\log N)^2}.
\end{aligned} \tag{66}
$$

Similarly, we have

$$
\begin{aligned}
S'_{21} \geqslant & \frac{N^\theta}{ab\theta \log N} \frac{2 \times 14 e^{-\gamma} C(abN)(1+o(1))}{\log N} \times \\
& \sum_{\substack{(\frac{N}{b})^{\frac{1}{14}} \leqslant p_1<p_2<(\frac{N}{b})^{\frac{1}{8.8}}\\ (p_1p_2, N)=1}} \frac{1}{p_1p_2} f\left(14\left(\frac{\theta}{2}-\frac{\log p_1p_2}{\log \frac{N}{b}}\right)\right) \\
\geqslant & \frac{N^\theta}{ab\theta \log N} \frac{2 \times 14 e^{-\gamma} C(abN)(1+o(1))}{\log N} \times \\
& \sum_{\substack{(\frac{N}{b})^{\frac{1}{14}} \leqslant p_1<p_2<(\frac{N}{b})^{\frac{1}{8.8}}\\ (p_1p_2, N)=1}} \frac{1}{p_1p_2} \frac{2e^\gamma \log\left(14\left(\frac{\theta}{2}-\frac{\log p_1p_2}{\log \frac{N}{b}}\right)-1\right)}{14\left(\frac{\theta}{2}-\frac{\log p_1p_2}{\log \frac{N}{b}}\right)} \\
\geqslant & (1+o(1)) \frac{4C(abN)N^\theta}{ab\theta(\log N)^2}\left(\int_{\frac{1}{14}}^{\frac{1}{8.8}} \int_{t_1}^{\frac{1}{8.8}} \frac{\log\left((7\theta-1)-14\left(t_1+t_2\right)\right)}{t_1 t_2\left(\frac{\theta}{2}-\left(t_1+t_2\right)\right)} dt_1 dt_2\right) \\
\geqslant & (1+o(1)) \frac{8C(abN)N^\theta}{ab\theta(\log N)^2}\left(\int_{\frac{1}{14}}^{\frac{1}{8.8}} \int_{t_1}^{\frac{1}{8.8}} \frac{\log\left((7\theta-1)-14\left(t_1+t_2\right)\right)}{t_1 t_2\left(\theta-2\left(t_1+t_2\right)\right)} dt_1 dt_2\right), \\
S'_{22} \geqslant & \frac{N^\theta}{ab\theta \log N} \frac{2 \times 14 e^{-\gamma} C(abN)(1+o(1))}{\log N} \times \\
& \sum_{\substack{(\frac{N}{b})^{\frac{1}{14}} \leqslant p_1<(\frac{N}{b})^{\frac{1}{8.8}} \leqslant p_2<(\frac{N}{b})^{\frac{4.5863}{14}} p_1^{-1}\\ (p_1p_2, N)=1}} \frac{1}{p_1p_2} f\left(14\left(\frac{\theta}{2}-\frac{\log p_1p_2}{\log \frac{N}{b}}\right)\right) \\
\geqslant & \frac{N^\theta}{ab\theta \log N} \frac{2 \times 14 e^{-\gamma} C(abN)(1+o(1))}{\log N} \times
\end{aligned}
$$

$$\begin{aligned}
&\sum_{\substack{(\frac{N}{b})^{\frac{1}{14}} \leqslant p_1 < (\frac{N}{b})^{\frac{1}{8.8}} \leqslant p_2 < (\frac{N}{b})^{\frac{4.5863}{14}} p_1^{-1} \\ (p_1 p_2, N)=1}} \frac{1}{p_1 p_2} \frac{2e^{\gamma} \log\left(14\left(\frac{\theta}{2} - \frac{\log p_1 p_2}{\log \frac{N}{b}}\right) - 1\right)}{14\left(\frac{\theta}{2} - \frac{\log p_1 p_2}{\log \frac{N}{b}}\right)} \\
\geqslant &(1+o(1)) \frac{4C(abN)N^{\theta}}{ab\theta(\log N)^2} \left(\int_{\frac{1}{14}}^{\frac{1}{8.8}} \int_{\frac{1}{8.8}}^{\frac{4.5863}{14}-t_1} \frac{\log\left((7\theta-1) - 14(t_1+t_2)\right)}{t_1 t_2 \left(\frac{\theta}{2} - (t_1+t_2)\right)} dt_1 dt_2\right) \\
\geqslant &(1+o(1)) \frac{8C(abN)N^{\theta}}{ab\theta(\log N)^2} \left(\int_{\frac{1}{14}}^{\frac{1}{8.8}} \int_{\frac{1}{8.8}}^{\frac{4.5863}{14}-t_1} \frac{\log\left((7\theta-1) - 14(t_1+t_2)\right)}{t_1 t_2 \left(\theta - 2(t_1+t_2)\right)} dt_1 dt_2\right), \\
S_2' = &S_{21}' + S_{22}' \\
\geqslant &(1+o(1)) \frac{8C(abN)N^{\theta}}{ab\theta(\log N)^2} \left(\int_{\frac{1}{14}}^{\frac{1}{8.8}} \int_{t_1}^{\frac{4.5863}{14}-t_1} \frac{\log\left((7\theta-1) - 14(t_1+t_2)\right)}{t_1 t_2 \left(\theta - 2(t_1+t_2)\right)} dt_1 dt_2\right) \\
\geqslant &5.914688 \frac{C(abN)N^{\theta}}{ab(\log N)^2},
\end{aligned} \tag{67}$$

$$\begin{aligned}
S_{31}' \leqslant &\frac{N^{\theta}}{ab\theta \log N} \frac{2 \times 14 e^{-\gamma} C(abN)(1+o(1))}{\log N} \sum_{\substack{(\frac{N}{b})^{\frac{1}{14}} \leqslant p < (\frac{N}{b})^{\frac{4.08631}{14}} \\ (p,N)=1}} \frac{1}{p} F\left(14\left(\frac{\theta}{2} - \frac{\log p}{\log \frac{N}{b}}\right)\right) \\
\leqslant &\frac{N^{\theta}}{ab\theta \log N} \frac{2 \times 14 e^{-\gamma} C(abN)(1+o(1))}{\log N} \int_{(\frac{N}{b})^{\frac{1}{14}}}^{(\frac{N}{b})^{\frac{4.08631}{14}}} \frac{1}{u \log u} F\left(14\left(\frac{\theta}{2} - \frac{\log u}{\log \frac{N}{b}}\right)\right) du \\
\leqslant &(1+o(1)) \frac{8C(abN)N^{\theta}}{ab\theta^2(\log N)^2} \left(\log \frac{4.08631(14\theta-2)}{14\theta - 8.17262}\right. \\
&+ \int_{2}^{7\theta-2} \frac{\log(s-1)}{s} \log \frac{(7\theta-1)(7\theta-1-s)}{s+1} ds \\
&\left.+ \int_{2}^{7\theta-4} \frac{\log(s-1)}{s} ds \int_{s+2}^{7\theta-2} \frac{1}{t} \log \frac{t-1}{s+1} \log \frac{(7\theta-1)(7\theta-1-t)}{t+1} dt\right) \\
\leqslant &24.63508 \frac{C(abN)N^{\theta}}{ab(\log N)^2}, \\
S_{32}' \leqslant &\frac{N^{\theta}}{ab\theta \log N} \frac{2 \times 14 e^{-\gamma} C(abN)(1+o(1))}{\log N} \sum_{\substack{(\frac{N}{b})^{\frac{1}{14}} \leqslant p < (\frac{N}{b})^{\frac{3.5863}{14}} \\ (p,N)=1}} \frac{1}{p} F\left(14\left(\frac{\theta}{2} - \frac{\log p}{\log \frac{N}{b}}\right)\right) \\
\leqslant &\frac{N^{\theta}}{ab\theta \log N} \frac{2 \times 14 e^{-\gamma} C(abN)(1+o(1))}{\log N} \int_{(\frac{N}{b})^{\frac{1}{14}}}^{(\frac{N}{b})^{\frac{3.5863}{14}}} \frac{1}{u \log u} F\left(14\left(\frac{\theta}{2} - \frac{\log u}{\log \frac{N}{b}}\right)\right) du \\
\leqslant &(1+o(1)) \frac{8C(abN)N^{\theta}}{ab\theta^2(\log N)^2} \left(\log \frac{3.5863(14\theta-2)}{14\theta - 7.1726}\right. \\
&+ \int_{2}^{7\theta-2} \frac{\log(s-1)}{s} \log \frac{(7\theta-1)(7\theta-1-s)}{s+1} ds \\
&\left.+ \int_{2}^{7\theta-4} \frac{\log(s-1)}{s} ds \int_{s+2}^{7\theta-2} \frac{1}{t} \log \frac{t-1}{s+1} \log \frac{(7\theta-1)(7\theta-1-t)}{t+1} dt\right) \\
\leqslant &21.808021 \frac{C(abN)N^{\theta}}{ab(\log N)^2}, \\
S_3' = &S_{31}' + S_{32}' \leqslant 46.443101 \frac{C(abN)N^{\theta}}{ab(\log N)^2}.
\end{aligned} \tag{68}$$

7.2. **Evaluation of S_4', S_7'.** Let $D_{\mathcal{B}_2} = N^{\theta-1/2}(\log N)^{-B}$. By Chen's switching principle and similar arguments as in [1], we know that

$$|\mathcal{E}_2| < \left(\frac{N}{b}\right)^{\frac{2}{3}}, \quad \left(\frac{N}{b}\right)^{\frac{1}{3}} < e \leqslant \left(\frac{N}{b}\right)^{\frac{2}{3}} \text{ for } e \in \mathcal{E}_2, \quad S_4' \leqslant S\left(\mathcal{B}_2; \mathcal{P}, D_{\mathcal{B}_2}^{\frac{1}{2}}\right) + O\left(N^{\frac{2}{3}}\right). \tag{69}$$

Then we can take

$$X_{\mathcal{B}_2} = \sum_{\substack{(\frac{N}{b})^{\frac{1}{14}} \leqslant p_1 \leqslant (\frac{N}{b})^{\frac{1}{3.1}} < p_2 \leqslant (\frac{N}{bp_1})^{\frac{1}{2}} \\ 0 \leqslant j \leqslant a-1, (j,a)=1}} \left(\pi\left(\frac{N}{bp_1p_2}; a^2, N(bp_1p_2)_{a^2}^{-1} + ja\right)\right.$$

$$-\pi\left(\frac{N-N^{\theta}}{bp_1p_2};a^2,N\left(bp_1p_2\right)^{-1}_{a^2}+ja\right)\Bigg) \tag{70}$$

so that $|\mathcal{B}_2|\sim X_{\mathcal{B}_2}$. By Lemma 3.5 for $z_{\mathcal{B}_2}=D^{\frac{1}{2}}_{\mathcal{B}_2}=N^{\frac{2\theta-1}{4}}(\log N)^{-B/2}$ we have

$$W(z_{\mathcal{B}_2})=\frac{8e^{-\gamma}C(abN)(1+o(1))}{(2\theta-1)\log N},\quad F(2)=e^{\gamma}. \tag{71}$$

By Huxley's prime number theorem in short intervals and integeration by parts we get that

$$\begin{aligned}
X_{\mathcal{B}_2}&=(1+o(1))\sum_{(\frac{N}{b})^{\frac{1}{14}}\leqslant p_1\leqslant(\frac{N}{b})^{\frac{1}{3.1}}<p_2\leqslant(\frac{N}{bp_1})^{\frac{1}{2}}}\frac{\varphi(a)\frac{N^{\theta}}{bp_1p_2}}{\varphi\left(a^2\right)\log\left(\frac{N}{bp_1p_2}\right)}\\
&=(1+o(1))\frac{N^{\theta}}{ab}\sum_{(\frac{N}{b})^{\frac{1}{14}}\leqslant p_1\leqslant(\frac{N}{b})^{\frac{1}{3.1}}<p_2\leqslant(\frac{N}{bp_1})^{\frac{1}{2}}}\frac{1}{p_1p_2\log\left(\frac{N}{p_1p_2}\right)}\\
&=(1+o(1))\frac{N^{\theta}}{ab}\int_{(\frac{N}{b})^{\frac{1}{14}}}^{(\frac{N}{b})^{\frac{1}{3.1}}}\frac{dt}{t\log t}\int_{(\frac{N}{b})^{\frac{1}{3.1}}}^{(\frac{N}{bt})^{\frac{1}{2}}}\frac{du}{u\log u\log\left(\frac{N}{ut}\right)}\\
&=(1+o(1))\frac{N^{\theta}}{ab\log N}\int_{2.1}^{13}\frac{\log\left(2.1-\frac{3.1}{s+1}\right)}{s}ds.
\end{aligned} \tag{72}$$

To deal with the error terms, for an integer n such that $(n,abN)>1$, similarly to the discussion for $\eta\left(X_{\mathcal{B}_1},n\right)$, we have $\eta\left(X_{\mathcal{B}_2},n\right)=0$. For a square-free integer n such that $(n,abN)=1$, if $n\mid\frac{N-bp_1p_2p_3}{a}$, then $(p_1,n)=1$ and $(p_2,n)=1$. Moreover, if $\left(\frac{N-bp_1p_2p_3}{an},a\right)=1$, then we have $bp_1p_2p_3\equiv N+jan\left(\bmod a^2n\right)$ for some j such that $0\leqslant j\leqslant a-1$ and $(j,a)=1$. Conversely, if $bp_1p_2p_3=N+jan+sa^2n$ for some integer j such that $0\leqslant j\leqslant a$ and $(j,a)=1$, some integer n relatively prime to p_1p_2 such that $an\mid(N-bp_1p_2p_3)$, and some integer s, then $\left(\frac{N-bp_1p_2p_3}{an},a\right)=(-j,a)=1$. Since jbp_1p_2 runs through the reduced residues modulo a when j runs through the reduced residues modulo a and $\pi(x;k,1,1)=\pi\left(\frac{x}{k};1,1\right)$, for square-free integers n such that $(n,abN)=1$, we have

$$\begin{aligned}
\left|\eta\left(X_{\mathcal{B}_2},n\right)\right|&=\left|\sum_{\substack{a\in\mathcal{B}_2\\ a\equiv 0(\bmod n)}}1-\frac{\omega(n)}{n}X_{\mathcal{B}_2}\right|=\left|\sum_{\substack{a\in\mathcal{B}_2\\ a\equiv 0(\bmod n)}}1-\frac{X_{\mathcal{B}_2}}{\varphi(n)}\right|\\
&=\Bigg|\sum_{\substack{(\frac{N}{b})^{\frac{1}{14}}\leqslant p_1\leqslant(\frac{N}{b})^{\frac{1}{3.1}}<p_2\leqslant(\frac{N}{bp_1})^{\frac{1}{2}},(p_1p_2,N)=1\\ (p_1p_2,n)=1,0\leqslant j\leqslant a-1,(j,a)=1}}\left(\pi\left(N;bp_1p_2,a^2n,N+jan\right)-\pi\left(N-N^{\theta};bp_1p_2,a^2n,N+jan\right)\right)\\
&\quad-\sum_{\substack{(\frac{N}{b})^{\frac{1}{14}}\leqslant p_1\leqslant(\frac{N}{b})^{\frac{1}{3.1}}<p_2\leqslant(\frac{N}{bp_1})^{\frac{1}{2}}\\ (p_1p_2,n)=1,0\leqslant j\leqslant a-1,(j,a)=1}}\frac{\left(\pi\left(\frac{N}{bp_1p_2};a^2,N\left(bp_1p_2\right)^{-1}_{a^2}+ja\right)-\pi\left(\frac{N-N^{\theta}}{bp_1p_2};a^2,N\left(bp_1p_2\right)^{-1}_{a^2}+ja\right)\right)}{\varphi(n)}\Bigg|\\
&\ll\Bigg|\sum_{\substack{(\frac{N}{b})^{\frac{1}{14}}\leqslant p_1\leqslant(\frac{N}{b})^{\frac{1}{3.1}}<p_2\leqslant(\frac{N}{bp_1})^{\frac{1}{2}},(p_1p_2,N)=1\\ (p_1p_2,n)=1,0\leqslant j\leqslant a-1,(j,a)=1}}\Bigg(\left(\pi\left(N;bp_1p_2,a^2n,N+jan\right)-\pi\left(N-N^{\theta};bp_1p_2,a^2n,N+jan\right)\right)\\
&\quad-\frac{\left(\pi\left(\frac{N}{bp_1p_2};a^2,N\left(bp_1p_2\right)^{-1}_{a^2}+ja\right)-\pi\left(\frac{N-N^{\theta}}{bp_1p_2};a^2,N\left(bp_1p_2\right)^{-1}_{a^2}+ja\right)\right)}{\varphi(n)}\Bigg)\Bigg|\\
&\quad+\sum_{\substack{(\frac{N}{b})^{\frac{1}{14}}\leqslant p_1\leqslant(\frac{N}{b})^{\frac{1}{3.1}}<p_2\leqslant(\frac{N}{bp_1})^{\frac{1}{2}}\\ (p_1p_2,nN)>1,0\leqslant j\leqslant a-1,(j,a)=1}}\frac{\left(\pi\left(\frac{N}{bp_1p_2};a^2,N\left(bp_1p_2\right)^{-1}_{a^2}+ja\right)-\pi\left(\frac{N-N^{\theta}}{bp_1p_2};a^2,N\left(bp_1p_2\right)^{-1}_{a^2}+ja\right)\right)}{\varphi(n)}
\end{aligned}$$

$$\ll \left| \sum_{\substack{(\frac{N}{b})^{\frac{1}{14}} \leqslant p_1 \leqslant (\frac{N}{b})^{\frac{1}{3.1}} < p_2 \leqslant (\frac{N}{bp_1})^{\frac{1}{2}}, (p_1p_2,N)=1 \\ (p_1p_2,n)=1, 0 \leqslant j \leqslant a-1, (j,a)=1}} \left(\left(\pi \left(N; bp_1p_2, a^2n, N+jan\right) - \pi \left(N-N^{\theta}; bp_1p_2, a^2n, N+jan\right) \right) \right. \right.$$
$$\left. \left. - \frac{\left(\pi \left(N; bp_1p_2, 1, 1\right) - \pi \left(N-N^{\theta}; bp_1p_2, 1, 1\right)\right)}{\varphi\left(a^2n\right)} \right) \right|$$
$$+ \left| \sum_{\substack{(\frac{N}{b})^{\frac{1}{14}} \leqslant p_1 \leqslant (\frac{N}{b})^{\frac{1}{3.1}} < p_2 \leqslant (\frac{N}{bp_1})^{\frac{1}{2}}, (p_1p_2,N)=1 \\ (p_1p_2,n)=1, 0 \leqslant j \leqslant a-1, (j,a)=1}} \right.$$
$$\left(\frac{\left(\pi \left(\frac{N}{bp_1p_2}; a^2, N\left(bp_1p_2\right)^{-1}_{a^2} + ja\right) - \pi \left(\frac{N-N^{\theta}}{bp_1p_2}; a^2, N\left(bp_1p_2\right)^{-1}_{a^2} + ja\right)\right)}{\varphi(n)} \right.$$
$$\left. \left. - \frac{\left(\pi \left(\frac{N}{bp_1p_2}; 1, 1\right) - \pi \left(\frac{N-N^{\theta}}{bp_1p_2}; 1, 1\right)\right)}{\varphi\left(a^2n\right)} \right) \right| + N^{\frac{13}{14}} (\log N)^2$$
$$\ll \left| \sum_{\substack{(\frac{N}{b})^{\frac{1}{14}} \leqslant p_1 \leqslant (\frac{N}{b})^{\frac{1}{3.1}} < p_2 \leqslant (\frac{N}{bp_1})^{\frac{1}{2}}, (p_1p_2,N)=1 \\ (p_1p_2,n)=1, 0 \leqslant j \leqslant a-1, (j,a)=1}} \left(\left(\pi \left(N; bp_1p_2, a^2n, N+jan\right) - \pi \left(N-N^{\theta}; bp_1p_2, a^2n, N+jan\right) \right) \right. \right.$$
$$\left. \left. - \frac{\left(\pi \left(N; bp_1p_2, 1, 1\right) - \pi \left(N-N^{\theta}; bp_1p_2, 1, 1\right)\right)}{\varphi\left(a^2n\right)} \right) \right|$$
$$+ \frac{1}{\varphi(n)} \left| \sum_{\substack{(\frac{N}{b})^{\frac{1}{14}} \leqslant p_1 \leqslant (\frac{N}{b})^{\frac{1}{3.1}} < p_2 \leqslant (\frac{N}{bp_1})^{\frac{1}{2}}, (p_1p_2,N)=1 \\ (p_1p_2,n)=1, 0 \leqslant j \leqslant a-1, (j,a)=1}} \right.$$
$$\left(\left(\pi \left(\frac{N}{bp_1p_2}; a^2, N\left(bp_1p_2\right)^{-1}_{a^2} + ja\right) - \pi \left(\frac{N-N^{\theta}}{bp_1p_2}; a^2, N\left(bp_1p_2\right)^{-1}_{a^2} + ja\right) \right) \right.$$
$$\left. \left. - \frac{\left(\pi \left(\frac{N}{bp_1p_2}; 1, 1\right) - \pi \left(\frac{N-N^{\theta}}{bp_1p_2}; 1, 1\right)\right)}{\varphi\left(a^2\right)} \right) \right| + N^{\frac{13}{14}} (\log N)^2. \tag{73}$$

By Lemma 4.3 with

$$g(k) = \begin{cases} 1, & \text{if } k \in \mathcal{E}_2 \\ 0, & \text{otherwise} \end{cases},$$

we have

$$\sum_{\substack{n \leqslant D_{\mathcal{B}_2} \\ n \mid P(z_{\mathcal{B}_2})}} \left| \eta\left(X_{\mathcal{B}_2}, n\right) \right| \ll N^{\theta} (\log N)^{-5}. \tag{74}$$

Then by (69)–(74) and some routine arguments we have

$$S'_{41} \leqslant (1+o(1)) \frac{8C(abN)N^{\theta}}{ab(2\theta-1)(\log N)^2} \int_{2.1}^{13} \frac{\log\left(2.1 - \frac{3.1}{s+1}\right)}{s} ds.$$

Similarly, we have

$$S'_{42} \leqslant (1+o(1)) \frac{8C(abN)N^{\theta}}{ab(2\theta-1)(\log N)^2} \int_{2.7}^{7.8} \frac{\log\left(2.7 - \frac{3.7}{s+1}\right)}{s} ds,$$

$$S'_{4} = S'_{41} + S'_{42} \leqslant (1+o(1)) \frac{8C(abN)N^{\theta}}{ab(2\theta-1)(\log N)^2} \left(\int_{2.1}^{13} \frac{\log\left(2.1 - \frac{3.1}{s+1}\right)}{s} ds + \int_{2.7}^{7.8} \frac{\log\left(2.7 - \frac{3.7}{s+1}\right)}{s} ds \right)$$
$$\leqslant 13.953531 \frac{C(abN)N^{\theta}}{ab(\log N)^2}, \tag{75}$$

$$
\begin{aligned}
S'_{71} &\leqslant (1+o(1))\frac{8C(abN)N^\theta}{ab(2\theta-1)(\log N)^2}\int_2^{2.1}\frac{\log(s-1)}{s}ds\\
S'_{72} &\leqslant (1+o(1))\frac{8C(abN)N^\theta}{ab(2\theta-1)(\log N)^2}\int_2^{2.7}\frac{\log(s-1)}{s}ds,\\
S'_7 = S'_{71}+S'_{72} &\leqslant (1+o(1))\frac{8C(abN)N^\theta}{ab(2\theta-1)(\log N)^2}\left(\int_2^{2.1}\frac{\log(s-1)}{s}ds+\int_2^{2.7}\frac{\log(s-1)}{s}ds\right)\\
&\leqslant 0.771273\frac{C(abN)N^\theta}{ab(\log N)^2}.
\end{aligned}
\tag{76}
$$

7.3. **Evaluation of S'_6.** Let $D_{\mathcal{C}_2} = N^{\theta-1/2}(\log N)^{-B}$. By Chen's switching principle and similar arguments as in [3], we know that

$$
S'_{61} \leqslant S\left(\mathcal{C}_2;\mathcal{P},D_{\mathcal{C}_2}^{\frac{1}{2}}\right)+O\left(D_{\mathcal{C}_2}^{\frac{1}{2}}\right). \tag{77}
$$

By Lemma 3.5 for $z_{\mathcal{C}_2} = D_{\mathcal{C}_2}^{\frac{1}{2}} = N^{\frac{2\theta-1}{4}}(\log N)^{-B/2}$ we have

$$
W(z_{\mathcal{C}_2}) = \frac{8e^{-\gamma}C(abN)(1+o(1))}{(2\theta-1)\log N},\quad F(2)=e^\gamma. \tag{78}
$$

By Lemma 3.4 we have

$$
\begin{aligned}
|\mathcal{C}_2| &= \sum_{\substack{mp_1p_2p_3p_4\in\mathcal{F}_2\\ mp_1p_2p_3p_4\equiv Nb_{a^2}^{-1}+ja(\bmod a^2)\\ 0\leqslant j\leqslant a-1,(j,a)=1}} 1\\
&= \sum_{\substack{(\frac{N}{b})^{\frac{1}{14}}\leqslant p_1<p_2<p_3<p_4<(\frac{N}{b})^{\frac{1}{8.8}}\\ (p_1p_2p_3p_4,N)=1}}\ \sum_{\substack{\frac{N-N^\theta}{bp_1p_2p_3p_4}\leqslant m\leqslant\frac{N}{bp_1p_2p_3p_4}\\ \left(m,p_1^{-1}abNP(p_2)\right)=1}}\frac{\varphi(a)}{\varphi(a^2)}\\
&< (1+o(1))\frac{N^\theta}{ab}\sum_{(\frac{N}{b})^{\frac{1}{14}}\leqslant p_1<p_2<p_3<p_4<(\frac{N}{b})^{\frac{1}{8.8}}}\frac{0.5617}{p_1p_2p_3p_4\log p_2}\\
&= (1+o(1))\frac{0.5617N^\theta}{ab\log N}\int_{\frac{1}{14}}^{\frac{1}{8.8}}\frac{dt_1}{t_1}\int_{t_1}^{\frac{1}{8.8}}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\frac{1}{8.8t_2}dt_2.
\end{aligned}
\tag{79}
$$

To deal with the error terms, for an integer n such that $(n,abN)>1$, similarly to the discussion for $\eta\left(X_{\mathcal{C}_1},n\right)$, we have $\eta\left(|\mathcal{C}_2|,n\right)=0$. For a square-free integer n that is relatively prime to abN, if $n \mid \frac{N-bmp_1p_2p_3p_4}{a}$, then $(p_1,n)=1,(p_2,n)=1,(p_3,n)=1$ and $(p_4,n)=1$. Moreover, if $\left(\frac{N-bmp_1p_2p_3p_4}{an},a\right)=1$, then we have $bmp_1p_2p_3p_4\equiv N+jan\left(\bmod a^2n\right)$ for some j such that $0\leqslant j\leqslant a-1$ and $(j,a)=1$. Conversely, if $bmp_1p_2p_3p_4=N+jan+sa^2n$ for some integer j such that $0\leqslant j\leqslant a$ and $(j,a)=1$, some integer n relatively prime to $p_1p_2p_3p_4$ such that $an\mid(N-bmp_1p_2p_3p_4)$, and some integer s, then $\left(\frac{N-bmp_1p_2p_3p_4}{an},a\right)=(-j,a)=1$. Since $jb_{a^2n}^{-1}$ runs through the reduced residues modulo a when j runs through the reduced residues modulo a, for a square-free integer n relatively prime to abN, we have

$$
\begin{aligned}
|\eta\left(|\mathcal{C}_2|,n\right)| &= \left|\sum_{\substack{a\in\mathcal{C}_2\\ a\equiv 0(\bmod n)}}1-\frac{\omega(n)}{n}|\mathcal{C}_2|\right| = \left|\sum_{\substack{a\in\mathcal{C}_2\\ a\equiv 0(\bmod n)}}1-\frac{|\mathcal{C}_2|}{\varphi(n)}\right|\\
&\ll \left|\sum_{\substack{e\in\mathcal{F}_2\\(e,n)=1\\ e\equiv Nb_{a^2n}^{-1}+jan(\bmod a^2n)\\ 0\leqslant j\leqslant a-1,(j,a)=1}}1-\frac{1}{\varphi(n)}\sum_{\substack{e\in\mathcal{F}_2\\(e,n)=1\\ e\equiv Nb_{a^2}^{-1}+ja(\bmod a^2)\\ 0\leqslant j\leqslant a-1,(j,a)=1}}1\right|+\frac{1}{\varphi(n)}\sum_{\substack{e\in\mathcal{F}_2\\(e,n)>1\\ e\equiv Nb_{a^2}^{-1}+ja(\bmod a^2)\\ 0\leqslant j\leqslant a-1,(j,a)=1}}1\\
&\ll \left|\sum_{\substack{e\in\mathcal{F}_2\\(e,n)=1\\ e\equiv Nb_{a^2n}^{-1}+jan(\bmod a^2n)\\ 0\leqslant j\leqslant a-1,(j,a)=1}}1-\frac{1}{\varphi(a^2n)}\sum_{\substack{e\in\mathcal{F}_2\\(e,n)=1}}1\right|
\end{aligned}
$$

$$+\left|\frac{1}{\varphi(n)}\sum_{\substack{e\in\mathcal{F}_2\\(e,n)=1\\e\equiv Nb_{a^2}^{-1}+ja(\bmod a^2)\\0\leqslant j\leqslant a-1,(j,a)=1}}1-\frac{1}{\varphi(a^2n)}\sum_{\substack{e\in\mathcal{F}_2\\(e,n)=1}}1\right|+N^{\theta-\frac{1}{14}}(\log N)^2$$

$$\ll\left|\sum_{\substack{e\in\mathcal{F}_2\\(e,n)=1\\e\equiv Nb_{a^2n}^{-1}+jan(\bmod a^2n)\\0\leqslant j\leqslant a-1,(j,a)=1}}1-\frac{1}{\varphi(a^2n)}\sum_{\substack{e\in\mathcal{F}_2\\(e,n)=1}}1\right|$$

$$+\frac{1}{\varphi(n)}\left|\sum_{\substack{e\in\mathcal{F}_2\\(e,n)=1\\e\equiv Nb_{a^2}^{-1}+ja(\bmod a^2)\\0\leqslant j\leqslant a-1,(j,a)=1}}1-\frac{1}{\varphi(a^2)}\sum_{\substack{e\in\mathcal{F}_2\\(e,n)=1}}1\right|+N^{\theta-\frac{1}{14}}(\log N)^2. \tag{80}$$

By Lemma 4.5, we have

$$\sum_{\substack{n\leqslant D_{\mathcal{C}_2}\\n\mid P(z_{\mathcal{C}_2})}}|\eta(|\mathcal{C}_2|,n)|\ll N^{\theta}(\log N)^{-5}. \tag{81}$$

Then by (77)–(81) and some routine arguments we have

$$\begin{aligned}S_{61}'&\leqslant(1+o(1))\frac{0.5617\times 8C(abN)N^{\theta}}{ab(2\theta-1)(\log N)^2}\int_{\frac{1}{14}}^{\frac{1}{8.8}}\frac{dt_1}{t_1}\int_{t_1}^{\frac{1}{8.8}}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\frac{1}{8.8t_2}dt_2\\&\leqslant 0.115227\frac{C(abN)N^{\theta}}{ab(\log N)^2}.\end{aligned} \tag{82}$$

Similarly, we have

$$\begin{aligned}S_{62}'=&\sum_{(\frac{N}{b})^{\frac{1}{14}}\leqslant p_1<p_2<p_3<(\frac{N}{b})^{\frac{1}{8.8}}\leqslant p_4<(\frac{N}{b})^{\frac{1.8}{8.8}}}S\left(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}\left(p_1\right),p_2\right)\\&+\sum_{(\frac{N}{b})^{\frac{1}{14}}\leqslant p_1<p_2<p_3<(\frac{N}{b})^{\frac{1}{8.8}}<(\frac{N}{b})^{\frac{1.8}{8.8}}\leqslant p_4<(\frac{N}{b})^{\frac{4.5863}{14}}p_3^{-1}}S\left(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}\left(p_1\right),p_2\right)\\\leqslant&(1+o(1))\frac{0.5617\times 8C(abN)N^{\theta}}{ab(2\theta-1)(\log N)^2}\left(22.8\log\frac{14}{8.8}-10.4\right)\log 1.8\\&+(1+o(1))\frac{0.5644\times 8C(abN)N^{\theta}}{ab(2\theta-1)(\log N)^2}\int_{\frac{1}{14}}^{\frac{1}{8.8}}\frac{dt_1}{t_1}\int_{t_1}^{\frac{1}{8.8}}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\left(\frac{8.8}{1.8}\left(\frac{4.5863}{14}-t_2\right)\right)dt_2\\\leqslant&0.654234\frac{C(abN)N^{\theta}}{ab(\log N)^2}.\end{aligned} \tag{83}$$

By (82) and (83) we have

$$\begin{aligned}S_6'=S_{61}'+S_{62}'&\leqslant 0.115227\frac{C(abN)N^{\theta}}{ab(\log N)^2}+0.654234\frac{C(abN)N^{\theta}}{ab(\log N)^2}\\&\leqslant 0.769461\frac{C(abN)N^{\theta}}{ab(\log N)^2}.\end{aligned} \tag{84}$$

7.4. **Evaluation of S_5'.** For $p\geqslant\left(\frac{N}{b}\right)^{\frac{4.08631}{14}}$ we have

$$\underline{p}'^{\frac{1}{2.5}}\leqslant\left(\frac{N}{b}\right)^{\frac{1}{14}},\quad S\left(\mathcal{A}_p;\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{14}}\right)\leqslant S\left(\mathcal{A}_p;\mathcal{P},\underline{p}'^{\frac{1}{2.5}}\right).$$

By Lemma 5.6 we have

$$S'_{51}=\sum_{\substack{(\frac{N}{b})^{\frac{4.08631}{14}}\leqslant p<(\frac{N}{b})^{\frac{1}{3.1}}\\(p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{14}}\right)$$

$$\leqslant \sum_{\substack{(\frac{N}{b})^{\frac{4.08631}{14}}\leqslant p<(\frac{N}{b})^{\frac{1}{3.1}}\\(p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P},\underline{p}'^{\frac{1}{2.5}}\right)\leqslant \Gamma'_1-\frac{1}{2}\Gamma'_2+\frac{1}{2}\Gamma'_3+O\left(N^{\theta-\frac{1}{20}}\right). \tag{85}$$

By Lemmas 3.1, 3.2, 3.5, 4.1 and some routine arguments we get

$$\Gamma'_1=\sum_{\substack{(\frac{N}{b})^{\frac{4.08631}{14}}\leqslant p<(\frac{N}{b})^{\frac{1}{3.1}}\\(p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P},\underline{p}'^{\frac{1}{3.675}}\right)$$

$$\leqslant (1+o(1))\frac{8C(abN)N^{\theta}}{ab\theta(\log N)^2}\left(\int_{\frac{4.08631}{14}}^{\frac{1}{3.1}}\frac{dt}{t(\theta-2t)}\right)\left(1+\int_2^{2.675}\frac{\log(t-1)}{t}dt\right), \tag{86}$$

$$\Gamma'_2=\sum_{\substack{(\frac{N}{b})^{\frac{4.08631}{14}}\leqslant p<(\frac{N}{b})^{\frac{1}{3.1}}\\(p,N)=1}}\ \sum_{\substack{\underline{p}'^{\frac{1}{3.675}}\leqslant p_1<\underline{p}'^{\frac{1}{2.5}}\\(p_1,N)=1}} S\left(\mathcal{A}_{pp_1};\mathcal{P},\underline{p}'^{\frac{1}{3.675}}\right)$$

$$\geqslant (1+o(1))\frac{8C(abN)N^{\theta}}{ab\theta(\log N)^2}\left(\int_{\frac{4.08631}{14}}^{\frac{1}{3.1}}\frac{dt}{t(\theta-2t)}\right)\left(\int_{1.5}^{2.675}\frac{\log\left(2.675-\frac{3.675}{t+1}\right)}{t}dt\right). \tag{87}$$

By arguments similar to the evaluation of S_8 in [4] we get that

$$\Gamma'_3=\sum_{\substack{(\frac{N}{b})^{\frac{4.08631}{14}}\leqslant p<(\frac{N}{b})^{\frac{1}{3.1}}\\(p,N)=1}}\ \sum_{\substack{\underline{p}'^{\frac{1}{3.675}}\leqslant p_1<p_2<p_3<\underline{p}'^{\frac{1}{2.5}}\\(p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{pp_1p_2p_3};\mathcal{P}(p_1),p_2\right)$$

$$\leqslant (1+o(1))\frac{8C(abN)}{(2\theta-1)\log N}\sum_{\substack{(\frac{N}{b})^{\frac{4.08631}{14}}\leqslant p<(\frac{N}{b})^{\frac{1}{3.1}}\\(p,N)=1}}\ \sum_{\substack{\underline{p}^{\frac{1}{3.675}}\leqslant p_1<p_2<p_3<\underline{p}^{\frac{1}{2.5}}\\(p_1p_2p_3,N)=1}}\ \sum_{\substack{m\leqslant\frac{N}{bpp_1p_2p_3}\\\left(m,p_1^{-1}abNP(p_2)\right)=1}}\frac{\varphi(a)}{\varphi(a^2)}$$

$$\leqslant (1+o(1))\frac{8C(abN)N^{\theta}}{1.763(2\theta-1)ab\log N}\sum_{\substack{(\frac{N}{b})^{\frac{4.08631}{14}}\leqslant p<(\frac{N}{b})^{\frac{1}{3.1}}\\(p,N)=1}}\frac{1}{p\log\underline{p}}\int_{\frac{1}{3.675}}^{\frac{1}{2.5}}\int_{t_1}^{\frac{1}{2.5}}\int_{t_2}^{\frac{1}{2.5}}\frac{dt_1dt_2dt_3}{t_1t_2^2t_3}$$

$$\leqslant (1+o(1))\frac{16C(abN)N^{\theta}}{1.763ab(2\theta-1)(\log N)^2}\left(\int_{\frac{4.08631}{14}}^{\frac{1}{3.1}}\frac{dt}{t(\theta-2t)}\right)\left(6.175\log\frac{3.675}{2.5}-2.35\right). \tag{88}$$

By (85)–(88) we have

$$S'_{51}=\sum_{\substack{(\frac{N}{b})^{\frac{4.08631}{14}}\leqslant p<(\frac{N}{b})^{\frac{1}{3.1}}\\(p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{14}}\right)$$

$$\leqslant (1+o(1))\frac{8C(abN)N^{\theta}}{ab\theta(\log N)^2}\left(\int_{\frac{4.08631}{14}}^{\frac{1}{3.1}}\frac{dt}{t(\theta-2t)}\right)\times$$

$$\left(1+\int_2^{2.675}\frac{\log(t-1)}{t}dt-\frac{1}{2}\int_{1.5}^{2.675}\frac{\log\left(2.675-\frac{3.675}{t+1}\right)}{t}dt+\frac{\theta}{1.763(2\theta-1)}\left(6.175\log\frac{3.675}{2.5}-2.35\right)\right).$$

Similarly, we have

$$S'_{52}=\sum_{\substack{(\frac{N}{b})^{\frac{3.5863}{14}}\leqslant p<(\frac{N}{b})^{\frac{1}{3.7}}\\(p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P},\left(\frac{N}{b}\right)^{\frac{1}{8.8}}\right)$$

$$\leqslant (1+o(1))\frac{8C(abN)N^{\theta}}{ab\theta(\log N)^2}\left(\int_{\frac{3.5863}{14}}^{\frac{1}{3.7}}\frac{dt}{t(\theta-2t)}\right)\times$$

$$\left(1+\int_2^{2.675}\frac{\log(t-1)}{t}dt-\frac{1}{2}\int_{1.5}^{2.675}\frac{\log\left(2.675-\frac{3.675}{t+1}\right)}{t}dt+\frac{\theta}{1.763(2\theta-1)}\left(6.175\log\frac{3.675}{2.5}-2.35\right)\right)$$

$$
\begin{aligned}
S_5' &= S_{51}' + S_{52}' \\
&\leqslant (1+o(1))\frac{8C(abN)N^\theta}{ab\theta(\log N)^2}\left(\int_{\frac{4.08631}{14}}^{\frac{1}{3.1}}\frac{dt}{t(\theta-2t)}+\int_{\frac{3.5863}{14}}^{\frac{1}{3.7}}\frac{dt}{t(\theta-2t)}\right)\times \\
&\left(1+\int_2^{2.675}\frac{\log(t-1)}{t}dt-\frac{1}{2}\int_{1.5}^{2.675}\frac{\log\left(2.675-\frac{3.675}{t+1}\right)}{t}dt+\frac{\theta}{1.763(2\theta-1)}\left(6.175\log\frac{3.675}{2.5}-2.35\right)\right) \\
&\leqslant 3.669999\frac{C(abN)N^\theta}{ab(\log N)^2}.
\end{aligned} \tag{89}
$$

7.5. **Proof of theorem 1.2.** By (66)–(68), (75)–(76), (84) and (89) we get

$$
\begin{aligned}
S_1' + S_2' &\geqslant 66.37909\frac{C(abN)N^\theta}{ab(\log N)^2}, \\
S_3' + S_4' + S_5' + S_6' + 2S_7' &\leqslant 66.378638\frac{C(abN)N^\theta}{ab(\log N)^2}, \\
4R_{a,b}^\theta(N) \geqslant (S_1'+S_2') - (S_3'+S_4'+S_5'+S_6'+2S_7') &\geqslant 0.000452\frac{C(abN)N^\theta}{ab(\log N)^2}, \\
R_{a,b}^\theta(N) &\geqslant 0.000113\frac{C(abN)N^\theta}{ab(\log N)^2}.
\end{aligned}
$$

Theorem 1.2 is proved.

8. An outline of the proof of Theorems 1.3–1.8

The proof of Theorems 1.3–1.8 is similar and even simpler than the proof of Theorems 1.1–1.2.

For Theorem 1.3, we only need Lemma 4.3 and Remark 4.4 to deal with the sieve error terms involved instead of Lemma 4.5 (i.e. $\frac{5*0.97-3}{2} = 0.925 > \frac{12.2}{13.2}$). For example, let $D_{\mathcal{A}_3} = \left(\frac{N}{b}\right)^{0.97-1/2}\left(\log\left(\frac{N}{b}\right)\right)^{-B}$ and by Huxley's prime number theorem in short intervals, we can take

$$
\begin{aligned}
X_{\mathcal{A}_3} &= \sum_{\substack{0\leqslant k\leqslant b-1\\(k,b)=1}}\left(\pi\left(\frac{N/2+N^{0.97}}{a};b^2,Na_{b^2}^{-1}+kb\right)-\pi\left(\frac{N/2-N^{0.97}}{a};b^2,Na_{b^2}^{-1}+kb\right)\right) \\
&\sim \frac{\varphi(b)\left(\pi\left(\frac{N/2+N^{0.97}}{a}\right)-\pi\left(\frac{N/2-N^{0.97}}{a}\right)\right)}{\varphi(b^2)} \sim \frac{2N^{0.97}}{ab\log N}
\end{aligned} \tag{90}
$$

and we can construct the sets $\mathcal{B}$, $\mathcal{C}$, $\mathcal{E}$ and $\mathcal{F}$ for Theorem 1.3 similar to those of Theorem 1.1 and [6].

The proof of Theorems 1.4–1.5 is very similar to that of Theorem 1.1. For example, let $D_{\mathcal{A}_4} = \left(\frac{N}{b}\right)^{1/2}\left(\log\left(\frac{N}{b}\right)\right)^{-B}$, we can take

$$
X_{\mathcal{A}_4} \sim \frac{1}{\varphi(c)}X_{\mathcal{A}_1} \sim \frac{N}{\varphi(c)ab\log N}. \tag{91}
$$

We can construct the sets $\mathcal{B}$, $\mathcal{C}$, $\mathcal{E}$ and $\mathcal{F}$ for Theorems 1.4–1.5 similar to those of Theorem 1.1. The infinite set of primes used in the proof of Theorems 1.4–1.7 is $\mathcal{P}' = \{p : (p, Nc) = 1\}$, so by using the similar arguments to those of Lemma 3.5, for $j = 4, 5, 6$ we have

$$
W'(z_{\mathcal{A}_j}) = \prod_{\substack{p<z\\(p,Nc)=1}}\left(1-\frac{\omega(p)}{p}\right) = \prod_{\substack{p\mid c\\p\nmid N\\p>2}}\left(\frac{p-1}{p-2}\right)\frac{2\alpha e^{-\gamma}C(abN)(1+o(1))}{\log N}. \tag{92}
$$

To deal with the error terms involved, we need to modify our Lemmas 4.1–4.2. We can do that by using the similar arguments to those of Kan and Shan's paper [23] and we refer the interested readers to check it. For Theorem 1.5, we need Lemma 4.6 to control the sieve error terms with "large" c.

The proof of Theorems 1.6–1.7 is like a combination of the proof of Theorems 1.2–1.3 and Theorem 1.4. For example, let $D_{\mathcal{A}_5} = \left(\frac{N}{b}\right)^{\theta/2}\left(\log\left(\frac{N}{b}\right)\right)^{-B}$ and $D_{\mathcal{A}_6} = \left(\frac{N}{b}\right)^{0.97-1/2}\left(\log\left(\frac{N}{b}\right)\right)^{-B}$, we can take

$$
X_{\mathcal{A}_5} \sim \frac{1}{\varphi(c)}X_{\mathcal{A}_2} \sim \frac{N^\theta}{\varphi(c)ab\theta\log N} \quad \text{and} \quad X_{\mathcal{A}_6} \sim \frac{1}{\varphi(c)}X_{\mathcal{A}_3} \sim \frac{2N^{0.97}}{\varphi(c)ab\log N}. \tag{93}
$$

We can construct the sets $\mathcal{B}$, $\mathcal{C}$, $\mathcal{E}$ and $\mathcal{F}$ for Theorems 1.6–1.7 similar to those of Theorem 1.2 and [6]. To deal with the sieve error terms involved, we also need to modify our Lemmas 4.3–4.5 by using the similar arguments to those of [23]. Our Lemmas 4.6–4.8 will help us if we want to combine Theorems 1.2–1.3 with Theorem 1.5 and get similar results to Theorems 1.6–1.7 with "large" c.

Finally, in order to prove Theorem 1.8, we need Lemma 5.7 to give an upper bound. Then we can treat Υ_1 and Υ_2 by arguments involved in evaluation of S_1, S_2, S_3, and Υ_3 by similar arguments involved in evaluation of S_6.

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