PRIMES IN ARITHMETIC PROGRESSIONS TO MODULI UP TO $x^{8/15}$

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ABSTRACT. We show that the primes up to x are equidistributed in the residue class $a \pmod{q_1q_2}$ on average over moduli q_1q_2 as large as $x^{8/15-\varepsilon}$, when the average is weighted by an arbitrary bilinear weight and the sizes of q_1 and q_2 lie in an explicit region. This improves the exponent $9/17$ for bilinear weights, and the exponent $17/32$ for trilinear weights, obtained recently by Li. The new ingredient is an estimate for the Type-II convolution sums with both factors have size about $x^{1/2}$. With this estimate, the level of distribution is no longer limited by the Type-II sums but by the Type-I_3 sums, or the current range in the ternary divisor problem.

1. INTRODUCTION

1.1. **Background.** For coprime integers a and $q > 1$, let $\pi(x; q, a)$ denote the number of primes $p \leqslant x$ with $p \equiv a \pmod{q}$. The famous Bombieri–Vinogradov theorem [2] [26] states that for every $A > 0$ there is $B = B(A)$ such that

$$\sum_{q \leqslant x^{1/2}(\log x)^{-B}} \max_{(a,q)=1} \left| \pi(x; q, a) - \frac{\pi(x)}{\varphi(q)} \right| \ll \frac{x}{(\log x)^A}. \tag{1}$$

The Elliott–Halberstam conjecture [9] predicts that the range of summation may be extended to $q \leqslant x^{1-\varepsilon}$, but no estimate of the strength of (1) is known with $x^{1/2}$ replaced by $x^{1/2+\delta}$ for any $\delta > 0$.

Beginning with the work of Fouvry and Iwaniec [13] [11] and of Bombieri, Friedlander and Iwaniec [3] [4] [5], one has nevertheless been able to go beyond $x^{1/2}$ for a *fixed* non-zero residue class a, provided that the absolute values in (1) are replaced by weights with suitable factorisation properties, or that the moduli are restricted. Thus one knows such results for well-factorable weights of level $x^{4/7-\varepsilon}$ [3], for triply well-factorable weights of level $x^{3/5-\varepsilon}$ [21] and $x^{5/8-\varepsilon}$ [22], for smooth moduli [27] [23] [25], and for moduli with a conveniently sized divisor: Maynard [20] proved that

$$\sum_{\substack{q_1 \leqslant Q_1 \\ q_2 \leqslant Q_2 \\ (q_1q_2, a)=1}} \left| \pi(x; q_1q_2, a) - \frac{\pi(x)}{\varphi(q_1q_2)} \right| \ll \frac{x}{(\log x)^A} \tag{2}$$

whenever

$$Q_1^2 Q_2 < x^{1-\varepsilon}, \quad Q_1^7 Q_2^{12} < x^{4-\varepsilon}, \quad Q_1^{19} Q_2^{20} < x^{10-\varepsilon},$$

which allows Q_1Q_2 to be as large as $x^{11/21-\varepsilon}$.

An intermediate situation between (2) and well-factorable weights is that of *bilinear weights*: one asks for estimates of the form

$$\sum_{\substack{q_1 \leqslant Q_1 \\ q_2 \leqslant Q_2 \\ (q_1q_2, a)=1}} \lambda_1(q_1)\lambda_2(q_2) \left(\pi(x; q_1q_2, a) - \frac{\pi(x)}{\varphi(q_1q_2)} \right) \ll \frac{x}{(\log x)^A} \tag{3}$$

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for arbitrary divisor-bounded sequences λ_1, λ_2. In 1986, Bombieri, Friedlander and Iwaniec [3] first showed (3) holds if

$$Q_1 < x^{1/3}, \quad Q_2 < x^{1/5}, \quad Q_1^5 Q_2^2 < x^2, \quad Q_1 Q_2 < x^{29/56}.$$

In 1987, Fouvry [12] proved that (3) holds if either

$$Q_1 Q_2^3 < x, \quad Q_1 Q_2 < x^{29/56}, \quad Q_1 < \max\left(x^{1/2} Q_2^{-1}, x^{2/5} Q_2^{-2/5}\right)$$

or

$$Q_1 Q_2^3 < x, \quad Q_1 Q_2 < x^{29/56}, \quad Q_1^4 Q_2 < x^{403/266}, \quad Q_1^{7/4} Q_2 < x^{403/532}.$$

In 1998, Baker and Harman [1] improved the result of Bombieri, Friedlander and Iwaniec [3] by removing the condition $Q_1^5 Q_2^2 < x^2$ in their result above. They showed that (3) holds if

$$Q_1 < x^{1/3}, \quad Q_2 < x^{1/5}, \quad Q_1 Q_2 < x^{29/56}.$$

Clearly (2) implies (3) in Maynard's region. Recently Li [17] showed, by combining the arithmetic information of [20], [12], [14] and [18] with refinements of Harman's sieve, that (3) holds whenever

$$Q_1^2 Q_2 < x^{1-\varepsilon}, \qquad Q_1^7 Q_2^{12} < x^{4-\varepsilon}, \tag{4}$$

so that $Q_1 Q_2$ may be as large as $x^{9/17-\varepsilon}$. For quadrilinear weights Lichtman [18] reached moduli of size $x^{17/32-\varepsilon}$, and Li [17] obtained the same exponent 17/32 for trilinear weights.

1.2. **Main Results.** The purpose of this paper is to show that for bilinear weights one can reach the exponent 8/15.

Theorem 1.1. *Let a be a fixed non-zero integer and let $\varepsilon > 0$, $A > 0$, $B_0 > 0$. Let $Q_1, Q_2 \geqslant 1$ satisfy*

$$Q_1^2 Q_2 \leqslant x^{1-\varepsilon}, \qquad Q_1 Q_2^5 \leqslant x^{1-\varepsilon}, \qquad Q_1^6 Q_2^{13} \leqslant x^{4-\varepsilon}, \qquad Q_1 Q_2 \leqslant x^{8/15-\varepsilon}. \tag{5}$$

Let λ_1, λ_2 be complex sequences with $|\lambda_i(q)| \leqslant \tau(q)^{B_0}$. Then

$$\sum_{\substack{q_1 \leqslant Q_1 \\ q_2 \leqslant Q_2 \\ (q_1 q_2, a) = 1}} \lambda_1(q_1)\lambda_2(q_2) \left(\pi(x; q_1 q_2, a) - \frac{\pi(x)}{\varphi(q_1 q_2)}\right) \ll_{a,\varepsilon,A,B_0} \frac{x}{(\log x)^A}.$$

Write $Q_i = x^{\theta_i}$. In terms of θ_1, θ_2, the region (5) is given by $2\theta_1 + \theta_2 < 1$, $\theta_1 + 5\theta_2 < 1$, $6\theta_1 + 13\theta_2 < 4$ and $\theta_1 + \theta_2 < 8/15$. The first condition forces $\theta_1 < 1/2$, so that q_1 is the "large" and q_2 the "small" factor whenever $\theta_1 + \theta_2 \geqslant 1/2$. Parametrising by the level, we obtain the following.

Corollary 1.2. *Let $0 < \delta < 1/30$ and let θ_2 satisfy*

$$2\delta < \theta_2 < \min\left(\frac{1-2\delta}{8}, \frac{1-6\delta}{7}\right).$$

Then (3) holds with $Q_1 = x^{1/2+\delta-\theta_2}$ and $Q_2 = x^{\theta_2}$. In particular, for every fixed $\theta_2 \in (1/15, 4/35)$ and every sufficiently small $\varepsilon > 0$, the estimate (3) holds with $Q_2 = x^{\theta_2}$ and $Q_1 Q_2 = x^{8/15-\varepsilon}$.

Indeed, with $\theta_1 = 1/2 + \delta - \theta_2$, the four conditions read $\theta_2 > 2\delta$, $\theta_2 < (1-2\delta)/8$, $\theta_2 < (1-6\delta)/7$ and $\delta < 1/30$, and Theorem 1.1 applies with a sufficiently small ε. A simple verification shows that Li's 9/17 region (4) with $Q_1 Q_2 \geqslant x^{1/2}$ (the part with $Q_1 Q_2 < x^{1/2-\varepsilon}$ being covered by the Bombieri–Vinogradov theorem) is contained in (5): if $2\theta_1 + \theta_2 < 1$, $7\theta_1 + 12\theta_2 < 4$ and $\theta_1 + \theta_2 \geqslant 1/2$, then $\theta_2 < 1/10$ and $\theta_1 > 2/5$, whence $\theta_1 + 5\theta_2 < (1+9\theta_2)/2 < 1$, $6\theta_1 + 13\theta_2 = (7\theta_1 + 12\theta_2) - (\theta_1 - \theta_2) < 4$ and $\theta_1 + \theta_2 < 9/17 < 8/15$. The known results are compared in Table 1.

Since a Dirichlet convolution of divisor-bounded sequences is also divisor-bounded, Theorem 1.1 immediately covers multilinear weights.

	Type of average	Conditions on θ_i	$\sup(\sum \theta_i)$
B–F–I [3]	bilinear weights	$\theta_1 < \frac{1}{3}$, $\theta_2 < \frac{1}{5}$, $5\theta_1 + 2\theta_2 < 2$, $\theta_1 + \theta_2 < \frac{29}{56}$	$29/56 \approx 0.5179$
Fouvry [12]	bilinear weights	$\theta_1 + 3\theta_2 < 1$, $\theta_1 + \theta_2 < \frac{29}{56}$, $\theta_1 < \max(\frac{1}{2} - \theta_2, \frac{2}{5}(1 - \theta_2))$;	$29/56 \approx 0.5179$
		$\theta_1 + 3\theta_2 < 1$, $\theta_1 + \theta_2 < \frac{29}{56}$, $4\theta_1 + \theta_2 < \frac{403}{266}$, $\frac{7}{4}\theta_1 + \theta_2 < \frac{403}{532}$	$29/56 \approx 0.5179$
B–H [1]	bilinear weights	$\theta_1 < \frac{1}{3}$, $\theta_2 < \frac{1}{5}$, $\theta_1 + \theta_2 < \frac{29}{56}$	$29/56 \approx 0.5179$
Maynard [20]	absolute values	$2\theta_1 + \theta_2 < 1$, $7\theta_1 + 12\theta_2 < 4$, $19\theta_1 + 20\theta_2 < 10$	$11/21 \approx 0.5238$
Lichtman [18]	quadrilinear weights	$\theta_1 + \theta_2 < \frac{1}{2}$, $\theta_1 + 2\theta_3 < \frac{1}{2}$, $2\theta_3 < \theta_2 < \frac{1}{32}$	$17/32 = 0.53125$
Li [17]	bilinear weights	$2\theta_1 + \theta_2 < 1$, $7\theta_1 + 12\theta_2 < 4$	$9/17 \approx 0.5294$
	trilinear weights	see [17]	$17/32 = 0.53125$
Theorem 1.1	bilinear weights	$2\theta_1 + \theta_2 < 1$, $\theta_1 + 5\theta_2 < 1$, $6\theta_1 + 13\theta_2 < 4$, $\theta_1 + \theta_2 < \frac{8}{15}$	$8/15 \approx 0.5333$

TABLE 1. Average equidistribution results for primes in a fixed residue class to factored moduli

Corollary 1.3. *Let $k \geqslant 2$, let $Q_1, \dots, Q_k \geqslant 1$ and let $\lambda_1, \dots, \lambda_k$ be complex sequences with $|\lambda_i(q)| \leqslant \tau(q)^{B_0}$. Suppose that for some $I \subset \{1, \dots, k\}$, the pair $\big(\prod_{i\in I} Q_i, \prod_{i\notin I} Q_i\big)$ satisfies (5). Then, for every fixed non-zero integer a and every $A > 0$,*

$$\sum_{\substack{q_i \leqslant Q_i \ (1\leqslant i\leqslant k)\\ (q_1\cdots q_k, a)=1}} \lambda_1(q_1)\cdots\lambda_k(q_k)\left(\pi(x; q_1\cdots q_k, a) - \frac{\pi(x)}{\varphi(q_1\cdots q_k)}\right) \ll \frac{x}{(\log x)^A}.$$

In particular this holds for suitable trilinear weights with $Q_1Q_2Q_3 = x^{8/15-\varepsilon}$, for every small $\varepsilon > 0$.

Proof. Put $\lambda^I(q) := \sum \prod_{i\in I} \lambda_i(q_i)$, the sum being over all factorizations $q = \prod_{i\in I} q_i$ with $q_i \leqslant Q_i$, and define λ^J similarly with respect to the complement of I. Then $|\lambda^I(q)|, |\lambda^J(q)| \leqslant \tau(q)^{k(B_0+1)}$, the sequences are supported on $q \leqslant \prod_{i\in I} Q_i$ and $q \leqslant \prod_{i\notin I} Q_i$ respectively, and the sum in question equals the sum of Theorem 1.1 with (λ_1, λ_2) replaced by (λ^I, λ^J). □

1.3. **A new Type-II estimate near $x^{1/2}$.** Results of the above type are proved by decomposing the indicator function of the primes into convolutions, and the range of moduli is limited by the available estimates for

$$\Delta(q) = \Delta_{\alpha,\beta}(q; a) := \sum_{m\sim M}\sum_{n\sim N} \alpha_n\beta_m\left(\mathbb{1}_{mn\equiv a \,(\mathrm{mod}\, q)} - \frac{\mathbb{1}_{(mn,q)=1}}{\varphi(q)}\right), \qquad MN \asymp x, \tag{6}$$

in various ranges of N. When the moduli have size $x^{1/2+\delta}$, the most delicate range is $x^{1/2-\delta} \leqslant N \leqslant x^{1/2+\delta}$, where no estimate is known for general moduli. In the works of Zhang [27], Polymath8a [23] and Maynard [20] this range is treated with the help of the Weil bound for (incomplete) Kloosterman sums, and for moduli q_1q_2 with $q_1 \sim Q$, $q_2 \sim R$ this requires $Q^7R^{12} < x^4$ (see [[20], Proposition 8.2]). This is the origin of the second condition in (4), which is the reason why the region (4) stops at 9/17. Our main new result removes this obstruction for bilinear weights.

Throughout the paper we put

$$z_0 := x^{1/(\log\log x)^3}, \qquad y_0 := x^{1/\log\log x},$$

and we say that a complex sequence (α_n) supported on $n \sim N$ satisfies the *Siegel–Walfisz condition* if for every $d \geqslant 1$, $k \geqslant 1$, $(b,k)=1$ and every $A>1$ we have

$$\left| \sum_{\substack{n\sim N\\ n\equiv b \pmod k\\ (n,d)=1}} \alpha_n - \frac{1}{\varphi(k)} \sum_{\substack{n\sim N\\ (n,dk)=1}} \alpha_n \right| \ll_A \frac{N\tau(d)^{B_0}}{(\log N)^A}. \tag{7}$$

Theorem 1.4 (Type-II estimate near $x^{1/2}$ for bilinear weights)**.** *Let a be a fixed non-zero integer and let $\varepsilon>0$, $A>0$, $B_0>0$. Let $M,N,Q,R \geqslant 1$ satisfy $MN \asymp x$ and*

$$Qx^{\varepsilon} \leqslant N \leqslant x^{1/2+\varepsilon/100}, \tag{8}$$

$$N^{1/2}Q^{1/2}R^4 \leqslant x^{1-\varepsilon}, \tag{9}$$

$$NQ^{1/2}R^{5/2} \leqslant x^{1-\varepsilon}. \tag{10}$$

Let $\alpha_n, \beta_m, \lambda_q, \nu_r$ be complex sequences with $|\alpha_n|, |\beta_n|, |\lambda_n|, |\nu_n| \leqslant \tau(n)^{B_0}$, and assume that α_n is supported on integers n with $P^-(n) \geqslant z_0$ and satisfies the Siegel–Walfisz condition (7). Then, with $\Delta(q)$ as in (6),

$$\sum_{q\sim Q} \sum_{\substack{r\sim R\\ (qr,a)=1}} \lambda_q \nu_r \Delta(qr) \ll_{a,\varepsilon,A,B_0} \frac{x}{(\log x)^A}.$$

When both α and β are of the above type the roles of m and n may be interchanged, and we obtain an estimate in a range of N which is symmetric about $x^{1/2}$.

Corollary 1.5. *Let $\varepsilon>0$ and let $Q,R \geqslant 1$ satisfy*

$$QR^5 \leqslant x^{1-3\varepsilon}, \qquad QR^8 \leqslant x^{3/2-3\varepsilon}.$$

Let $MN \asymp x$ with $Qx^{\varepsilon} \leqslant N \leqslant x^{1-\varepsilon}/Q$, and let $\alpha_n, \beta_m, \lambda_q, \nu_r$ be divisor-bounded as in Theorem 1.4, where now both α_n and β_m are supported on integers without prime factors below z_0 and satisfy the Siegel–Walfisz condition. Then

$$\sum_{q\sim Q} \sum_{\substack{r\sim R\\ (qr,a)=1}} \lambda_q \nu_r \Delta(qr) \ll \frac{x}{(\log x)^A}.$$

For instance, suppose that $Q^2R = x^{1-3\varepsilon}$, so that the range $Qx^{\varepsilon} \leqslant N \leqslant x^{1-\varepsilon}/Q$ contains the whole interval from $x^{1+\varepsilon}/(QR)$ to $QRx^{-\varepsilon}$. Then the corollary applies for $R \leqslant x^{(1-3\varepsilon)/9}$, that is, for moduli qr of size almost $x^{5/9}$. By comparison, the Zhang-type estimate [[20], Proposition 8.2] (valid with absolute values) requires $Q^7R^{12} < x^4$, which under $Q^2R \approx x$ means $R < x^{1/17}$ and $QR < x^{9/17}$. Theorem 1.4 should also be compared with [[3], Theorem 1], which in the present notation (and under similar assumptions on the sequences) gives an estimate of the same type when

$$Qx^{\varepsilon} < N, \qquad N^2R < x^{1-2\varepsilon}, \qquad NQR^5 < x^{2-\varepsilon}, \qquad NQ^{1/2}R^2 < x^{1-\varepsilon}.$$

The condition $N^2R < x^{1-2\varepsilon}$ excludes precisely the critical range $x/(QR) \leqslant N \leqslant x^{1/2}$ (note that $N^2R \geqslant x$ there, if $Q \leqslant N$). Theorem 1.4 trades this condition for $N \leqslant x^{1/2+\varepsilon/100}$, at the cost of replacing $NQR^5 < x^2$ and $NQ^{1/2}R^2 < x$ by the slightly stronger conditions (9) and (10).

1.4. **Outline of the method.** We describe the proof of Theorem 1.4 in a simplified situation, ignoring coprimality conditions and factors $x^{o(1)}$. Following Linnik's dispersion method, we apply Cauchy's inequality to the sum over q and m; this removes the unknown coefficients λ_q and β_m and makes q and m *smooth* variables. After an application of Poisson summation to the sum over m, and after the removal of the expected main terms, one has to show that

$$\sum_{q\sim Q}\sum_{r_1,r_2\sim R}\frac{\nu_{r_1}\overline{\nu_{r_2}}}{qr_1r_2}\sum_{\substack{n_1,n_2\sim N\\ n_1\equiv n_2 \pmod q}}\alpha_{n_1}\overline{\alpha_{n_2}}\sum_{1\leqslant|h|\leqslant H}e\left(ah\frac{n_1-n_2}{q}\cdot\frac{\overline{n_2r_1}}{n_1r_2}\right),\qquad H\approx\frac{QR^2}{M},\tag{11}$$

is smaller than the trivial bound N^2H/Q by a factor a little larger than H.

The treatments of sums like (11) in the literature differ in the choice of the variables which are kept outside in a second application of Cauchy's inequality. In the first method of [[3], Section 8] one replaces the smooth variable q by the complementary divisor $f=(n_1-n_2)/q$ of n_1-n_2, and keeps (n_1r_2,n_2,f) outside and (h,r_1) inside; the off-diagonal terms are sums of Kloosterman fractions over the two smooth variables $c=n_1r_2$ and $d=n_2$ and are estimated by the bounds of Deshouillers and Iwaniec [7], but the diagonal terms are admissible only if $N^2R<x$. In the work of Zhang [27] (see [[20], Chapter 17]) the variables (q,n_1,n_2) are kept outside and (h,r_1,r_2) inside; the diagonal terms are then admissible for all N up to about $x^{1/2}$, but the coefficients depend on q (as they must in a result with absolute values), and the off-diagonal terms are estimated by completing the sums over n_1,n_2 and applying the Weil bound, which leads to $Q^7R^{12}<x^4$.

Our observation is that for bilinear weights the two approaches can be combined. Since q is smooth we may pass to $f=(n_1-n_2)/q$, which is a *short* variable, $|f|\leqslant N/Q$. We then keep (f,n_1,n_2) outside and (h,r_1,r_2) inside, and we drop by positivity the congruence $n_1\equiv n_2 \pmod f$, so that n_1 and n_2 become free smooth variables of length N. After expanding the square, and after the usual decomposition of the pairs (r_1,r_2) into classes within which all coprimality conditions hold [11], the off-diagonal terms are

$$\sum_f\sum_{\substack{(h,r_1,r_2)\\(h',r_1',r_2')}}\xi(h,r_1,r_2)\overline{\xi(h',r_1',r_2')}\sum_{n_1,n_2}\Psi\left(\frac{n_1}{N}\right)\Psi\left(\frac{n_2}{N}\right)e\left(afk\frac{\overline{n_2r_1r_1'}}{n_1[r_2,r_2']}\right)$$

with certain bounded coefficients ξ and with

$$k=\frac{hr_1'r_2'-h'r_1r_2}{(r_2,r_2')}\neq 0.$$

This is exactly a sum of the type considered in [[7], Theorem 12], with both "level" variables $r=r_1r_1'$ and $s=[r_2,r_2']$ present. The resulting bound is admissible under (9) and (10); these two conditions come from the regular and from the (possible) exceptional spectrum respectively. The diagonal terms $k=0$ are admissible as long as $N\leqslant x^{1/2-\varepsilon}$. In order to reach $N=x^{1/2+\varepsilon/100}$ we use the de-amplification device of Maynard [[20], Chapter 14]: before the first application of Cauchy's inequality, the variable n is split into residue classes modulo e for auxiliary moduli $e\sim E=x^{O(\varepsilon)}$, which produces the stronger congruence $n_1\equiv n_2 \pmod{qe}$ in (11). The congruence modulo e cannot be dropped; we retain it in the outer summation and appeal to the generalization of the Deshouillers–Iwaniec bound to sums with congruence conditions due to Drappeau [8].

1.5. **Scope and limitations.** With Theorem 1.4 in hand the Type-II sums are under control for moduli of size up to about $x^{5/9}$ (if $Q^2R\approx x$). The exponent 8/15 in Theorem 1.1 comes from a different source: products of three primes of size about $x^{1/3}$ are handled by means of estimates for the ternary divisor function [[20], Chapter 20], which are currently available only for moduli up to $x^{8/15}$. The conditions $Q_1Q_2^5\leqslant x^{1-\varepsilon}$ and $Q_1^6Q_2^{13}\leqslant x^{4-\varepsilon}$ in (5) are needed for a Type-II

estimate with a small factor due to Fouvry [12] (the first of them is also needed in Theorem 1.4), and $Q_1^2 Q_2 \leqslant x^{1-\varepsilon}$ guarantees that the range of Theorem 1.4 covers the critical interval around $x^{1/2}$.

The method relies in an essential way on the fact that the coefficient of the large modulus is removed by the first application of Cauchy's inequality. It therefore gives nothing new for general moduli (the exponent 1/2 in (1), for either fixed or uniform residue classes), nor for estimates with absolute values such as (2); the region of [3] [1], where q_1 and q_2 are of comparable size, is also disjoint from (5), and 29/56 remains the best exponent there. For well-factorable weights Theorem 1.4 provides new information when $x^{3/7} < N \leqslant x^{1/2}$, but it does not by itself improve the exponent 4/7 of [3]; see Section 9 for a detailed explanation.

Sections 2–6 contain the proof of Theorem 1.4. The new arguments are in Section 5; the reduction to exponential sums in Section 3 is a variant of [[20], Proposition 14.4], and we indicate precisely where it differs. In Section 7 we deduce Theorem 1.1, following the architecture of [20]: the only new arithmetic information input is Theorem 1.4, the remaining ingredients are taken from [4], [12], [14], [20] and [18]. In Section 8 we provide several applications of Theorem 1.1.

Notation. Throughout this paper, we always suppose that x is a sufficiently large real number, a is a fixed non-zero integer and ε, ε_0 are sufficiently small positive real numbers. We use both Landau and Vinogradov notations, and implied constants may depend on a, ε, and on the exponents A, B_0. We write $e(t) := \exp(2\pi i t)$, and $n \sim N$ for $N < n \leqslant 2N$. As usual, τ, φ, μ denote the divisor function, Euler's totient function and the Möbius function, $P^-(n)$ and $P^+(n)$ denote the least and the greatest prime factor of n, and (m, n), $[m, n]$ denote the greatest common divisor and the least common multiple. If $(n, c) = 1$, then $\overline{n}$ denotes an inverse of n modulo c; for complex coefficients a bar denotes complex conjugation. A sequence is called 1-*bounded* if all its terms have modulus at most 1. We write $\mathrm{sm}(n; z) := \prod_{p \leqslant z} p^{v_p(n)}$ for the z-smooth part of n. Following [20], we let $\psi_0 : \mathbb{R} \to \mathbb{R}$ be the fixed smooth function in [[20], Chapter 6]; it is supported on $[1/2, 5/2]$ and equal to 1 on $[1, 2]$, and $\widehat{\psi_0}(\xi) = \int \psi_0(t) e(-\xi t) \mathrm{d}t$ is its Fourier transform. Finally we fix a smooth function $\Psi : \mathbb{R} \to [0, 1]$ supported on $[1/2, 5/2]$ with $\Psi = 1$ on $[1, 2]$.

2. Preliminary Lemmas

In this section we establish various lemmas which will be used in later sections.

2.1. Reciprocity.

Lemma 2.1 (Bezout's identity)**.** *Let $(q_1, q_2) = 1$. Then we have*

$$\frac{a}{q_1 q_2} \equiv \frac{a\overline{q_1}}{q_2} + \frac{a\overline{q_2}}{q_1} \pmod 1.$$

The next lemma will be used to compute the phase arising from Poisson summation in the dispersion method. In [3] and [20] the corresponding computation is carried out with Lemma 2.1 under the additional assumption $(\ell r_1, r_2) = 1$; it is convenient for us to avoid this assumption.

Lemma 2.2. *Let $\ell, r_1, r_2, n_1, n_2 \geqslant 1$ and b be integers with*

$$(r_1, r_2) = (n_1, n_2) = (n_1, r_1) = (n_2, r_2) = 1 \qquad \text{and} \qquad n_1 \equiv n_2 \pmod \ell,$$

and put $f := (n_1 - n_2)/\ell$. Then $(n_1 n_2, \ell) = 1$, the system of congruences

$$m n_1 \equiv b \pmod{\ell r_1}, \qquad m n_2 \equiv b \pmod{\ell r_2}$$

has a unique solution $m \equiv \mu \pmod{\ell r_1 r_2}$, and

$$\frac{\mu}{\ell r_1 r_2} \equiv b f \frac{\overline{n_2 r_1}}{n_1 r_2} + \frac{b}{\ell r_1 r_2 n_1} \pmod 1.$$

Proof. A prime dividing ℓ and n_1 divides $n_2 = n_1 - \ell f$, which is impossible; hence $(n_1 n_2, \ell) = 1$. Consequently $(n_1, \ell r_1) = (n_2, \ell r_2) = 1$, so that two solutions of the system agree modulo ℓr_1 and modulo ℓr_2, hence modulo $[\ell r_1, \ell r_2] = \ell r_1 r_2$. Our assumptions give $(n_2 r_1, n_1 r_2) = 1$. Let X be an integer with $X n_2 r_1 \equiv bf \pmod{n_1 r_2}$, say $X n_2 r_1 = bf + t n_1 r_2$. Multiplying by ℓ and using $\ell f = n_1 - n_2$ we get

$$\ell r_1 X n_2 + b n_2 = b n_1 + \ell t n_1 r_2.$$

Thus n_1 divides $(\ell r_1 X + b) n_2$, and since $(n_1, n_2) = 1$ the number $\mu := (\ell r_1 X + b)/n_1$ is an integer. By construction $\mu n_1 = \ell r_1 X + b \equiv b \pmod{\ell r_1}$, and the displayed identity gives $\mu n_2 = b + \ell t r_2 \equiv b \pmod{\ell r_2}$. Finally

$$\frac{\mu}{\ell r_1 r_2} = \frac{\ell r_1 X + b}{\ell r_1 r_2 n_1} = \frac{X}{n_1 r_2} + \frac{b}{\ell r_1 r_2 n_1}.$$

□

Lemma 2.3. *Let $n_1, n_2, r_1, r_1', r_2, r_2' \geqslant 1$ be integers with $(n_2 r_1 r_1', n_1 r_2 r_2') = 1$, and let $h, h' \in \mathbb{Z}$. Then*

$$h\frac{\overline{n_2 r_1}}{n_1 r_2} - h'\frac{\overline{n_2 r_1'}}{n_1 r_2'} \equiv k\frac{\overline{n_2 r_1 r_1'}}{n_1[r_2, r_2']} \pmod 1, \qquad k := \frac{h r_1' r_2' - h' r_1 r_2}{(r_2, r_2')}.$$

Proof. Let $s := [r_2, r_2']$ and let Y be an inverse of $n_2 r_1 r_1'$ modulo $n_1 s$. Then $r_1' Y$ is an inverse of $n_2 r_1$ modulo $n_1 r_2$ and $r_1 Y$ is an inverse of $n_2 r_1'$ modulo $n_1 r_2'$. Hence the left-hand side is congruent modulo 1 to

$$\frac{h r_1' Y}{n_1 r_2} - \frac{h' r_1 Y}{n_1 r_2'} = \frac{Y}{n_1 s}\left(h r_1' \frac{s}{r_2} - h' r_1 \frac{s}{r_2'}\right),$$

and $s/r_2 = r_2'/(r_2, r_2')$, $s/r_2' = r_2/(r_2, r_2')$. □

2.2. **Splitting into coprime sets.**

Lemma 2.4. *([[20], Lemma 13.2]). Let $\mathcal{N}$ be a set of pairs (u, v) of positive integers with $u, v \leqslant x$, $(u, v) = 1$ and $P^-(uv) \geqslant z_0$. Then there is a partition $\mathcal{N} = \mathcal{N}_1 \sqcup \cdots \sqcup \mathcal{N}_J$ with*

$$J \ll \exp\left(O(\log\log x)^7\right)$$

such that $(u, v') = (u', v) = 1$ whenever (u, v) and (u', v') lie in the same class $\mathcal{N}_j$.

2.3. **Kloosterman sums.**

Lemma 2.5 (Incomplete Kloosterman sums)**.** *Let $C_0 > 0$ and let $f : \mathbb{R} \to \mathbb{C}$ be a smooth function supported on $[1/2, 5/2]$ with $\|f^{(j)}\|_\infty \ll_j (\log x)^{jC_0}$ for all $j \geqslant 0$. Then for all integers b and $1 \leqslant c \leqslant x^{C_0}$ and all real $0 < Y \leqslant x^{C_0}$ we have*

$$\sum_{\substack{n \\ (n,c)=1}} f\left(\frac{n}{Y}\right) e\left(\frac{b\overline{n}}{c}\right) \ll x^{o(1)}\left(c^{1/2} + \frac{Y}{c}(b, c)\right).$$

Proof. If $Y < 1$, the sum has $O(1)$ terms and the bound is trivial. For $Y \geqslant 1$, this estimate follows from completion of the sum and the Weil bound; see [[20], Lemma 16.1]. □

The following deep estimate for sums of Kloosterman sums is the main tool of this paper. It originates in the work of Deshouillers and Iwaniec [[7], Theorem 12], where it is deduced from the Kuznetsov trace formula. The version with congruence conditions is due to Drappeau [8].

Theorem 2.6. *([[8], Theorem 2.1], with a correction from [6]). Let $\mathsf{C}, \mathsf{D}, \mathsf{N}, \mathsf{R}, \mathsf{S} \geqslant 1$, and let $\mathsf{q}, c_0, d_0 \geqslant 1$ be integers with $(c_0 d_0, \mathsf{q}) = 1$. Let $(b_{n,r,s})$ be a complex sequence supported on $1 \leqslant$*

$n \leqslant \mathsf{N}$, $\mathsf{R} < r \leqslant 2\mathsf{R}$, $\mathsf{S} < s \leqslant 2\mathsf{S}$. *Let* $g : \mathbb{R}^5_{>0} \to \mathbb{C}$ *be a smooth function, compactly supported in* $(\mathsf{C}, 2\mathsf{C}] \times (\mathsf{D}, 2\mathsf{D}] \times \mathbb{R}^3_{>0}$, *such that*

$$\frac{\partial^{\nu_1+\cdots+\nu_5} g}{\partial c^{\nu_1}\partial d^{\nu_2}\partial n^{\nu_3}\partial r^{\nu_4}\partial s^{\nu_5}}(c,d,n,r,s) \ll_{\nu_1,\dots,\nu_5} \left(c^{-\nu_1}d^{-\nu_2}n^{-\nu_3}r^{-\nu_4}s^{-\nu_5}\right)^{1-\epsilon_0}$$

for some small $\epsilon_0 > 0$ *and all fixed* $\nu_j \geqslant 0$. *Then*

$$\sum_{\substack{c,d,n,r,s \\ c\equiv c_0, d\equiv d_0 \pmod{\mathsf{q}} \\ (\mathsf{q}rd, sc)=1}} b_{n,r,s} g(c,d,n,r,s) e\left(n\frac{\overline{rd}}{sc}\right) \ll_{\epsilon,\epsilon_0} (\mathsf{qCDNRS})^{\epsilon+O(\epsilon_0)}\mathsf{q}^{3/2}\mathsf{K}\|b\|_2,$$

where $\|b\|_2 := \left(\sum_{n,r,s} |b_{n,r,s}|^2\right)^{1/2}$ *and*

$$\mathsf{K}^2 := \mathsf{qCS}(\mathsf{RS}+\mathsf{N})(\mathsf{C}+\mathsf{RD}) + \mathsf{C}^2\mathsf{DS}\sqrt{(\mathsf{RS}+\mathsf{N})\mathsf{R}} + \mathsf{D}^2\mathsf{NR}. \tag{12}$$

3. The dispersion method

In this section we reduce the estimation of $\sum \lambda_q \nu_r \Delta(qr)$ to that of two exponential sums. The reduction is that of [[20], Chapter 14], which incorporates the de-amplification by an auxiliary modulus e; we need it in a slightly different form.

Let $M, N, Q, R, D, E \geqslant 1$ with $MN \asymp x$, and put

$$H_1 := \frac{QDR}{M}(\log x)^5, \qquad H_2 := \frac{QDR^2}{M}(\log x)^5.$$

For complex sequences (α_n), $(\lambda_{d,r})$ and a constant $C \geqslant 0$ we define

$$\mathcal{E}_1 := \sum_{e\sim E}\mu^2(e)\sum_{\substack{q \\ (q,a)=1}}\psi_0\left(\frac{q}{Q}\right)\sum_{\substack{d\sim D \\ (d,a)=1}}\sum_{\substack{r_1,r_2\sim R \\ (r_1r_2,a)=1}}\frac{\lambda_{d,r_1}\overline{\lambda_{d,r_2}}}{\varphi(qder_2)\,qdr_1}$$
$$\times \sum_{\substack{n_1,n_2\sim N \\ (n_1,qder_1)=1 \\ (n_2,qder_2)=1}}\alpha_{n_1}\overline{\alpha_{n_2}}\sum_{1\leqslant|h|\leqslant H_1}\widehat{\psi}_0\left(\frac{hM}{qdr_1}\right)e\left(\frac{ah\overline{n_1}}{qdr_1}\right)$$

and

$$\mathcal{E}_2 := \sum_{e\sim E}\mu^2(e)\sum_{\substack{q \\ (q,a)=1}}\psi_0\left(\frac{q}{Q}\right)\sum_{\substack{d\sim D \\ (d,a)=1}}\sum_{\substack{r_1,r_2\sim R \\ (r_1,r_2)=1 \\ (r_1r_2,ae)=1}}\frac{\lambda_{d,r_1}\overline{\lambda_{d,r_2}}}{qdr_1r_2}\sum_{\substack{n_1,n_2\sim N \\ n_1\equiv n_2 \pmod{qde} \\ (n_1,n_2)=(n_1,r_1)=(n_2,r_2)=1 \\ |n_1-n_2|\geqslant N/(\log x)^C}}\alpha_{n_1}\overline{\alpha_{n_2}}$$
$$\times \sum_{1\leqslant|h|\leqslant H_2}\widehat{\psi}_0\left(\frac{hM}{qdr_1r_2}\right)e\left(ah\frac{n_1-n_2}{qd}\cdot\frac{\overline{n_2r_1}}{n_1r_2}\right)e\left(\frac{ah}{qdr_1r_2n_1}\right).$$

Note that in $\mathcal{E}_2$ the conditions $(n_1, n_2) = 1$ and $n_1 \equiv n_2 \pmod{qde}$ imply $(n_1n_2, qde) = 1$.

Proposition 3.1 (Reduction to exponential sums)**.** *Let* $\alpha_n, \beta_m, \gamma_{q,d}, \lambda_{d,r}$ *be complex sequences with* $|\alpha_n|, |\beta_n| \leqslant \tau(n)^{B_0}$, $|\gamma_{q,d}| \leqslant \tau(qd)^{B_0}$ *and* $|\lambda_{d,r}| \leqslant \tau(dr)^{B_0}$. *Suppose that* α_n *is supported on* $P^-(n) \geqslant z_0$ *and satisfies the Siegel–Walfisz condition* (7), *and that* $\lambda_{d,r}$ *is supported on* $P^-(r) \geqslant z_0$. *Let*

$$\mathcal{S} := \sum_{\substack{d\sim D \\ (d,a)=1}}\sum_{\substack{q\sim Q \\ (q,a)=1}}\sum_{\substack{r\sim R \\ (r,a)=1}}\lambda_{d,r}\gamma_{q,d}\sum_{m\sim M}\beta_m\sum_{n\sim N}\alpha_n\left(\mathbb{1}_{mn\equiv a \pmod{qrd}} - \frac{\mathbb{1}_{(mn,qrd)=1}}{\varphi(qrd)}\right).$$

Let $A > 0$, $1 \leqslant E \leqslant x$, and let $C = C(A, B_0)$ be sufficiently large. If

$$N > QDE(\log x)^C \qquad \text{and} \qquad M > (\log x)^C,$$

then

$$|\mathcal{S}| \ll_{A,B_0} \frac{x}{(\log x)^A} + MD^{1/2}Q^{1/2}(\log x)^{O_{B_0}(1)} \left(|\mathcal{E}_1|^{1/2} + |\mathcal{E}_2|^{1/2}\right).$$

Proof. This is [[20], Proposition 14.4], in the special case that the coefficients $\lambda_{q,d,r}$ of [20] do not depend on q, except that our sum $\mathcal{E}_2$ differs from the corresponding sum (denoted E_2) of [20] in three aspects: the condition $(r_1r_2, e) = 1$ is present; the condition $(r_2, qd) = 1$ is absent; and the phase is written in a different form. We indicate the necessary changes in the proof.

As in the proof of [[20], Proposition 14.4], one inserts the conditions $(n, e) = (r, e) = 1$ at a negligible cost, splits the sum over n into residue classes $b \pmod{qde}$, averages over square-free $e \sim E$ and applies Cauchy's inequality to the sum over e, b, m, q, d. This gives

$$|\mathcal{S}|^2 \ll MQD(\log x)^{O(1)} \left(\mathcal{S}_1 - 2\operatorname{Re}\mathcal{S}_2 + \mathcal{S}_3\right) + O\left(\frac{x^2}{(\log x)^{2A}}\right),$$

where $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are the sums evaluated in [[20], Lemmas 14.1–14.3] (with $\eta_{q,d,r} = \lambda_{d,r}$). By [[20], Lemmas 14.1–14.2], we have

$$\mathcal{S}_1 = \mathcal{S}_{MT} + O\left(\frac{MN^2}{QD(\log x)^{2A}}\right), \qquad \mathcal{S}_2 = \mathcal{S}_{MT} + O(M|\mathcal{E}_1|) + O\left(\frac{MN^2}{QD(\log x)^{2A}}\right),$$

with the main term $\mathcal{S}_{MT}$ of [[20], Lemma 14.1]. It remains to show that

$$\mathcal{S}_3 = \mathcal{S}_{MT} + O(M|\mathcal{E}_2|) + O\left(\frac{MN^2}{QD(\log x)^{2A}}\right), \tag{13}$$

where we recall that

$$\mathcal{S}_3 = \sum_{e\sim E} \mu^2(e) \sum_{\substack{q \\ (q,a)=1}} \psi_0\left(\frac{q}{Q}\right) \sum_{\substack{d\sim D \\ (d,a)=1}} \sum_{\substack{r_1,r_2\sim R \\ (r_1r_2,ae)=1}} \lambda_{d,r_1}\overline{\lambda_{d,r_2}} \times \sum_{\substack{n_1,n_2\sim N \\ n_1\equiv n_2 \pmod{qde} \\ (n_1,qder_1)=(n_2,qder_2)=1}} \alpha_{n_1}\overline{\alpha_{n_2}} \sum_{\substack{m \\ mn_1\equiv a \pmod{qdr_1} \\ mn_2\equiv a \pmod{qdr_2}}} \psi_0\left(\frac{m}{M}\right).$$

We follow the proof of [[20], Lemma 14.3]. Exactly as there, one may restrict the summation to $(r_1, r_2) = 1$, to $(n_1, n_2) = 1$ and to $|n_1 - n_2| \geqslant N/(\log x)^C$, at the cost of an error $O(MN^2/(QD(\log x)^{2A}))$. We deviate from [20] in two places:

(1) We do not remove the condition $(e, r_1r_2) = 1$. In [20] this condition is removed at the end of the first part of the proof and re-introduced later in the main term; neither step is needed.

(2) We do not discard the terms with $(qd, r_2) > 1$. In [20] this is done only in order to apply the Chinese remainder theorem in the form of Lemma 2.1; we use Lemma 2.2 instead. Indeed, after the above restrictions the hypotheses of Lemma 2.2 are satisfied with $\ell = qd$ and $b = a$, so the two congruences for m are equivalent to $m \equiv \mu \pmod{qdr_1r_2}$, and Poisson summation [[20], Lemma 13.4] gives

$$\sum_{m\equiv\mu \pmod{qdr_1r_2}} \psi_0\left(\frac{m}{M}\right) = \frac{M\widehat{\psi}_0(0)}{qdr_1r_2} + \frac{M}{qdr_1r_2} \sum_{1\leqslant|h|\leqslant H_2} \widehat{\psi}_0\left(\frac{hM}{qdr_1r_2}\right) e\left(\frac{h\mu}{qdr_1r_2}\right) + O(x^{-100}),$$

where, by Lemma 2.2,

$$e\left(\frac{h\mu}{qdr_1r_2}\right) = e\left(ah\frac{n_1 - n_2}{qd} \cdot \frac{\overline{n_2r_1}}{n_1r_2}\right) e\left(\frac{ah}{qdr_1r_2n_1}\right).$$

The terms with $h \neq 0$ contribute $M\mathcal{E}_2$ to $\mathcal{S}_3$. In the contribution of the terms with $h = 0$ one inserts, as in [20], the factor

$$\frac{\varphi(qde)^2\varphi(qdr_1r_2)}{\varphi(qd)\varphi(qder_1)\varphi(qder_2)} = \frac{r_1r_2\prod_{p\mid r_1r_2,\ p\nmid qd}\frac{p-1}{p}}{r_1\prod_{p\mid r_1,\ p\nmid qde}\frac{p-1}{p}\cdot r_2\prod_{p\mid r_2,\ p\nmid qde}\frac{p-1}{p}},$$

which in [20] is $1+O(\log x/z_0)$ and in our situation is exactly equal to 1, since $(r_1, r_2) = (r_1r_2, e) = 1$. From this point on the argument of [20] applies verbatim: the conditions $(r_1, r_2) = 1$, $(n_1, n_2) = 1$ and $|n_1 - n_2| \geqslant N/(\log x)^C$ are removed from the terms with $h = 0$ at the cost of an admissible error, and the condition $n_1 \equiv n_2 \pmod{qde}$ is replaced by the factor $\frac{1}{\varphi(qde)}$ by means of the Siegel–Walfisz condition and the large sieve [[20], Lemma 13.8]; here the hypothesis $N > QDE(\log x)^C$ is used. This gives (13), and Proposition 3.1 follows. □

4. The first exponential sum

The sum $\mathcal{E}_1$ is easily handled, because the sum over q is an incomplete Kloosterman sum with a smooth weight. The following lemma is a variant of [[21], Lemma 5.10].

Lemma 4.1. *Let $\eta > 0$ and $1 \leqslant E \leqslant x$, and suppose that*

$$DRN^{3/2} \leqslant x^{1-2\eta} \qquad \text{and} \qquad QDR \leqslant x^{1-2\eta}. \tag{14}$$

Then for all 1-bounded sequences (α_n) and $(\lambda_{d,r})$, we have

$$\mathcal{E}_1 \ll_\eta \frac{N^2}{QDx^\eta}.$$

Proof. Fix $e, d, r_1, r_2, n_1, n_2, h$ with $(n_1, der_1) = (n_2, der_2) = 1$ and put $m_0 := der_2$. The sum over q in $\mathcal{E}_1$ is

$$\Sigma := \sum_{(q,an_1n_2)=1} \psi_0\left(\frac{q}{Q}\right)\frac{1}{q\varphi(qm_0)}\widehat{\psi}_0\left(\frac{hM}{qdr_1}\right)e\left(\frac{ah\overline{n_1}}{qdr_1}\right).$$

By Lemma 2.1 with $(u, v) = (n_1, qdr_1)$ we have

$$e\left(\frac{ah\overline{n_1}}{qdr_1}\right) = e\left(-\frac{ah\overline{qdr_1}}{n_1}\right)e\left(\frac{ah}{qdr_1n_1}\right).$$

Moreover

$$\frac{1}{\varphi(qm_0)} = \frac{1}{q\varphi(m_0)}\sum_{\substack{g\mid q\\(g,m_0)=1}}\frac{\mu^2(g)}{\varphi(g)}, \qquad \mathbb{1}_{(q,an_2)=1} = \sum_{t\mid(q,an_2)}\mu(t).$$

Hence, writing $q = [g, t]q'$,

$$\Sigma = \frac{1}{\varphi(m_0)}\sum_{(g,m_0n_1)=1}\frac{\mu^2(g)}{\varphi(g)}\sum_{\substack{t\mid an_2\\(t,n_1)=1}}\mu(t)\sum_{(q',n_1)=1}W\left([g,t]q'\right)e\left(\frac{b\overline{q'}}{n_1}\right),$$

where $b := -ah\overline{dr_1[g,t]} \pmod{n_1}$ and

$$W(q) := \psi_0\left(\frac{q}{Q}\right)\frac{1}{q^2}\widehat{\psi}_0\left(\frac{hM}{qdr_1}\right)e\left(\frac{ah}{qdr_1n_1}\right).$$

For $q \asymp Q$ we have $|hM/(qdr_1)| \ll H_1M/(QDR) = (\log x)^5$ and $|ah/(qdr_1n_1)| \ll 1$, so that the function $f(u) := Q^2W(Qu)$ is supported on $[1/2, 5/2]$ and satisfies $\|f^{(j)}\|_\infty \ll_j (\log x)^{5j}$. Since $(b, n_1) = (ah, n_1)$, Lemma 2.5 (with $Y = Q/[g,t]$) shows that the innermost sum is

$$\ll x^{o(1)}Q^{-2}\left(N^{1/2} + \frac{Q}{N}(ah, n_1)\right).$$

Summing over $g \leqslant 3Q$ and over the divisors t of an_2 we obtain

$$\Sigma \ll x^{o(1)}\frac{1}{\varphi(m_0)Q^2}\left(N^{1/2} + QN^{-1}(ah, n_1)\right).$$

Therefore, using $\sum_{1\leqslant|h|\leqslant H_1}(ah, n_1) \ll H_1\tau(n_1)$,

$$\begin{aligned}\mathcal{E}_1 &\ll x^{o(1)}\sum_{e\sim E}\sum_{d\sim D}\sum_{r_1,r_2\sim R}\frac{1}{dr_1\varphi(der_2)}\sum_{n_1,n_2\sim N}\sum_{1\leqslant|h|\leqslant H_1}\frac{1}{Q^2}\left(N^{1/2} + \frac{Q}{N}(ah, n_1)\right)\\ &\ll x^{o(1)}\frac{N^2H_1}{DQ^2}\left(N^{1/2} + \frac{Q}{N}\right)\\ &= x^{o(1)}\frac{N^2}{QD}\cdot\frac{DRN^{3/2} + QDR}{MN},\end{aligned}$$

and the lemma follows from (14) and $MN \asymp x$. □

5. The second exponential sum

In this section we prove an upper bound for $\mathcal{E}_2$, which is the most crucial estimate in the proof of Theorem 1.4 and Theorem 1.1.

Proposition 5.1. *Let $\eta > 0$ and $C \geqslant 0$ be fixed. Let $M, N, Q, R, D, E \geqslant 1$ satisfy $MN \asymp x$ and*

$$Q \leqslant M, \qquad D \leqslant y_0, \qquad QDE \leqslant N, \tag{15}$$

$$ND \leqslant x^{-3\eta}ME, \tag{16}$$

$$E^{7/2}\left(1 + \frac{N}{M}\right)^{1/2}N^{1/2}Q^{1/2}R^4 \leqslant x^{1-4\eta}, \tag{17}$$

$$E^3\left(1 + \frac{N}{M}\right)^{1/2}NQ^{1/2}R^{5/2} \leqslant x^{1-4\eta}. \tag{18}$$

Let (α_n) and $(\lambda_{d,r})$ be 1-bounded complex sequences such that $\lambda_{d,r}$ is supported on $P^-(r) \geqslant z_0$. Then

$$\mathcal{E}_2 \ll_{a,\eta} \frac{N^2}{QDx^\eta}.$$

The rest of this section is devoted to the proof. We note that (17) implies $R \leqslant x^{1/4}$, and that $H_2 = QDR^2(\log x)^5/M \leqslant DR^2(\log x)^5 \leqslant x^{1/2+o(1)}$ by (15).

5.1. Passage to the complementary divisor.

The trivial bound for $\mathcal{E}_2$ is

$$\sum_{e\sim E}\sum_{d\sim D}\sum_{Q/2\leqslant q\leqslant 5Q/2}\frac{1}{qd}\sum_{r_1,r_2\sim R}\frac{1}{r_1r_2}\sum_{\substack{n_1,n_2\sim N\\ n_1\equiv n_2 \pmod{qde}}}\sum_{1\leqslant|h|\leqslant H_2}1 \ll \frac{N^2H_2}{QD},$$

since $QDE \leqslant N$. In $\mathcal{E}_2$ we have

$$\left|\frac{ah}{qdr_1r_2n_1}\right| \ll \frac{H_2}{QDR^2N} \ll \frac{(\log x)^5}{x}.$$

Hence the factor $e(ah/(qdr_1r_2n_1))$ may be replaced by 1 at the cost of an error

$$O\left(\frac{N^2H_2(\log)^5}{QDx}\right) = O\left(\frac{N^2}{QDx^{1/3}}\right).$$

Next, for d, e, n_1, n_2, as in $\mathcal{E}_2$ we have $n_1 \neq n_2$, and the variable q runs over positive integers such that qde divides $n_1 - n_2$. We write

$$n_1 - n_2 = q,\ d,\ e,\ f, \qquad f \in \mathbb{Z}\backslash\{0\},$$

and use f as a variable in place of q; thus $q = (n_1 - n_2)/(def)$. Since $q \geqslant Q/2$, $d > D$, $e > E$ and $|n_1 - n_2| < N$, we have

$$1 \leqslant |f| \leqslant F := \frac{2N}{QDE},$$

and $(n_1 - n_2)/(qd) = ef$. Therefore

$$\mathcal{E}_2 = \mathcal{E}_2' + O\left(\frac{N^2}{QDx^{1/3}}\right), \tag{19}$$

where

$$\mathcal{E}_2' := \sum_{e\sim E}\mu^2(e)\sum_{\substack{d\sim D\\(d,a)=1}}\frac{1}{d}\sum_{1\leqslant|f|\leqslant F}\sum_{n_1,n_2}^{\mathbf{N}}\alpha_{n_1}\overline{\alpha_{n_2}}$$
$$\times\sum_{r_1,r_2}^{\mathbf{R}}\frac{\lambda_{d,r_1}\overline{\lambda_{d,r_2}}}{r_1r_2}\sum_{1\leqslant|h|\leqslant H_2}\frac{1}{q}\psi_0\left(\frac{q}{Q}\right)\widehat{\psi}_0\left(\frac{hM}{qdr_1r_2}\right)e\left(ahef\frac{\overline{n_2r_1}}{n_1r_2}\right), \qquad q := \frac{n_1-n_2}{def}.$$

Here $\sum^{\mathbf{N}}$ denotes a sum over $n_1, n_2 \sim N$ with

$$(n_1, n_2) = 1,\ n_1 \equiv n_2 (\mathrm{mod}\, def),\ \frac{n_1 - n_2}{def} > 0,\ \left(\frac{n_1-n_2}{def}, a\right) = 1,\ |n_1 - n_2| \geqslant \frac{N}{(\log x)^C}, \tag{20}$$

and $\sum^{\mathbf{R}}$ denotes a sum over $r_1, r_2 \sim R$ with

$$(r_1, r_2) = 1, \qquad (r_1r_2, ae) = 1, \qquad (r_1, n_1) = (r_2, n_2) = 1. \tag{21}$$

Note that (20) implies $(n_1n_2, e) = 1$.

5.2. **Separation of variables and Cauchy's inequality.** The variable q, which is now a function of n_1, n_2, d, e, f, is entangled with h, r_1, r_2 through the factor $\widehat{\psi}_0$. We separate them as follows. By the definition of the Fourier transform and the change of variables $t = q\tau$,

$$\widehat{\psi}_0\left(\frac{hM}{qdr_1r_2}\right) = \int\psi_0(t)e\left(-\frac{thM}{qdr_1r_2}\right)\mathrm{d}t = q\int\psi_0(q\tau)e\left(-\frac{\tau hM}{dr_1r_2}\right)\mathrm{d}\tau.$$

Since $\psi_0(q/Q)\psi_0(q\tau)$ vanishes unless $1/(5Q) \leqslant \tau \leqslant 5/Q$, we get

$$\frac{1}{q}\psi_0\left(\frac{q}{Q}\right)\widehat{\psi}_0\left(\frac{hM}{qdr_1r_2}\right) = \int_{1/(5Q)}^{5/Q}\psi_0\left(\frac{q}{Q}\right)\psi_0(q\tau)e\left(-\frac{\tau hM}{dr_1r_2}\right)\mathrm{d}\tau. \tag{22}$$

For $\tau \in [1/(5Q), 5/Q]$ and $d \sim D$ we define the 1-bounded coefficients

$$\xi(h, r_1, r_2) = \xi_{\tau,d}(h, r_1, r_2) := \frac{R^2\lambda_{d,r_1}\overline{\lambda_{d,r_2}}}{r_1r_2}e\left(-\frac{\tau hM}{dr_1r_2}\right),$$

which are supported on $r_1, r_2 \sim R$ with $P^-(r_1 r_2) \geqslant z_0$. Let $\mathcal{N}$ be the set of pairs (r_1, r_2) with $r_1, r_2 \sim R$, $(r_1, r_2) = 1$ and $P^-(r_1 r_2) \geqslant z_0$, and let $\mathcal{N} = \mathcal{N}_1 \sqcup \cdots \sqcup \mathcal{N}_J$ be the partition given by Lemma 2.4. For $1 \leqslant j \leqslant J$ and e, f, n_1, n_2 with $(n_1, n_2) = 1$ put

$$I_j(f, n_1, n_2) := \sum_{\substack{(r_1,r_2)\in\mathcal{N}_j \\ (r_1 r_2, ae)=1 \\ (r_1,n_1)=(r_2,n_2)=1}} \sum_{1\leqslant|h|\leqslant H_2} \xi(h, r_1, r_2) e\left(ahef \frac{\overline{n_2 r_1}}{n_1 r_2}\right).$$

Inserting (22) into the definition of $\mathcal{E}_2'$ we obtain

$$|\mathcal{E}_2'| \leqslant \frac{1}{R^2} \int_{1/(5Q)}^{5/Q} \sum_{e\sim E} \mu^2(e) \sum_{d\sim D} \frac{1}{d} \sum_{j\leqslant J} \sum_{1\leqslant|f|\leqslant F} \sum_{n_1,n_2}^{\mathbf{N}} \|\psi_0\|_\infty^2 \, |I_j(f, n_1, n_2)| \, \mathrm{d}\tau.$$

For fixed d and e the number of triples (f, n_1, n_2) with $1 \leqslant |f| \leqslant F$ and n_1, n_2 satisfying (20) with $Q/2 \leqslant q \leqslant 5Q/2$ is $\ll N^2/(DE)$, because such a triple is determined by $n_1 \sim N$, $q \in [Q/2, 5Q/2]$ and f. By Cauchy's inequality, and enlarging the range of summation by positivity, we infer that

$$\mathcal{E}_2' \ll \frac{J}{R^2} \cdot \frac{1}{Q} \cdot E \cdot D \cdot \frac{1}{D} \cdot \left(\frac{N^2}{DE}\right)^{1/2} \sup \mathcal{Z}^{1/2} = \frac{JNE^{1/2}}{QD^{1/2}R^2} \sup \mathcal{Z}^{1/2}, \tag{23}$$

where the supremum is over $\tau \in [1/(5Q), 5/Q]$, $d \sim D$, square-free $e \sim E$ and $j \leqslant J$, and

$$\mathcal{Z} = \mathcal{Z}(\tau, d, e, j) := \sum_{1\leqslant|f|\leqslant F} \sum_{\substack{n_1,n_2 \\ (n_1,n_2)=1 \\ n_1\equiv n_2 \,(\mathrm{mod}\, e) \\ (n_1 n_2, e)=1}} \Psi\left(\frac{n_1}{N}\right) \Psi\left(\frac{n_2}{N}\right) |I_j(f, n_1, n_2)|^2. \tag{24}$$

In passing from (20) to the conditions in (24) we have dropped the congruence $n_1 \equiv n_2 \pmod{df}$ and the conditions involving q, but we have kept the congruence modulo e. We shall prove that

$$\mathcal{Z} \ll \frac{R^4 N^2}{DE x^{5\eta/2 - o(1)}} \tag{25}$$

uniformly in τ, d, e, j. By (23), (19) and $J = x^{o(1)}$ this gives $\mathcal{E}_2 \ll N^2/(QDx^{5\eta/4 - o(1)})$, which proves Proposition 5.1.

5.3. **The diagonal terms.** We expand the square in (24). Let (h, r_1, r_2) and (h', r_1', r_2') be two triples occurring in $I_j(f, n_1, n_2)$. Since (r_1, r_2) and (r_1', r_2') belong to the same class $\mathcal{N}_j$ we have $(r_1, r_2') = (r_1', r_2) = 1$ in addition to $(r_1, r_2) = (r_1', r_2') = 1$; hence

$$(r_1 r_1', r_2 r_2') = 1.$$

Together with $(n_1, n_2) = (n_1, r_1 r_1') = (n_2, r_2 r_2') = 1$ this gives $(n_2 r_1 r_1', n_1 r_2 r_2') = 1$, and Lemma 2.3 yields

$$\begin{aligned} \mathcal{Z} = \sum_{\substack{(h,r_1,r_2) \\ (h',r_1',r_2')}} \xi(h, r_1, r_2) \overline{\xi(h', r_1', r_2')} \\ \times \sum_{1\leqslant|f|\leqslant F} \sum_{\substack{n_1,n_2 \\ (n_1,n_2)=1 \\ n_1\equiv n_2 \,(\mathrm{mod}\, e), (n_1 n_2, e)=1 \\ (n_1, r_1 r_1')=(n_2, r_2 r_2')=1}} \Psi\left(\frac{n_1}{N}\right) \Psi\left(\frac{n_2}{N}\right) e\left(aefk \frac{\overline{n_2 \rho}}{n_1 \sigma}\right), \end{aligned} \tag{26}$$

where

$$\rho := r_1 r_1', \qquad \sigma := [r_2, r_2'], \qquad k := \frac{h r_1' r_2' - h' r_1 r_2}{(r_2, r_2')},$$

and the outer sum is over all h, h' with $1 \leqslant |h|, |h'| \leqslant H_2$ and over all pairs (r_1, r_2), (r_1', r_2') in $\mathcal{N}_j$ which satisfy $(r_1 r_2 r_1' r_2', ae) = 1$. We write $\mathcal{Z} = \mathcal{Z}_= + \mathcal{Z}_{\neq}$ according to whether $k = 0$ or $k \neq 0$. If $k = 0$, then $h r_1' r_2' = h' r_1 r_2$. Given (h, r_1', r_2'), the integer $h' r_1 r_2$ is determined, and so (h', r_1, r_2) is determined up to $x^{o(1)}$ possibilities by the divisor bound. Thus there are at most $x^{o(1)} H_2 R^2$ pairs of triples with $k = 0$. For each of them the sum over f, n_1, n_2 in (26) is trivially $\ll FN(N/E + 1) \ll FN^2/E$. Hence

$$\mathcal{Z}_= \ll x^{o(1)} H_2 R^2 \frac{FN^2}{E} = x^{o(1)} \frac{R^4 N^3}{ME^2},$$

because $H_2 F = 2R^2 N (\log)^5/(ME)$. By (16) we have $N/(ME) \leqslant x^{-3\eta}/D$, so that

$$\mathcal{Z}_= \ll \frac{R^4 N^2}{DE x^{3\eta - o(1)}}. \tag{27}$$

This is the only place where the auxiliary modulus e is essential: for $E = 1$ the bound (27) would require $N \leqslant x^{-3\eta} M$.

5.4. **The off-diagonal terms.** We now estimate $\mathcal{Z}_{\neq}$ by means of Theorem 2.6. We split the summation in (26) according to the sign $\vartheta \in \{\pm 1\}$ of afk and according to the sizes of ρ and σ. Since $R^2 < \rho \leqslant 4R^2$ and $R < \sigma \leqslant 4R^2$, it suffices to consider $O(\log x)$ pairs (R^*, S^*) with

$$R^2 \leqslant R^* \leqslant 2R^2, \qquad R \leqslant S^* < 4R^2, \qquad R^* < \rho \leqslant 2R^*, \qquad S^* < \sigma \leqslant 2S^*.$$

Fix ϑ, R^* and S^*. In the corresponding terms we have $(r_2, r_2') = r_2 r_2'/\sigma > R^2/(2S^*)$, so that $|k| \leqslant 8H_2 R^2/(r_2, r_2') < 16 H_2 S^*$, and therefore

$$1 \leqslant |aefk| \leqslant N^* := 32|a|EFH_2 S^* = 64|a|(\log x)^5 \frac{N}{M} R^2 S^*.$$

For integers $n \geqslant 1$, $\rho \sim R^*$, $\sigma \sim S^*$ we define

$$b_{n,\rho,\sigma} := \sum \xi(h, r_1, r_2) \overline{\xi(h', r_1', r_2')},$$

where the sum is over all f with $1 \leqslant |f| \leqslant F$ and all pairs of triples as in (26) such that

$$r_1 r_1' = \rho, \qquad [r_2, r_2'] = \sigma, \qquad aefk = \vartheta n.$$

On the support of $b_{n,\rho,\sigma}$ we have $(e, \sigma) = (\rho, \sigma) = 1$. Hence, for $(n_1, e) = 1$, the conditions $(n_1, n_2) = (n_1, r_1 r_1') = (n_2, r_2 r_2') = 1$ in (26) are equivalent to the single condition $(e\rho n_2, \sigma n_1) = 1$. Splitting $n_1 \equiv n_2 \pmod e$ into the reduced residue classes u modulo e, we see that the part of $\mathcal{Z}_{\neq}$ under consideration equals

$$\sum_{\substack{u \,(\mathrm{mod}\, e) \\ (u,e)=1}} \sum_{n \leqslant N^*} \sum_{\rho \sim R^*} \sum_{\sigma \sim S^*} b_{n,\rho,\sigma} \sum_{\substack{n_1 \equiv u, n_2 \equiv u \,(\mathrm{mod}\, e) \\ (e\rho n_2, \sigma n_1) = 1}} \Psi\left(\frac{n_1}{N}\right) \Psi\left(\frac{n_2}{N}\right) e\left(\vartheta n \frac{\overline{\rho n_2}}{\sigma n_1}\right). \tag{28}$$

If $\vartheta = -1$ we pass to the complex conjugate, which replaces b by $\bar{b}$. Let $\Psi = \Psi_1 + \Psi_2 + \Psi_3$ be a smooth partition with Ψ_i supported in $[c_i, 1.9c_i]$ for $c_1 = 0.45$, $c_2 = 0.8$, $c_3 = 1.4$, and let $w(n, \rho, \sigma)$ be a smooth function, compactly supported in $\mathbb{R}^3_{>0}$, which is equal to 1 on $[1, N^*] \times [R^*, 2R^*] \times [S^*, 2S^*]$ and satisfies $\partial^\nu w \ll_\nu n^{-\nu_3} \rho^{-\nu_4} \sigma^{-\nu_5}$. Then, for each u and each pair (i, i'), the inner part of (28) is a sum of the type considered in Theorem 2.6, with

$$(c, d, n, r, s) = (n_1, n_2, n, \rho, \sigma), \qquad g = \Psi_i\left(\frac{c}{N}\right) \Psi_{i'}\left(\frac{d}{N}\right) w(n, r, s),$$

$$(\mathsf{C}, \mathsf{D}, \mathsf{N}, \mathsf{R}, \mathsf{S}) = (c_i N, c_{i'} N, N^*, R^*, S^*), \qquad \mathsf{q} = e, \qquad c_0 = d_0 = u.$$

We now compute the quantity K of (12). Using $R^* \leqslant 2R^2$, $S^* < 4R^2$ and $e \leqslant 2E$, we find

$$\mathsf{RS} + \mathsf{N} \ll (\log x)^5 \left(1 + \frac{N}{M}\right) R^2 S^*,$$

and hence

$$\mathsf{qCS}(\mathsf{RS} + \mathsf{N})(\mathsf{C} + \mathsf{RD}) \ll (\log x)^5 \left(1 + \frac{N}{M}\right) EN^2 R^8,$$

$$\mathsf{C}^2\mathsf{DS}\sqrt{(\mathsf{RS} + \mathsf{N})\mathsf{R}} \ll (\log x)^{5/2} \left(1 + \frac{N}{M}\right)^{1/2} N^3 R^5,$$

$$\mathsf{D}^2\mathsf{NR} \ll (\log x)^5 \left(1 + \frac{N}{M}\right) N^2 R^6.$$

The third term is dominated by the first, and so

$$\mathsf{K} \ll (\log x)^{5/2} \left(1 + \frac{N}{M}\right)^{1/2} \left(E^{1/2} N R^4 + N^{3/2} R^{5/2}\right). \tag{29}$$

Next we estimate $\|b\|_2$. Write $\mathbf{r} := (r_1, r_1', r_2, r_2')$. Since ξ is 1-bounded, $|b_{n,\rho,\sigma}|$ is at most the number $\nu(n,\rho,\sigma)$ of tuples $(f, h, h', \mathbf{r})$ counted in its definition. Given (n,ρ,σ) there are at most $x^{o(1)}$ possibilities for $(f, k, \mathbf{r})$, because f divides n, $r_1 r_1' = \rho$, and r_2, r_2' divide σ. For given $(k, \mathbf{r})$ let $\nu_0(k;\mathbf{r})$ be the number of pairs (h, h') with $1 \leqslant |h|, |h'| \leqslant H_2$ and

$$hr_1' r_2' - h' r_1 r_2 = k(r_2, r_2').$$

The solutions $(h, h') \in \mathbb{Z}^2$ of this linear equation form a coset of the lattice generated by the vector $(r_1 r_2/g, r_1' r_2'/g)$, where

$$g := (r_1 r_2, r_1' r_2') = (r_1, r_1')(r_2, r_2')$$

in view of $(r_1 r_1', r_2 r_2') = 1$. As $r_1 r_2 > R^2$, we get

$$\nu_0(k;\mathbf{r}) \leqslant 1 + \frac{2H_2 (r_1, r_1')(r_2, r_2')}{R^2}.$$

By Cauchy's inequality $\nu(n,\rho,\sigma)^2 \leqslant x^{o(1)} \sum \nu_0(k;\mathbf{r})^2$, the sum being over the tuples $(f, k, \mathbf{r})$ compatible with (n,ρ,σ). Summing over (n,ρ,σ) and using that $\sum_k \nu_0(k;\mathbf{r}) \leqslant (2H_2)^2$, we obtain

$$\begin{aligned}
\|b\|_2^2 &\ll x^{o(1)} \sum_{1 \leqslant |f| \leqslant F} \sum_{\substack{r_1, r_1', r_2, r_2' \sim R \\ (r_1 r_1', r_2 r_2') = 1}} \left(\max_k \nu_0(k;\mathbf{r})\right) \sum_k \nu_0(k;\mathbf{r}) \\
&\ll x^{o(1)} F H_2^2 \sum_{\substack{r_1, r_1', r_2, r_2' \sim R \\ (r_1 r_1', r_2 r_2') = 1}} \left(1 + \frac{H_2 (r_1, r_1')(r_2, r_2')}{R^2}\right) \\
&\leqslant x^{o(1)} F H_2^2 \sum_{r_1, r_1', r_2, r_2' \sim R} \left(1 + \frac{H_2 (r_1, r_1')(r_2, r_2')}{R^2}\right) \\
&\ll x^{o(1)} F H_2^2 R^4 \left(1 + \frac{H_2}{R^2}\right),
\end{aligned}$$

since $\sum_{r, r' \sim R} (r, r') \ll R^2 (\log x)$. By (15) we have $H_2/R^2 = QD(\log x)^5/M \leqslant x^{o(1)}$. Inserting the values of F and H_2 we conclude that

$$\|b\|_2 \ll x^{o(1)} F^{1/2} H_2 R^2 = x^{o(1)} \frac{(NQD)^{1/2} R^4}{M E^{1/2}}. \tag{30}$$

We apply Theorem 2.6 with ϵ and ϵ_0 sufficiently small such that $(\mathsf{qCDNRS})^{\epsilon+O(\epsilon_0)} \leqslant x^{\eta/2}$. Summing over the $\varphi(e) \leqslant 2E$ classes u, over (i,i'), over ϑ and over the $O(\log x)$ pairs (R^*,S^*), we deduce from (29) and (30) that

$$\begin{aligned}\mathcal{Z}_{\neq} &\ll x^{\eta/2+o(1)}E^{5/2}\left(1+\frac{N}{M}\right)^{1/2}\left(E^{1/2}NR^4+N^{3/2}R^{5/2}\right)\frac{(NQD)^{1/2}R^4}{ME^{1/2}}\\ &= x^{\eta/2+o(1)}\frac{R^4N^2}{DE}\cdot\frac{D^{3/2}(1+N/M)^{1/2}}{MN}\left(E^{7/2}N^{1/2}Q^{1/2}R^4+E^3NQ^{1/2}R^{5/2}\right).\end{aligned}$$

By (17), (18), $D \leqslant y_0$ and $MN \asymp x$, the last factor is $\ll x^{-4\eta+o(1)}$. Hence

$$\mathcal{Z}_{\neq} \ll \frac{R^4N^2}{DEx^{7\eta/2-o(1)}},$$

and together with (27) this proves (25). The proof of Proposition 5.1 is now complete.

6. Proof of Theorem 1.4 and Corollary 1.5

Proof of Theorem 1.4. Let $\mathcal{S}_0$ denote the sum to be estimated. Note that (8) and $MN \asymp x$ give

$$\frac{N}{M} \ll x^{\varepsilon/50}, \qquad Q \leqslant Nx^{-\varepsilon} \leqslant M, \qquad R \leqslant x^{1/4}, \qquad QR \leqslant x^{3/4+\varepsilon/100}, \tag{31}$$

where the bound for R follows from (9). If $R \leqslant x^{\varepsilon/2}$, then all moduli qr are at most $4QR \leqslant 4Nx^{-\varepsilon/2} \leqslant x^{1/2-\varepsilon/3}$, and the claim follows from the generalized Bombieri–Vinogradov theorem (see [[3], Theorem 0]). We may therefore assume that $R > x^{\varepsilon/2}$. We write each $r \sim R$ uniquely as $r = dr'$ with $d = \mathrm{sm}(r;z_0)$, so that $P^+(d) \leqslant z_0 < P^-(r')$. The moduli qr with $d \geqslant y_0$ satisfy $\mathrm{sm}(qr;z_0) \geqslant y_0$, and since the coefficient $\sum_{qr=n}\lambda_q\nu_r$ of a modulus n is bounded by $\tau(n)^{2B_0+1}$, [[20], Lemma 13.10] shows that these moduli contribute $O(x/(\log x)^A)$ to $\mathcal{S}_0$. We split the remaining range of d into $O(\log x)$ dyadic intervals $d \sim D$, $D \leqslant y_0$ (the same results also hold for $d=1$); then $r' \sim R'$ for some $R/(2D) \leqslant R' \leqslant R/D$ and $R' \geqslant 1$. For each such (D,R'), the corresponding part of $\mathcal{S}_0$ is a sum $\mathcal{S}$ as in Proposition 3.1, with R' in place of R and with

$$\gamma_{q,d} := \lambda_q, \qquad \lambda_{d,r'} := \nu_{dr'}\mathbb{1}_{P^+(d)\leqslant z_0<P^-(r')}\mathbb{1}_{R<dr'\leqslant 2R}.$$

It suffices to show that each of these sums is $O(x/(\log x)^{A+1})$.

We apply Proposition 3.1 with $E := x^{\varepsilon/10}$. Its hypotheses hold, since $N \geqslant Qx^{\varepsilon} > QDE(\log x)^C$ and $M \gg x^{1/2-\varepsilon/100}$. As $|\alpha_n|, |\lambda_{d,r'}| \leqslant \tau(\cdot)^{B_0} \leqslant x^{o(1)}$, it remains to verify that Lemma 4.1 and Proposition 5.1 (applied to suitably normalized sequences, with R' in place of R and with $\eta := \varepsilon/100$) yield

$$\mathcal{E}_1, \mathcal{E}_2 \ll \frac{N^2}{QDx^{\eta-o(1)}}, \tag{32}$$

because then Proposition 3.1 gives $\mathcal{S} \ll x/(\log x)^{A+1} + MNx^{-\eta/2+o(1)}$.

For Lemma 4.1 we note that by (8), (9) and (31)

$$\begin{aligned}DR'N^{3/2} \leqslant RN^{3/2} &\leqslant \left(x^{1-\varepsilon}N^{-1/2}\right)^{1/4}N^{3/2} \leqslant x^{1/4}N^{11/8} \leqslant x^{15/16+\varepsilon/50},\\ QDR' &\leqslant QR \leqslant x^{3/4+\varepsilon/100},\end{aligned}$$

so that (14) holds. For Proposition 5.1 we check (15)–(18). Condition (15) follows from (31) and $QDE \leqslant Qy_0x^{\varepsilon/10} \leqslant N$. By (31),

$$ND \leqslant x^{\varepsilon/50+o(1)}M \leqslant x^{-3\eta}ME,$$

which is (16). Finally, by (9), (10) and (31),

$$E^{7/2}\left(1+\frac{N}{M}\right)^{1/2}N^{1/2}Q^{1/2}R'^4 \ll x^{7\varepsilon/20+\varepsilon/100}\cdot x^{1-\varepsilon}\leqslant x^{1-4\eta},$$
$$E^{3}\left(1+\frac{N}{M}\right)^{1/2}NQ^{1/2}R'^{5/2} \ll x^{3\varepsilon/10+\varepsilon/100}\cdot x^{1-\varepsilon}\leqslant x^{1-4\eta}.$$

This proves (32) and hence Theorem 1.4. □

Proof of Corollary 1.5. If $N\leqslant x^{1/2+\varepsilon/200}$, we apply Theorem 1.4 with $\varepsilon/2$ in place of ε: condition (8) holds, and we have

$$N^{1/2}Q^{1/2}R^4\leqslant x^{1/4+\varepsilon/400}(QR^8)^{1/2}\leqslant x^{1-\varepsilon/2},\qquad NQ^{1/2}R^{5/2}\leqslant x^{1/2+\varepsilon/200}(QR^5)^{1/2}\leqslant x^{1-\varepsilon/2}.$$

If $N>x^{1/2+\varepsilon/200}$, then $M\ll x^{1/2-\varepsilon/200}$ and $M\gg x/N\geqslant Qx^{\varepsilon}$, and we apply Theorem 1.4 in the same way with the roles of M and N reversed. □

7. Proof of Theorem 1.1

In this section we deduce Theorem 1.1 from Theorem 1.4 and other available arithmetic information. Following [20], we define

$$S_n:=\mathbb{1}_{n\equiv a(\mathrm{mod}\, q_1q_2)}-\frac{\mathbb{1}_{(n,q_1q_2)=1}}{\varphi(q_1q_2)},\qquad S_d(z):=\sum_{\substack{n\sim x/d\\ P^-(n)\geqslant z}}S_{dn}.$$

Let $\mathcal{Q}=\mathcal{Q}(\varepsilon_0)$ be the set of pairs (Q_1,Q_2) of real numbers $Q_1,Q_2\geqslant 1$ with

$$Q_1^2Q_2\leqslant x^{1-\varepsilon_0},\qquad Q_1Q_2^5\leqslant x^{1-\varepsilon_0},\qquad Q_1^6Q_2^{13}\leqslant x^{4-\varepsilon_0},\qquad x^{1/2-\varepsilon}\leqslant Q_1Q_2\leqslant x^{8/15-\varepsilon_0}. \tag{33}$$

For $(Q_1,Q_2)\in\mathcal{Q}$, we also have

$$Q_2=\left(\frac{Q_1Q_2^5}{Q_1Q_2}\right)^{1/4}\leqslant x^{1/8},\qquad Q_1^3Q_2^2=(Q_1^2Q_2)(Q_1Q_2)\leqslant x^{23/15-2\varepsilon_0},$$
$$Q_1^{11}Q_2^{12}=(Q_1Q_2)^{11}Q_2\leqslant x^{6-1/120}. \tag{34}$$

By a Harman's sieve argument as in [[20], Chapter 7], we only need to show the following four propositions:

Proposition 7.1 (Type-II estimate)**.** *Let $A>0$ and $(Q_1,Q_2)\in\mathcal{Q}$. Let $P_1,\dots,P_J\geqslant x^{1/7+10\varepsilon}$ be such that $P_1\cdots P_J\ll x$ and*

$$x^{3/7+\varepsilon}\leqslant\prod_{j\in\mathcal{J}}P_j\leqslant x^{4/7-\varepsilon}$$

for some subset $\mathcal{J}\subseteq\{1,\dots,J\}$. Let λ_q,ν_q be complex sequences with $|\lambda_q|,|\nu_q|\leqslant\tau(q)^{B_0}$. Then we have

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\lambda_{q_1}\nu_{q_2}\sideset{}{^*}\sum_{\substack{p_1,\dots,p_J\\ p_i\sim P_i\forall i}}S_{p_1\cdots p_J}(p_J)\ll_A\frac{x}{(\log x)^A}.$$

Here $\sum^*$ indicates that the summation is restricted by $O(1)$ inequalities of the form $p_1^{\alpha_1}\cdots p_J^{\alpha_J}\leqslant B$. The implied constant may depend on all such exponents α_i, but none of the quantities B.

Proposition 7.2 (Sieve asymptotics). *Let $A > 0$ and $(Q_1, Q_2) \in \mathcal{Q}$. Let $x^{3/7+\varepsilon} \geqslant P_1 \geqslant \cdots \geqslant P_r \geqslant x^{1/7+10\varepsilon}$ be such that $P_1 \cdots P_r \leqslant x^{5/7}$ and such that either $r = 1$ or $P_r \leqslant x^{1/4+\varepsilon}$. Let λ_q, ν_q be complex sequences with $|\lambda_q|, |\nu_q| \leqslant \tau(q)^{B_0}$. Then we have*

$$\sum_{\substack{q_1 \sim Q_1 \\ q_2 \sim Q_2 \\ (q_1 q_2, a) = 1}} \lambda_{q_1} \nu_{q_2} \sideset{}{^*}\sum_{\substack{p_1, \dots, p_r \\ p_i \sim P_i \forall i}} S_{p_1 \cdots p_r}(x^{1/7+10\varepsilon}) \ll_A \frac{x}{(\log x)^A}.$$

Here $\sum^*$ means that the summation is restricted by $O(1)$ inequalities of the form $p_1^{\alpha_1} \cdots p_r^{\alpha_r} \leqslant B$ for some constants $\alpha_1, \dots, \alpha_r$. The implied constant may depend on all such exponents α_i, but none of the quantities B. Moreover, we also have the related estimate

$$\sum_{\substack{q_1 \sim Q_1 \\ q_2 \sim Q_2 \\ (q_1 q_2, a) = 1}} \lambda_{q_1} \nu_{q_2} S_1(x^{1/7+10\varepsilon}) \ll_A \frac{x}{(\log x)^A}.$$

Proposition 7.3 (Numbers with four or more prime factors). *Let $A > 0$ and $(Q_1, Q_2) \in \mathcal{Q}$. Let $J \geqslant 4$ and $P_1 \geqslant \cdots \geqslant P_J \geqslant x^{1/7+10\varepsilon}$ with $P_1 \cdots P_J \asymp x$. Let λ_q, ν_q be complex sequences with $|\lambda_q|, |\nu_q| \leqslant \tau(q)^{B_0}$. Then we have*

$$\sum_{\substack{q_1 \sim Q_1 \\ q_2 \sim Q_2 \\ (q_1 q_2, a) = 1}} \lambda_{q_1} \nu_{q_2} \sideset{}{^*}\sum_{\substack{p_1, \dots, p_J \\ p_i \sim P_i \forall i}} S_{p_1 \cdots p_J} \ll_A \frac{x}{(\log x)^A}.$$

Here $\sum^*$ indicates that the summation is restricted by $O(1)$ inequalities of the form $p_1^{\alpha_1} \cdots p_J^{\alpha_J} \leqslant B$. The implied constant may depend on all such exponents α_i, but none of the quantities B.

Proposition 7.4 (Numbers with three prime factors). *Let $A > 0$ and $(Q_1, Q_2) \in \mathcal{Q}$. Let $P_1, P_2, P_3 \in [x^{1/4}, x^{3/7+\varepsilon}]$ with $P_1 P_2 P_3 \asymp x$. Let λ_q, ν_q be complex sequences with $|\lambda_q|, |\nu_q| \leqslant \tau(q)^{B_0}$. Then we have*

$$\sum_{\substack{q_1 \sim Q_1 \\ q_2 \sim Q_2 \\ (q_1 q_2, a) = 1}} \lambda_{q_1} \nu_{q_2} \sideset{}{^*}\sum_{\substack{p_1, p_2, p_3 \\ p_i \sim P_i \forall i}} S_{p_1 p_2 p_3} \ll_A \frac{x}{(\log x)^A}.$$

Here $\sum^*$ means that the summation is restricted by $O(1)$ inequalities of the form $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \leqslant B$ for some constants $\alpha_1, \alpha_2, \alpha_3$. The implied constant may depend on all such exponents α_i, but none of the quantities B.

We mention that when applying Theorem 1.4 and other arithmetic information proved elsewhere, we often need [[20], Lemma 8.11] to remove the dependencies (cross conditions) between variables.

Proof of Proposition 7.1. We can write $S_{p_1 \cdots p_J}(p_J)$ as a sum of $S_{p_1 \cdots p_{J'}}$ with $J \leqslant J' \leqslant 6$ by a splitting argument, hence we can assume that $P_1 \cdots P_J \asymp x$. Since $Q_1 Q_2 < x^{8/15} < x^{127/224-\varepsilon}$, [[20], Lemma 8.13] gives the desired result for $Q_1 Q_2 x^{\varepsilon} < \prod_{j \in \mathcal{J}} P_j \leqslant x^{4/7-\varepsilon}$. Now we can assume that $x^{1/2} \ll \prod_{j \in \mathcal{J}} P_j \leqslant Q_1 Q_2 x^{\varepsilon}$ by symmetry. In this case we apply Theorem 1.4 with $Q = Q_1$, $R = Q_2$, $m = \prod_{j \in \mathcal{J}} p_j$ and $n = \prod_{j \notin \mathcal{J}} p_j$. By (33) and (34), we have

$$N \gg \frac{x^{1-\varepsilon}}{Q_1 Q_2} \geqslant Q_1 x^{\varepsilon_0 - \varepsilon}, \qquad N \ll x^{1/2},$$
$$N^{1/2} Q_1^{1/2} Q_2^4 \ll x^{1/4} (Q_1 Q_2^5)^{1/2} Q_2^{3/2} \leqslant x^{1/4+1/2+3/16} = x^{15/16},$$

$$NQ_1^{1/2}Q_2^{5/2} \ll x^{1/2}(Q_1Q_2^5)^{1/2} \leqslant x^{1-\varepsilon_0/2}.$$

Hence (8)–(10) hold and Proposition 7.1 follows. □

Proof of Proposition 7.2. Note that Proposition 7.2 is a combination of a Type-I/II estimate and a small factor Type-II estimate. We first establish an improved small factor Type-II estimate using Type-II estimates of Fouvry [12] and Iwaniec–Pomykała [14].

Lemma 7.5 (Small factor Type-II estimate)**.** *Let $A > 0$ and $Q_1, Q_2 \geqslant 1$ satisfy the first three conditions of (33). Let $MN \asymp x$ with $x^{\varepsilon/2} \leqslant N \leqslant x^{1/7+11\varepsilon}$. Let λ_{q_1}, ν_{q_2} be 1-bounded and let α_n, β_m be complex sequences with $|\alpha_n|, |\beta_n| \leqslant \tau(n)^{B_0}$. Suppose that α_n is supported on $P^-(n) \geqslant z_0$ and satisfies the Siegel–Walfisz condition. Then we have*

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}} \lambda_{q_1}\nu_{q_2}\Delta(q_1q_2) \ll_{a,\varepsilon,A,B_0} \frac{x}{(\log x)^A},$$

where Δ is given by (6).

Proof of Lemma 7.5. Suppose first that $x^{\varepsilon/2} \leqslant N \leqslant Q_2x^{2\varepsilon}$. We apply [[12], Théorème(C.4)] with $R = Q_1$ and $S = Q_2$. We need

$$\max\left(Q_1^2Q_2, N^{5/2}Q_1Q_2^{5/2}, N^{5/4}Q_1^{3/2}Q_2^2\right) \leqslant x^{1-\varepsilon/3}.$$

By (33), we have

$$N^{5/2}Q_1Q_2^{5/2} \leqslant (Q_2x^{2\varepsilon})^{5/2}Q_1Q_2^{5/2} = Q_1Q_2^5x^{5\varepsilon} \leqslant x^{1-\varepsilon_0+5\varepsilon},$$
$$N^{5/4}Q_1^{3/2}Q_2^2 \leqslant (Q_2x^{2\varepsilon})^{5/4}Q_1^{3/2}Q_2^2 = (Q_1^6Q_2^{13})^{1/4}x^{5\varepsilon/2} \leqslant x^{1-\varepsilon_0/4+5\varepsilon/2},$$

and the required condition is satisfied.

Suppose now that $Q_2x^{2\varepsilon} < N \leqslant x^{1/7+11\varepsilon}$. We apply [[14], Proposition] in the form of [[18], Proposition 11.1] with $R = Q_1$ and $Q = Q_2$. We need

$$N^6Q_1^8Q_2^4 < x^{5-\varepsilon}, \qquad Q_1^2Q_2 < Nx^{1-\varepsilon}.$$

By (33), we have

$$N^6Q_1^8Q_2^4 = N^6(Q_1^2Q_2)^4 \leqslant x^{4+6/7-4\varepsilon_0+66\varepsilon}, \qquad Q_1^2Q_2 \leqslant x^{1-\varepsilon_0},$$

and the required condition is satisfied. Now the proof of Lemma 7.5 is completed. □

With Proposition 7.1 and Lemma 7.5, Proposition 7.2 can be proved in a same way of [[20], Proposition 7.2] with several modifications. The absolute values are replaced by bilinear weights, and the conditions on (Q_1, Q_2) in [20] are replaced by $(Q_1, Q_2) \in \mathcal{Q}$. In [[20], Lemma 10.4] we replace the condition $N \leqslant x^{3/7+\varepsilon}x/(Q_1Q_2)^{15/8}$ by $N \leqslant x^{3/7+2\varepsilon}$, and we have

$$KL \gg \frac{x}{N} \geqslant x^{4/7-2\varepsilon} \gg Q_1Q_2x^{\varepsilon},$$
$$K^3Q_1Q_2 = \frac{K^4Q_1Q_2N}{x^{2-2\varepsilon}} \cdot \frac{x^{2-2\varepsilon}}{NK} \ll \frac{x^{1+8/15+3/7-\varepsilon_0+6\varepsilon}}{x^{2-2\varepsilon}}Lx^{1-2\varepsilon} < Lx^{1-2\varepsilon},$$
$$K(Q_1Q_2)^2 = \frac{K^2(Q_1Q_2)^2N}{x^{2-2\varepsilon}} \cdot \frac{x^{2-2\varepsilon}}{NK} \ll \frac{x^{1/2+16/15+3/7-2\varepsilon_0+4\varepsilon}}{x^{2-2\varepsilon}}Lx^{1-2\varepsilon} < Lx^{1-2\varepsilon}.$$

In [[20], Proposition 10.1] we replace the condition $DEP > x^{1-2\varepsilon}/(Q_1Q_2)^{15/8}$ by $DEP > x^{\varepsilon}$. Proposition 7.2 now follows from the arguments in [[20], Chapter 10]. □

Proof of Proposition 7.3. Since $Q_1Q_2 < x^{8/15} < x^{4/7-5\varepsilon}$, Proposition 7.3 follows easily from the arguments in [[20], Chapter 9] using Proposition 7.1 and [[20], Proposition 8.3], with only two modifications needed: In *Case 1* ($J = 4$) we have

$$Q_1Q_2KL^4 < x^{8/15}P_1P_4(P_2P_3)^{3/2} \ll x^{8/15+1+3/14+2\varepsilon} = x^{367/210+2\varepsilon} < x^{57/32-10\varepsilon}.$$

In *Case 2* ($J = 5$) we have

$$Q_1Q_2KL^4 < x^{8/15}\frac{(P_1P_2P_3P_4P_5)^{4/3}}{(P_4P_5)^{1/3}} \ll x^{8/15+4/3-2/21} = x^{62/35} < x^{57/32-10\varepsilon}.$$

□

Proof of Proposition 7.4. Since we have (34), Proposition 7.4 follows from [[18], Proposition 12.2], Proposition 7.3, a variant of [[20], Proposition 10.1] proved using Lemma 7.5 (with absolute values replaced by bilinear weights, $DEP > x^{1-2\varepsilon}/(Q_1Q_2)^{15/8}$ replaced by $DEP > x^{\varepsilon}$, and conditions on (Q_1, Q_2) in [20] replaced by $(Q_1, Q_2) \in \mathcal{Q}$) and the arguments in [[20], Chapter 11]. □

8. Applications

Theorem 1.1 has many interesting applications, and one of them is to show the infinitude of shifted primes without large prime factors. Using the arguments in [[18], Section 3], one can easily deduce the following result from Theorem 1.1.

Theorem 8.1. *For any fixed non-zero integer a and any $\beta > 7/(15\sqrt{e}) \approx 0.283$, there exists $C \geqslant 1$ such that*

$$\sum_{\substack{p\sim x\\ P^+(p-a)\leqslant x^\beta}} 1 \gg \frac{x}{(\log x)^C}.$$

By the method of Erdős [10] and Pomerance [24], Theorem 8.1 implies the following result on the distribution of values of the φ function. Note that $0.7169 < 1 - 7/(15\sqrt{e})$.

Corollary 8.2. *Denote by $m_1 < m_2 < \cdots$ the integers $m \in \mathbb{Z}$ for which $m = \varphi(n)$ admits at least $m^{0.7169}$ solutions $n \in \mathbb{Z}$. Then the sequence (m_i) is infinite, and satisfies*

$$\lim_{i\to\infty} \frac{\log(m_{i+1})}{\log(m_i)} = 1.$$

Theorem 8.1 also implies the following result on the number of Carmichael numbers up to x, which improve the results 0.34 of Li [16] and 0.3389 of Lichtman [18]. Note that $0.3406 < (1 - 0.5248)(1 - 7/(15\sqrt{e}))$, where 0.5248 is from [[16], Theorem 9].

Corollary 8.3. *There are at least $x^{0.3406}$ Carmichael numbers up to x.*

9. Limitations and Connections

9.1. **Limitations of the method.** The four conditions in (5) have different origins. The first condition $Q_1^2Q_2 \leqslant x^{1-\varepsilon}$ ensures that the range $[Q_1x^{\varepsilon}, x^{1-\varepsilon}/Q_1]$ of Corollary 1.5 contains the interval $[x^{1-\varepsilon}/(Q_1Q_2), x^{\varepsilon}Q_1Q_2]$. The second condition $Q_1Q_2^5 \leqslant x^{1-\varepsilon}$ is needed in both Theorem 1.4 (see (10) with $N = x^{1/2}$) and in Fouvry's Type-II estimate [12], and the third condition $Q_1^6Q_2^{13} \leqslant x^{4-\varepsilon}$ is needed only in the latter. The last condition $Q_1Q_2 \leqslant x^{8/15-\varepsilon}$ is needed only in [[18], Proposition 12.2] in the proof of Proposition 7.4. Thus, any extension of the existing Type-I_3 estimates (see [[17], Lemmas 3.6–3.8]) would extend Theorem 1.1. Beyond the region (5), Theorem 1.4 and Corollary 1.5 still provide extra Type-II information for Q_1Q_2 up to $x^{5/9}$, which can be fed into Harman's sieve to produce majorants and minorants of $\mathbb{1}_{\mathbb{P}}(n)$ of the kind constructed in [17]; we have not pursued the corresponding numerical work.

9.2. **The exceptional spectrum.** Condition (10) comes from the second term of (12), which accounts for possible exceptional eigenvalues of the Laplacian. If one uses a version of Theorem 2.6 with the dependence on the spectral parameter θ is explicit, such as [[19], Theorem 1.8], together with the bound $\theta \leqslant 7/32$ of Kim and Sarnak [15], one expects that (10) can be relaxed to a condition of the shape $N^{1/2+\theta}Q^{1/2}R^{4-3\theta} \leqslant x^{1-\varepsilon}$. We have not carried this out since it would not affect Theorem 1.1: we still need the condition $Q_1Q_2^5 \leqslant x^{1-\varepsilon}$ in order to use Fouvry's Type-II estimate [12] to prove Lemma 7.5.

9.3. **Well-factorable weights, Absolute values and General moduli.** A well-factorable weight of level Q_0 can be written as a bilinear weight $\lambda_q\nu_r$ with $QR = Q_0$ for any choice of Q and R. Choosing $Q = Nx^{-\varepsilon}$ in Theorem 1.4, we see that the Type-II sums (6) with $x^{3/7} \leqslant N \leqslant x^{1/2}$ are under control for well-factorable weights of level

$$Q_0 = x^{-2\varepsilon} \min\left(N^{3/4}x^{1/4}, N^{2/5}x^{2/5}\right), \tag{35}$$

which equals $x^{3/5-2\varepsilon}$ at $N = x^{1/2}$. In this range, the first method of [3] gives the level $x^{1-\varepsilon}/N$, which decreases from $x^{4/7}$ to $x^{1/2}$. However, at $N = x^{3/7}$, both terms in (35) are equal to $x^{4/7}$, in agreement with [[3], Theorem 1]. Thus, for every level beyond $x^{4/7}$, there remains a range of N around $x^{3/7}$ that neither [[3], Theorem 1] nor Theorem 1.4 applies, and for this reason we cannot obtain an asymptotic result beyond the exponent 4/7 of [3] for well-factorable weights. Whether (35) can be combined with Harman's sieve to yield upper and lower bounds with correct order of magnitude slightly beyond $x^{4/7}$ is a question we have not investigated. Since the level (35) never exceeds $x^{3/5}$, Theorem 1.4 gives no information at the level $x^{5/8}$ in [22] for triply well-factorable weights.

Our argument starts by applying Cauchy's inequality to the sum over the large modulus q, and it is essential that after this step no unknown coefficient depending on q remains: only then can q be exchanged for the complementary divisor f, which makes n_1 and n_2 free smooth variables. In an estimate with absolute values, such as (2), the coefficients are of the form $c_{q,r}$ and this step is not available; one is then led back to the arrangement of Zhang, in which the congruence $n_1 \equiv n_2 \pmod q$ has to be kept for each individual q. For the same reason the method has nothing to say about general moduli.

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Claude Fable 5.2 (Anthropic) was used in the proof of parts of Theorem 1.4 and in the preparation of the manuscript, including language editing and formatting.

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