Proof of a weighted sieve inequality

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Let p, p_i denote prime numbers, $\mathcal{A}$ be a finite set of integers, $\mathcal{A}_d = \{a : a \in \mathcal{A},\ d \mid a\}$ and $S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, \prod_{p<z} p)=1}} 1$. In 2017, Brady [1] proposed the following weighted sieve inequality without a proof in his PhD thesis.

Theorem 1. *([[1], Theorem 42]). Let $z \geqslant 2$ and $z^{\frac{12}{5}} < y < z^{\frac{5}{2}}$. If every element of $\mathcal{A}$ is square-free and has size at most $y^{\frac{13}{12}}$, then we have*

$$
\begin{aligned}
S(\mathcal{A}, z) \leqslant\ & S\left(\mathcal{A}, \frac{y}{z^2}\right) - \frac{4}{5} \sum_{\frac{y}{z^2} \leqslant p_1 < \frac{z^3}{y}} S\left(\mathcal{A}_{p_1}, \frac{y}{z^2}\right) - \frac{2}{3} \sum_{\frac{z^3}{y} \leqslant p_1 < \frac{y^2}{z^4}} S\left(\mathcal{A}_{p_1}, \frac{y}{z^2}\right) - \frac{8}{15} \sum_{\frac{y^2}{z^4} \leqslant p_1 < z} S\left(\mathcal{A}_{p_1}, \frac{y}{z^2}\right) \\
& + \frac{3}{5} \sum_{\frac{y}{z^2} \leqslant p_2 < p_1 < \frac{z^3}{y}} S\left(\mathcal{A}_{p_1 p_2}, \frac{y}{z^2}\right) + \frac{7}{15} \sum_{\frac{y}{z^2} \leqslant p_2 < \frac{z^3}{y} \leqslant p_1 < \frac{y^2}{z^4}} S\left(\mathcal{A}_{p_1 p_2}, \frac{y}{z^2}\right) \\
& + \frac{1}{3} \sum_{\frac{y}{z^2} \leqslant p_2 < \frac{z^3}{y} < \frac{y^2}{z^4} \leqslant p_1 < z} S\left(\mathcal{A}_{p_1 p_2}, \frac{y}{z^2}\right) + \frac{1}{3} \sum_{\frac{z^3}{y} \leqslant p_2 < p_1 < \frac{y^2}{z^4}} S\left(\mathcal{A}_{p_1 p_2}, \frac{y}{z^2}\right) \\
& + \frac{4}{15} \sum_{\frac{z^3}{y} \leqslant p_2 < \frac{y^2}{z^4} \leqslant p_1 < z} S\left(\mathcal{A}_{p_1 p_2}, \frac{y}{z^2}\right) + \frac{1}{5} \sum_{\frac{y^2}{z^4} \leqslant p_2 < p_1 < z} S\left(\mathcal{A}_{p_1 p_2}, \frac{y}{z^2}\right) \\
& - \frac{2}{5} \sum_{\substack{\frac{y}{z^2} \leqslant p_3 < p_2 < p_1 < \frac{z^3}{y} \\ p_1 p_2 p_3^2 < z^2}} S\left(\mathcal{A}_{p_1 p_2 p_3}, \frac{y}{z^2}\right) \\
& - \frac{4}{15} \sum_{\frac{y}{z^2} \leqslant p_3 < p_2 < \frac{z^3}{y} \leqslant p_1 < \frac{y^2}{z^4}} \left(1 - \frac{3 \log(p_2 p_3)}{8 \log(y/p_1)}\right) S\left(\mathcal{A}_{p_1 p_2 p_3}, \frac{y}{z^2}\right) \\
& + \frac{1}{5} \sum_{\substack{\frac{y}{z^2} \leqslant p_4 < p_3 < p_2 < p_1 < \frac{z^3}{y} \\ p_1 p_2 p_3^2 < z^2}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, \frac{y}{z^2}\right) \\
& + \frac{1}{10} \sum_{\frac{y}{z^2} \leqslant p_4 < p_3 < p_2 < \frac{z^3}{y} \leqslant p_1 < \frac{y^2}{z^4}} \left(1 - \frac{\log(p_2 p_3 p_4)}{\log(y/p_1)}\right) S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, \frac{y}{z^2}\right).
\end{aligned}
$$

We shall give a proof of this inequality in this note.

Proof of Theorem 1. Since $z^{\frac{12}{5}} < y < z^{\frac{5}{2}}$, we have

$$z^{\frac{2}{5}} < \frac{y}{z^2} < \frac{z^3}{y} < \frac{y^2}{z^4} < z.$$

Let $a \in \mathcal{A}$. If a has any prime factor below $\frac{y}{z^2}$, then both quantities are clearly zero. Assume that a has no prime factors below $\frac{y}{z^2}$ and has exactly n prime factors between $\frac{y}{z^2}$ and z. Since $a \leqslant y^{\frac{13}{12}} < z^{\frac{65}{24}}$ and $\frac{y}{z^2} > z^{\frac{2}{5}}$, we know that $n \leqslant 6$. If $n = 0$, then $RHS = S\left(\mathcal{A}, \frac{y}{z^2}\right) = S\left(\mathcal{A}, z\right) = LHS$. Assume that $1 \leqslant n \leqslant 6$. Now we have $LHS = S\left(\mathcal{A}, z\right) = 0$, and we only need to prove that $RHS \geqslant 0$. For every a, let $RHSC$ denote the contribution of a to RHS. Thus, $RHSC \geqslant 0$ for every a implies $RHS \geqslant 0$.

Case 1. $n = 1$**.** Write $a = p_1\beta$, where $\frac{y}{z^2} \leqslant p_1 < z$ and $(\beta, \prod_{p<z} p) = 1$.

Subcase 1.1. $\frac{y}{z^2} \leqslant p_1 < \frac{z^3}{y}$. For every a in this case, we have

$$RHSC = 1 - \frac{4}{5} = \frac{1}{5} > 0.$$

Subcase 1.2. $\frac{z^3}{y} \leqslant p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$RHSC = 1 - \frac{2}{3} = \frac{1}{3} > 0.$$

Subcase 1.3. $\frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$RHSC = 1 - \frac{8}{15} = \frac{7}{15} > 0.$$

Case 2. $n = 2$**.** Write $a = p_1p_2\beta$, where where $\frac{y}{z^2} \leqslant p_2 < p_1 < z$ and $(\beta, \prod_{p<z} p) = 1$.

Subcase 2.1. $\frac{y}{z^2} \leqslant p_2 < p_1 < \frac{z^3}{y}$. For every a in this case, we have

$$RHSC = 1 - 2 \cdot \frac{4}{5} + \frac{3}{5} = 0.$$

Subcase 2.2. $\frac{y}{z^2} \leqslant p_2 < \frac{z^3}{y} \leqslant p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$RHSC = 1 - \frac{4}{5} - \frac{2}{3} + \frac{7}{15} = 0.$$

Subcase 2.3. $\frac{y}{z^2} \leqslant p_2 < \frac{z^3}{y} < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$RHSC = 1 - \frac{4}{5} - \frac{8}{15} + \frac{1}{3} = 0.$$

Subcase 2.4. $\frac{z^3}{y} \leqslant p_2 < p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$RHSC = 1 - 2 \cdot \frac{2}{3} + \frac{1}{3} = 0.$$

Subcase 2.5. $\frac{z^3}{y} \leqslant p_2 < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$RHSC = 1 - \frac{2}{3} - \frac{8}{15} + \frac{4}{15} = \frac{1}{15} > 0.$$

Subcase 2.6. $\frac{y^2}{z^4} \leqslant p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - 2 \cdot \frac{8}{15} + \frac{1}{5} = \frac{2}{15} > 0.$$

Case 3. $n = 3$**.** Write $a = p_1 p_2 p_3 \beta$, where $\frac{y}{z^2} \leqslant p_3 < p_2 < p_1 < z$ and $(\beta, \prod_{p<z} p) = 1$.

Subcase 3.1. $\frac{y}{z^2} \leqslant p_3 < p_2 < p_1 < \frac{z^3}{y}$. For every a in this case, we have

$$RHSC \geqslant 1 - 3 \cdot \frac{4}{5} + 3 \cdot \frac{3}{5} - \frac{2}{5} = 0.$$

Subcase 3.2. $\frac{y}{z^2} \leqslant p_3 < p_2 < \frac{z^3}{y} \leqslant p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$\begin{aligned} RHSC &= 1 - 2 \cdot \frac{4}{5} - \frac{2}{3} + \frac{3}{5} + 2 \cdot \frac{7}{15} - \frac{4}{15} \left(1 - \frac{3 \log(p_2 p_3)}{8 \log(y/p_1)}\right) \\ &= \frac{\log(p_2 p_3)}{10 \log(y/p_1)} \geqslant \frac{\log(\frac{y^2}{z^4})}{10 \log(\frac{y^2}{z^3})} > \frac{\log(z^{\frac{4}{5}})}{10 \log(z^2)} = \frac{1}{25} > 0. \end{aligned}$$

Subcase 3.3. $\frac{y}{z^2} \leqslant p_3 < p_2 < \frac{z^3}{y} < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$RHSC = 1 - 2 \cdot \frac{4}{5} - \frac{8}{15} + \frac{3}{5} + 2 \cdot \frac{1}{3} = \frac{2}{15} > 0.$$

Subcase 3.4. $\frac{y}{z^2} \leqslant p_3 < \frac{z^3}{y} \leqslant p_2 < p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$RHSC = 1 - \frac{4}{5} - 2 \cdot \frac{2}{3} + 2 \cdot \frac{7}{15} + \frac{1}{3} = \frac{2}{15} > 0.$$

Subcase 3.5. $\frac{y}{z^2} \leqslant p_3 < \frac{z^3}{y} \leqslant p_2 < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$RHSC = 1 - \frac{4}{5} - \frac{2}{3} - \frac{8}{15} + \frac{7}{15} + \frac{1}{3} + \frac{4}{15} = \frac{1}{15} > 0.$$

Subcase 3.6. $\frac{y}{z^2} \leqslant p_3 < \frac{z^3}{y} < \frac{y^2}{z^4} \leqslant p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - \frac{4}{5} - 2 \cdot \frac{8}{15} + 2 \cdot \frac{1}{3} + \frac{1}{5} = 0.$$

Subcase 3.7. $\frac{z^3}{y} \leqslant p_3 < p_2 < p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$RHSC = 1 - 3 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} = 0.$$

Subcase 3.8. $\frac{z^3}{y} \leqslant p_3 < p_2 < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$RHSC = 1 - 2 \cdot \frac{2}{3} - \frac{8}{15} + \frac{1}{3} + 2 \cdot \frac{4}{15} = 0.$$

Subcase 3.9. $\frac{z^3}{y} \leqslant p_3 < \frac{y^2}{z^4} \leqslant p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - \frac{2}{3} - 2 \cdot \frac{8}{15} + 2 \cdot \frac{4}{15} + \frac{1}{5} = 0.$$

Subcase 3.10. $\frac{y^2}{z^4} \leqslant p_3 < p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - 3 \cdot \frac{8}{15} + 3 \cdot \frac{1}{5} = 0.$$

Case 4. $n = 4$. Write $a = p_1 p_2 p_3 p_4 \beta$, where $\frac{y}{z^2} \leqslant p_4 < p_3 < p_2 < p_1 < z$ and $(\beta, \prod_{p<z} p) = 1$.

Subcase 4.1. $\frac{y}{z^2} \leqslant p_4 < p_3 < p_2 < p_1 < \frac{z^3}{y}$. If $p_1 p_2 p_3^2 < z^2$, then we have

$$RHSC = 1 - 4 \cdot \frac{4}{5} + 6 \cdot \frac{3}{5} - 4 \cdot \frac{2}{5} + \frac{1}{5} = 0.$$

If $p_1 p_2 p_3^2 \geqslant z^2$, then we have

$$RHSC \geqslant 1 - 4 \cdot \frac{4}{5} + 6 \cdot \frac{3}{5} - 3 \cdot \frac{2}{5} = \frac{1}{5} > 0.$$

Subcase 4.2. $\frac{y}{z^2} \leqslant p_4 < p_3 < p_2 < \frac{z^3}{y} \leqslant p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$\begin{aligned}
RHSC \geqslant\ & 1 - 3 \cdot \frac{4}{5} - \frac{2}{3} + 3 \cdot \frac{3}{5} + 3 \cdot \frac{7}{15} - \frac{2}{5} - \frac{4}{15}\left(1 - \frac{3\log(p_2p_3)}{8\log(y/p_1)}\right) \\
& - \frac{4}{15}\left(1 - \frac{3\log(p_2p_4)}{8\log(y/p_1)}\right) - \frac{4}{15}\left(1 - \frac{3\log(p_3p_4)}{8\log(y/p_1)}\right) + \frac{1}{10}\left(1 - \frac{\log(p_2p_3p_4)}{\log(y/p_1)}\right) \\
=\ & \frac{1}{30} + \frac{\log(p_2p_3)}{10\log(y/p_1)} + \frac{\log(p_2p_4)}{10\log(y/p_1)} + \frac{\log(p_3p_4)}{10\log(y/p_1)} - \frac{\log(p_2p_3p_4)}{10\log(y/p_1)} \\
>\ & \frac{1}{30} + \frac{3}{25} - \frac{\log(\frac{z^9}{y^3})}{10\log(\frac{z^4}{y})} > \frac{1}{30} + \frac{3}{25} - \frac{\log(z^{\frac{9}{5}})}{10\log(z^{\frac{3}{2}})} = \frac{1}{30} + \frac{3}{25} - \frac{3}{25} = \frac{1}{30} > 0.
\end{aligned}$$

Subcase 4.3. $\frac{y}{z^2} \leqslant p_4 < p_3 < p_2 < \frac{z^3}{y} < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$RHSC \geqslant 1 - 3 \cdot \frac{4}{5} - \frac{8}{15} + 3 \cdot \frac{3}{5} + 3 \cdot \frac{1}{3} - \frac{2}{5} = \frac{7}{15} > 0.$$

Subcase 4.4. $\frac{y}{z^2} \leqslant p_4 < p_3 < \frac{z^3}{y} \leqslant p_2 < p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$RHSC = 1 - 2 \cdot \frac{4}{5} - 2 \cdot \frac{2}{3} + \frac{3}{5} + 4 \cdot \frac{7}{15} + \frac{1}{3}$$

$$
\begin{aligned}
&- \frac{4}{15}\left(1-\frac{3\log(p_3p_4)}{8\log(y/p_1)}\right) - \frac{4}{15}\left(1-\frac{3\log(p_3p_4)}{8\log(y/p_2)}\right)\\
&> \frac{1}{3}+\frac{2}{25}=\frac{31}{75}>0.
\end{aligned}
$$

Subcase 4.5. $\frac{y}{z^2} \leqslant p_4 < p_3 < \frac{z^3}{y} \leqslant p_2 < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$
\begin{aligned}
RHSC &= 1-2\cdot\frac{4}{5}-\frac{2}{3}-\frac{8}{15}+\frac{3}{5}+2\cdot\frac{7}{15}+2\cdot\frac{1}{3}+\frac{4}{15}-\frac{4}{15}\left(1-\frac{3\log(p_3p_4)}{8\log(y/p_2)}\right)\\
&> \frac{2}{5}+\frac{1}{25}=\frac{11}{25}>0.
\end{aligned}
$$

Subcase 4.6. $\frac{y}{z^2} \leqslant p_4 < p_3 < \frac{z^3}{y} < \frac{y^2}{z^4} \leqslant p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1-2\cdot\frac{4}{5}-2\cdot\frac{8}{15}+\frac{3}{5}+4\cdot\frac{1}{3}+\frac{1}{5}=\frac{7}{15}>0.$$

Subcase 4.7. $\frac{y}{z^2} \leqslant p_4 < \frac{z^3}{y} \leqslant p_3 < p_2 < p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$RHSC = 1-\frac{4}{5}-3\cdot\frac{2}{3}+3\cdot\frac{7}{15}+3\cdot\frac{1}{3}=\frac{3}{5}>0.$$

Subcase 4.8. $\frac{y}{z^2} \leqslant p_4 < \frac{z^3}{y} \leqslant p_3 < p_2 < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$RHSC = 1-\frac{4}{5}-2\cdot\frac{2}{3}-\frac{8}{15}+2\cdot\frac{7}{15}+\frac{1}{3}+\frac{1}{3}+2\cdot\frac{4}{15}=\frac{7}{15}>0.$$

Subcase 4.9. $\frac{y}{z^2} \leqslant p_4 < \frac{z^3}{y} \leqslant p_3 < \frac{y^2}{z^4} \leqslant p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1-\frac{4}{5}-\frac{2}{3}-2\cdot\frac{8}{15}+\frac{7}{15}+2\cdot\frac{1}{3}+2\cdot\frac{4}{15}+\frac{1}{5}=\frac{1}{3}>0.$$

Subcase 4.10. $\frac{y}{z^2} \leqslant p_4 < \frac{z^3}{y} < \frac{y^2}{z^4} \leqslant p_3 < p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1-\frac{4}{5}-3\cdot\frac{8}{15}+3\cdot\frac{1}{3}+3\cdot\frac{1}{5}=\frac{1}{5}>0.$$

Subcase 4.11. $\frac{z^3}{y} \leqslant p_4 < p_3 < p_2 < p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$RHSC = 1-4\cdot\frac{2}{3}+6\cdot\frac{1}{3}=\frac{1}{3}>0.$$

Subcase 4.12. $\frac{z^3}{y} \leqslant p_4 < p_3 < p_2 < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$RHSC = 1-3\cdot\frac{2}{3}-\frac{8}{15}+3\cdot\frac{1}{3}+3\cdot\frac{4}{15}=\frac{4}{15}>0.$$

Subcase 4.13. $\frac{z^3}{y} \leqslant p_4 < p_3 < \frac{y^2}{z^4} \leqslant p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - 2\cdot\frac{2}{3} - 2\cdot\frac{8}{15} + \frac{1}{3} + 4\cdot\frac{4}{15} + \frac{1}{5} = \frac{1}{5} > 0.$$

Subcase 4.14. $\frac{z^3}{y} \leqslant p_4 < \frac{y^2}{z^4} \leqslant p_3 < p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - \frac{2}{3} - 3\cdot\frac{8}{15} + 3\cdot\frac{4}{15} + 3\cdot\frac{1}{5} = \frac{2}{15} > 0.$$

Subcase 4.15. $\frac{y^2}{z^4} \leqslant p_4 < p_3 < p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - 4\cdot\frac{8}{15} + 6\cdot\frac{1}{5} = \frac{1}{15} > 0.$$

Case 5. $n = 5$**.** Write $a = p_1p_2p_3p_4p_5\beta$, where $\frac{y}{z^2} \leqslant p_5 < p_4 < p_3 < p_2 < p_1 < z$ and $(\beta, \prod_{p<z} p) = 1$.

Subcase 5.1. $\frac{y}{z^2} \leqslant p_5 < p_4 < p_3 < p_2 < p_1 < \frac{z^3}{y}$. If $p_1p_2p_3^2 < z^2$, then we have

$$RHSC = 1 - 5\cdot\frac{4}{5} + 10\cdot\frac{3}{5} - 10\cdot\frac{2}{5} + 5\cdot\frac{1}{5} = 0.$$

If $p_1p_2p_4^2 < z^2 \leqslant p_1p_2p_3^2$, then we have

$$RHSC = 1 - 5\cdot\frac{4}{5} + 10\cdot\frac{3}{5} - 9\cdot\frac{2}{5} + 3\cdot\frac{1}{5} = 0.$$

If $p_2p_3p_4^2 < z^2 \leqslant p_1p_2p_4^2$, then we have

$$RHSC \geqslant 1 - 5\cdot\frac{4}{5} + 10\cdot\frac{3}{5} - 8\cdot\frac{2}{5} + \frac{1}{5} = 0.$$

If $p_2p_3p_4^2 \geqslant z^2$, then we have

$$RHSC \geqslant 1 - 5\cdot\frac{4}{5} + 10\cdot\frac{3}{5} - 6\cdot\frac{2}{5} = \frac{3}{5} > 0.$$

Subcase 5.2. $\frac{y}{z^2} \leqslant p_5 < p_4 < p_3 < p_2 < \frac{z^3}{y} \leqslant p_1 < \frac{y^2}{z^4}$. If $p_2p_3p_4^2 < z^2$, then we have

$$\begin{aligned}
RHSC = 1 &- 4\cdot\frac{4}{5} - \frac{2}{3} + 6\cdot\frac{3}{5} + 4\cdot\frac{7}{15} - 4\cdot\frac{2}{5} + \frac{1}{5} - \frac{4}{15}\left(1 - \frac{3\log(p_2p_3)}{8\log(y/p_1)}\right)\\
&- \frac{4}{15}\left(1 - \frac{3\log(p_2p_4)}{8\log(y/p_1)}\right) - \frac{4}{15}\left(1 - \frac{3\log(p_2p_5)}{8\log(y/p_1)}\right) - \frac{4}{15}\left(1 - \frac{3\log(p_3p_4)}{8\log(y/p_1)}\right)\\
&- \frac{4}{15}\left(1 - \frac{3\log(p_3p_5)}{8\log(y/p_1)}\right) - \frac{4}{15}\left(1 - \frac{3\log(p_4p_5)}{8\log(y/p_1)}\right) + \frac{1}{10}\left(1 - \frac{\log(p_2p_3p_4)}{\log(y/p_1)}\right)\\
&+ \frac{1}{10}\left(1 - \frac{\log(p_2p_3p_5)}{\log(y/p_1)}\right) + \frac{1}{10}\left(1 - \frac{\log(p_2p_4p_5)}{\log(y/p_1)}\right) + \frac{1}{10}\left(1 - \frac{\log(p_3p_4p_5)}{\log(y/p_1)}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\log(p_2p_3)}{10\log(y/p_1)} + \frac{\log(p_2p_4)}{10\log(y/p_1)} + \frac{\log(p_2p_5)}{10\log(y/p_1)} + \frac{\log(p_3p_4)}{10\log(y/p_1)} + \frac{\log(p_3p_5)}{10\log(y/p_1)} \\
&\quad + \frac{\log(p_4p_5)}{10\log(y/p_1)} - \frac{\log(p_2p_3p_4)}{10\log(y/p_1)} - \frac{\log(p_2p_3p_5)}{10\log(y/p_1)} - \frac{\log(p_2p_4p_5)}{10\log(y/p_1)} - \frac{\log(p_3p_4p_5)}{10\log(y/p_1)} \\
&= \frac{\log(p_2^3p_3^3p_4^3p_5^3) - \log(p_2^3p_3^3p_4^3p_5^3)}{10\log(y/p_1)} = 0.
\end{aligned}$$

If $p_2p_3p_4^2 \geqslant z^2$, then we have

$$\begin{aligned}
RHSC &\geqslant 1 - 4\cdot\frac{4}{5} - \frac{2}{3} + 6\cdot\frac{3}{5} + 4\cdot\frac{7}{15} - 3\cdot\frac{2}{5} - \frac{4}{15}\left(1 - \frac{3\log(p_2p_3)}{8\log(y/p_1)}\right) \\
&\quad - \frac{4}{15}\left(1 - \frac{3\log(p_2p_4)}{8\log(y/p_1)}\right) - \frac{4}{15}\left(1 - \frac{3\log(p_2p_5)}{8\log(y/p_1)}\right) - \frac{4}{15}\left(1 - \frac{3\log(p_3p_4)}{8\log(y/p_1)}\right) \\
&\quad - \frac{4}{15}\left(1 - \frac{3\log(p_3p_5)}{8\log(y/p_1)}\right) - \frac{4}{15}\left(1 - \frac{3\log(p_4p_5)}{8\log(y/p_1)}\right) + \frac{1}{10}\left(1 - \frac{\log(p_2p_3p_4)}{\log(y/p_1)}\right) \\
&\quad + \frac{1}{10}\left(1 - \frac{\log(p_2p_3p_5)}{\log(y/p_1)}\right) + \frac{1}{10}\left(1 - \frac{\log(p_2p_4p_5)}{\log(y/p_1)}\right) + \frac{1}{10}\left(1 - \frac{\log(p_3p_4p_5)}{\log(y/p_1)}\right) \\
&= \frac{1}{5} + \frac{\log(p_2^3p_3^3p_4^3p_5^3) - \log(p_2^3p_3^3p_4^3p_5^3)}{10\log(y/p_1)} = \frac{1}{5} > 0.
\end{aligned}$$

Subcase 5.3. $\frac{y}{z^2} \leqslant p_5 < p_4 < p_3 < p_2 < \frac{z^3}{y} < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$RHSC = 1 - 4\cdot\frac{4}{5} - \frac{8}{15} + 6\cdot\frac{3}{5} + 4\cdot\frac{1}{3} - 4\cdot\frac{2}{5} = \frac{3}{5} > 0.$$

Subcase 5.4. $\frac{y}{z^2} \leqslant p_5 < p_4 < p_3 < \frac{z^3}{y} \leqslant p_2 < p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$\begin{aligned}
RHSC &\geqslant 1 - 3\cdot\frac{4}{5} - 2\cdot\frac{2}{3} + 3\cdot\frac{3}{5} + 6\cdot\frac{7}{15} + \frac{1}{3} - \frac{2}{5} \\
&\quad - \frac{4}{15}\left(1 - \frac{3\log(p_3p_4)}{8\log(y/p_1)}\right) - \frac{4}{15}\left(1 - \frac{3\log(p_3p_5)}{8\log(y/p_1)}\right) - \frac{4}{15}\left(1 - \frac{3\log(p_4p_5)}{8\log(y/p_1)}\right) \\
&\quad - \frac{4}{15}\left(1 - \frac{3\log(p_3p_4)}{8\log(y/p_2)}\right) - \frac{4}{15}\left(1 - \frac{3\log(p_3p_5)}{8\log(y/p_2)}\right) - \frac{4}{15}\left(1 - \frac{3\log(p_4p_5)}{8\log(y/p_2)}\right) \\
&\quad + \frac{1}{10}\left(1 - \frac{\log(p_3p_4p_5)}{\log(y/p_1)}\right) + \frac{1}{10}\left(1 - \frac{\log(p_3p_4p_5)}{\log(y/p_2)}\right) \\
&> \frac{2}{5} + 6\cdot\frac{1}{25} - 2\cdot\frac{3}{25} = \frac{2}{5} > 0.
\end{aligned}$$

Subcase 5.5. $\frac{y}{z^2} \leqslant p_5 < p_4 < p_3 < \frac{z^3}{y} \leqslant p_2 < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$RHSC \geqslant 1 - 3\cdot\frac{4}{5} - \frac{2}{3} - \frac{8}{15} + 3\cdot\frac{3}{5} + 3\cdot\frac{7}{15} + 3\cdot\frac{1}{3} + \frac{4}{15} - \frac{2}{5}$$

$$
\begin{aligned}
&- \frac{4}{15}\left(1-\frac{3\log(p_3p_4)}{8\log(y/p_2)}\right) - \frac{4}{15}\left(1-\frac{3\log(p_3p_5)}{8\log(y/p_2)}\right) - \frac{4}{15}\left(1-\frac{3\log(p_4p_5)}{8\log(y/p_2)}\right)\\
&+ \frac{1}{10}\left(1-\frac{\log(p_3p_4p_5)}{\log(y/p_2)}\right)\\
&> \frac{23}{30} + 3\cdot\frac{1}{25} - \frac{3}{25} = \frac{23}{30} > 0.
\end{aligned}
$$

Subcase 5.6. $\frac{y}{z^2} \leqslant p_5 < p_4 < p_3 < \frac{z^3}{y} < \frac{y^2}{z^4} \leqslant p_2 < p_1 < z$. For every a in this case, we have

$$RHSC \geqslant 1 - 3\cdot\frac{4}{5} - 2\cdot\frac{8}{15} + 3\cdot\frac{3}{5} + 6\cdot\frac{1}{3} + \frac{1}{5} - \frac{2}{5} = \frac{17}{15} > 0.$$

Subcase 5.7. $\frac{y}{z^2} \leqslant p_5 < p_4 < \frac{z^3}{y} \leqslant p_3 < p_2 < p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$
\begin{aligned}
RHSC &= 1 - 2\cdot\frac{4}{5} - 3\cdot\frac{2}{3} + \frac{3}{5} + 6\cdot\frac{7}{15} + 3\cdot\frac{1}{3}\\
&- \frac{4}{15}\left(1-\frac{3\log(p_4p_5)}{8\log(y/p_1)}\right) - \frac{4}{15}\left(1-\frac{3\log(p_4p_5)}{8\log(y/p_2)}\right) - \frac{4}{15}\left(1-\frac{3\log(p_4p_5)}{8\log(y/p_3)}\right)\\
&> 1 + 3\cdot\frac{1}{25} = \frac{28}{25} > 0.
\end{aligned}
$$

Subcase 5.8. $\frac{y}{z^2} \leqslant p_5 < p_4 < \frac{z^3}{y} \leqslant p_3 < p_2 < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$
\begin{aligned}
RHSC &= 1 - 2\cdot\frac{4}{5} - 2\cdot\frac{2}{3} - \frac{8}{15} + \frac{3}{5} + 4\cdot\frac{7}{15} + 2\cdot\frac{1}{3} + \frac{1}{3} + 2\cdot\frac{4}{15}\\
&- \frac{4}{15}\left(1-\frac{3\log(p_4p_5)}{8\log(y/p_2)}\right) - \frac{4}{15}\left(1-\frac{3\log(p_4p_5)}{8\log(y/p_3)}\right)\\
&> 1 + 2\cdot\frac{1}{25} = \frac{27}{25} > 0.
\end{aligned}
$$

Subcase 5.9. $\frac{y}{z^2} \leqslant p_5 < p_4 < \frac{z^3}{y} \leqslant p_3 < \frac{y^2}{z^4} \leqslant p_2 < p_1 < z$. For every a in this case, we have

$$
\begin{aligned}
RHSC &= 1 - 2\cdot\frac{4}{5} - \frac{2}{3} - 2\cdot\frac{8}{15} + \frac{3}{5} + 2\cdot\frac{7}{15} + 4\cdot\frac{1}{3} + 2\cdot\frac{4}{15} + \frac{1}{5} - \frac{4}{15}\left(1-\frac{3\log(p_4p_5)}{8\log(y/p_3)}\right)\\
&> 1 + \frac{1}{25} = \frac{26}{25} > 0.
\end{aligned}
$$

Subcase 5.10. $\frac{y}{z^2} \leqslant p_5 < p_4 < \frac{z^3}{y} < \frac{y^2}{z^4} \leqslant p_3 < p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - 2\cdot\frac{4}{5} - 3\cdot\frac{8}{15} + \frac{3}{5} + 6\cdot\frac{1}{3} + 3\cdot\frac{1}{5} = 1 > 0.$$

Subcase 5.11. $\frac{y}{z^2} \leqslant p_5 < \frac{z^3}{y} \leqslant p_4 < p_3 < p_2 < p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$RHSC = 1 - \frac{4}{5} - 4\cdot\frac{2}{3} + 4\cdot\frac{7}{15} + 6\cdot\frac{1}{3} = \frac{7}{5} > 0.$$

Subcase 5.12. $\frac{y}{z^2} \leqslant p_5 < \frac{z^3}{y} \leqslant p_4 < p_3 < p_2 < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$RHSC = 1 - \frac{4}{5} - 3 \cdot \frac{2}{3} - \frac{8}{15} + 3 \cdot \frac{7}{15} + \frac{1}{3} + 3 \cdot \frac{1}{3} + 3 \cdot \frac{4}{15} = \frac{6}{5} > 0.$$

Subcase 5.13. $\frac{y}{z^2} \leqslant p_5 < \frac{z^3}{y} \leqslant p_4 < p_3 < \frac{y^2}{z^4} \leqslant p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - \frac{4}{5} - 2 \cdot \frac{2}{3} - 2 \cdot \frac{8}{15} + 2 \cdot \frac{7}{15} + 2 \cdot \frac{1}{3} + \frac{1}{3} + 4 \cdot \frac{4}{15} + \frac{1}{5} = 1 > 0.$$

Subcase 5.14. $\frac{y}{z^2} \leqslant p_5 < \frac{z^3}{y} \leqslant p_4 < \frac{y^2}{z^4} \leqslant p_3 < p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - \frac{4}{5} - \frac{2}{3} - 3 \cdot \frac{8}{15} + \frac{7}{15} + 3 \cdot \frac{1}{3} + 3 \cdot \frac{4}{15} + 3 \cdot \frac{1}{5} = \frac{4}{5} > 0.$$

Subcase 5.15. $\frac{y}{z^2} \leqslant p_5 < \frac{z^3}{y} < \frac{y^2}{z^4} \leqslant p_4 < p_3 < p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - \frac{4}{5} - 4 \cdot \frac{8}{15} + 4 \cdot \frac{1}{3} + 6 \cdot \frac{1}{5} = \frac{3}{5} > 0.$$

Subcase 5.16. $\frac{z^3}{y} \leqslant p_5 < p_4 < p_3 < p_2 < p_1 < \frac{y^2}{z^4}$. For every a in this case, we have

$$RHSC = 1 - 5 \cdot \frac{2}{3} + 10 \cdot \frac{1}{3} = 1 > 0.$$

Subcase 5.17. $\frac{z^3}{y} \leqslant p_5 < p_4 < p_3 < p_2 < \frac{y^2}{z^4} \leqslant p_1 < z$. For every a in this case, we have

$$RHSC = 1 - 4 \cdot \frac{2}{3} - \frac{8}{15} + 6 \cdot \frac{1}{3} + 4 \cdot \frac{4}{15} = \frac{13}{15} > 0.$$

Subcase 5.18. $\frac{z^3}{y} \leqslant p_5 < p_4 < p_3 < \frac{y^2}{z^4} \leqslant p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - 3 \cdot \frac{2}{3} - 2 \cdot \frac{8}{15} + 3 \cdot \frac{1}{3} + 6 \cdot \frac{4}{15} + \frac{1}{5} = \frac{11}{15} > 0.$$

Subcase 5.19. $\frac{z^3}{y} \leqslant p_5 < p_4 < \frac{y^2}{z^4} \leqslant p_3 < p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - 2 \cdot \frac{2}{3} - 3 \cdot \frac{8}{15} + \frac{1}{3} + 6 \cdot \frac{4}{15} + 3 \cdot \frac{1}{5} = \frac{3}{5} > 0.$$

Subcase 5.20. $\frac{z^3}{y} \leqslant p_5 < \frac{y^2}{z^4} \leqslant p_4 < p_3 < p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - \frac{2}{3} - 4 \cdot \frac{8}{15} + 4 \cdot \frac{4}{15} + 6 \cdot \frac{1}{5} = \frac{7}{15} > 0.$$

Subcase 5.21. $\frac{y^2}{z^4} \leqslant p_5 < p_4 < p_3 < p_2 < p_1 < z$. For every a in this case, we have

$$RHSC = 1 - 5 \cdot \frac{8}{15} + 10 \cdot \frac{1}{5} = \frac{1}{3} > 0.$$

Case 6. $n = 6$. Write $a = p_1p_2p_3p_4p_5p_6\beta$, where $\frac{y}{z^2} \leqslant p_6 < p_5 < p_4 < p_3 < p_2 < p_1 < z$ and $(\beta, \prod_{p<z} p) = 1$. Since $z^{\frac{12}{5}} < y$, we have

$$y^{\frac{35}{12}} > z^7.$$

If $p_1 \geqslant \frac{z^3}{y}$, then we have

$$a \geqslant p_1p_2p_3p_4p_5p_6 \geqslant \frac{z^3}{y} \cdot \left(\frac{y}{z^2}\right)^5 = \frac{y^4}{z^7} > y^{\frac{13}{12}},$$

making a contradiction with the fact $a \leqslant y^{\frac{13}{12}}$. Thus, we have $p_1 < \frac{z^3}{y}$.

Now we have $\frac{y}{z^2} \leqslant p_6 < p_5 < p_4 < p_3 < p_2 < p_1 < \frac{z^3}{y}$. If $p_1p_2p_3^2 < z^2$, then we have

$$RHSC = 1 - 6 \cdot \frac{4}{5} + 15 \cdot \frac{3}{5} - 20 \cdot \frac{2}{5} + 15 \cdot \frac{1}{5} = \frac{1}{5} > 0.$$

If $p_1p_2p_4^2 < z^2 \leqslant p_1p_2p_3^2$, then we have

$$RHSC = 1 - 6 \cdot \frac{4}{5} + 15 \cdot \frac{3}{5} - 19 \cdot \frac{2}{5} + 12 \cdot \frac{1}{5} = 0.$$

If $p_1p_2p_5^2 < z^2 \leqslant p_1p_2p_4^2$ and $p_1p_3p_4^2 < z^2$, then we have

$$RHSC = 1 - 6 \cdot \frac{4}{5} + 15 \cdot \frac{3}{5} - 18 \cdot \frac{2}{5} + 10 \cdot \frac{1}{5} = 0.$$

If $p_1p_2p_5^2 < z^2 \leqslant p_1p_2p_4^2$ and $p_2p_3p_4^2 < z^2 \leqslant p_1p_3p_4^2$, then we have

$$RHSC = 1 - 6 \cdot \frac{4}{5} + 15 \cdot \frac{3}{5} - 17 \cdot \frac{2}{5} + 8 \cdot \frac{1}{5} = 0.$$

If $p_1p_2p_5^2 < z^2 \leqslant p_1p_2p_4^2$ and $p_2p_3p_4^2 \geqslant z^2$, then we have

$$RHSC = 1 - 6 \cdot \frac{4}{5} + 15 \cdot \frac{3}{5} - 16 \cdot \frac{2}{5} + 6 \cdot \frac{1}{5} = 0.$$

If $p_1p_3p_5^2 < z^2 \leqslant p_1p_2p_5^2$ and $p_1p_3p_4^2 < z^2$, then we have

$$RHSC = 1 - 6 \cdot \frac{4}{5} + 15 \cdot \frac{3}{5} - 17 \cdot \frac{2}{5} + 9 \cdot \frac{1}{5} = \frac{1}{5} > 0.$$

If $p_1p_3p_5^2 < z^2 \leqslant p_1p_2p_5^2$ and $p_2p_3p_4^2 < z^2 \leqslant p_1p_3p_4^2$, then we have

$$RHSC = 1 - 6 \cdot \frac{4}{5} + 15 \cdot \frac{3}{5} - 16 \cdot \frac{2}{5} + 7 \cdot \frac{1}{5} = \frac{1}{5} > 0.$$

If $p_1p_3p_5^2 < z^2 \leqslant p_1p_2p_5^2$ and $p_2p_3p_4^2 \geqslant z^2$, then we have

$$RHSC = 1 - 6 \cdot \frac{4}{5} + 15 \cdot \frac{3}{5} - 15 \cdot \frac{2}{5} + 5 \cdot \frac{1}{5} = \frac{1}{5} > 0.$$

If $p_2p_3p_5^2 < z^2 \leqslant p_1p_3p_5^2$ and $p_2p_3p_4^2 < z^2$, then we have

$$RHSC = 1 - 6 \cdot \frac{4}{5} + 15 \cdot \frac{3}{5} - 15 \cdot \frac{2}{5} + 5 \cdot \frac{1}{5} = \frac{1}{5} > 0.$$

If $p_2p_3p_5^2 < z^2 \leqslant p_1p_3p_5^2$ and $p_2p_3p_4^2 \geqslant z^2$, then we have

$$RHSC = 1 - 6 \cdot \frac{4}{5} + 15 \cdot \frac{3}{5} - 14 \cdot \frac{2}{5} + 3 \cdot \frac{1}{5} = \frac{1}{5} > 0.$$

If $p_2p_3p_5^2 \geqslant z^2$, then we have

$$RHSC \geqslant 1 - 6 \cdot \frac{4}{5} + 15 \cdot \frac{3}{5} - 13 \cdot \frac{2}{5} = 0.$$

Combining all cases and subcases above, Theorem 1 is proved.

References

[1] Z. E. Brady. Sieves and iteration rules. *Ph.D. Thesis*, Stanford University, 2017.

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