NOTE ON VARIABLE ROLE-REVERSALS

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ABSTRACT. In [2], the author uses variable role-reversals only when $\theta < \frac{45}{89} - \varepsilon$. In this note, we shall discuss how to use variable role-reversals when $\frac{45}{89} - \varepsilon \leqslant \theta < \frac{11}{20} - \varepsilon$.

In a recent paper [2], the author studied the average distribution of majorants and minorants of $\mathbb{1}_p(x)$ in arithmetic progressions to moduli larger than $\sqrt{x}$. During the constructing process of those majorants and minorants, the author applied the *variable role-reversal*, which is also a very powerful technique in many other applications of Harman's sieve. For the sake of simplicity, the author only uses role-reversals when $\theta < \frac{45}{89} - \varepsilon$. Here, we generalize the application of role-reversals in [2] to a wider range $\theta < \frac{11}{20} - \varepsilon$. We mention that further extension to $\theta < \frac{4}{7} - \varepsilon$ may still be possible.

From here we suppose that $\frac{1}{2} \leqslant \theta < \frac{11}{20} - \varepsilon$. Almost all of the notations and definitions in this note are same as those in [2]. We only change the definitions of κ and τ in order to state our Lemma 0.2 and Lemma 0.3 below simpler. Note that κ here equals to κ in [2] when $\theta \leqslant \frac{7}{13} - \varepsilon$ and becomes κ' in [2] when $\theta > \frac{7}{13} - \varepsilon$. The definitions of some sets below are changed correspondingly.

$$\kappa = \kappa(\theta) = \begin{cases} \frac{5-8\theta}{6} - \varepsilon, & \theta \leqslant \frac{17}{32} - \varepsilon, \\ \frac{5-8\theta}{12} - 3\varepsilon, & \frac{17}{32} - \varepsilon < \theta \leqslant \frac{7}{13} - \varepsilon, \\ \frac{11-20\theta}{6} - 2\varepsilon, & \frac{7}{13} - \varepsilon < \theta \leqslant \frac{11}{20} - \varepsilon, \end{cases}$$

$$\tau = \tau(\theta) = \begin{cases} \frac{3(1-\theta)}{5} - \varepsilon, & \theta \leqslant \frac{11}{21}, \\ \frac{2}{7} - \varepsilon, & \frac{11}{21} < \theta \leqslant \frac{6}{11} - \varepsilon, \\ \frac{5-6\theta}{7} - \varepsilon, & \frac{6}{11} - \varepsilon < \theta. \end{cases}$$

$$\begin{aligned}
\boldsymbol{S} &= \boldsymbol{S}(\theta) = \{(s,t) : s \leqslant 1-\theta-\varepsilon,\ s+2t \leqslant 2-2\theta-\varepsilon,\ s+4t \leqslant 2-\theta-\varepsilon\}, \\
\boldsymbol{S}_j &= \boldsymbol{S}_j(\theta) = \{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \text{ partitions exactly into } \boldsymbol{S}\}, \\
\boldsymbol{A}_j &= \boldsymbol{A}_j(\theta) = \left\{\boldsymbol{\alpha}_j : (\log\log x)^{-3} \leqslant \alpha_j < \cdots < \alpha_1 < \tau,\ \alpha_1 + \cdots + \alpha_j \leqslant 1\right\}, \\
\boldsymbol{A}'_j &= \boldsymbol{A}'_j(\theta) = \left\{\boldsymbol{\alpha}_j : (\log\log x)^{-3} \leqslant \alpha_j < \cdots < \alpha_1 < \frac{3}{7} + \varepsilon,\ \alpha_1 + \cdots + \alpha_j \leqslant 1\right\}, \\
\boldsymbol{U}_j &= \boldsymbol{U}_j(\theta) = \{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \in \boldsymbol{A}_j,\ (\alpha_1, \ldots, \alpha_j, 2\theta-1+\varepsilon) \in \boldsymbol{S}_{j+1}\}, \\
\boldsymbol{U}'_j &= \boldsymbol{U}'_j(\theta) = \left\{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \in \boldsymbol{A}'_j,\ (\alpha_1, \ldots, \alpha_j, 2\theta-1+\varepsilon) \in \boldsymbol{S}_{j+1}\right\}.
\end{aligned}$$

The next three lemmas come directly from [2].

Lemma 0.1. *Let $\frac{1}{2} \leqslant \theta \leqslant \frac{45}{89} - \varepsilon$, $m_1 \sim M_1$ and $m_2 \sim M_2$. Then*

$$\sum_{\left(\frac{\log M_1}{\log x}, \frac{\log M_2}{\log x}\right) \in \boldsymbol{S}} a_{1,m_1} a_{2,m_2} S\left(\mathcal{A}^q_{m_1 m_2}, x^{\kappa}\right)$$

has an asymptotic formula.

Proof. This lemma is [[2], Lemma 2.14]. □

Lemma 0.2. *Let $\frac{1}{2} \leqslant \theta \leqslant \frac{17}{32} - \varepsilon$. Then*

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}'_j} S\left(\mathcal{A}^q_{p_1 \cdots p_j}, x^{\kappa}\right)$$

has an asymptotic formula.

Proof. This lemma can be easily deduced from [[2], Lemma 2.13]. □

Lemma 0.3. *Let* $\frac{1}{2} \leqslant \theta \leqslant \frac{11}{20} - \varepsilon$. *Then*

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j} S\left(\mathcal{A}^q_{p_1\cdots p_j}, x^\kappa\right)$$

has an asymptotic formula.

Proof. This lemma can be easily deduced from [[2], Lemma 2.13]. □

Next we shall briefly introduce the use of role-reversals in this problem. Suppose that we want to give a nontrivial estimate for the sum

$$\sum_{\boldsymbol{\alpha}_j} S\left(\mathcal{A}^q_{p_1\cdots p_j}, p_j\right). \tag{1}$$

Assume that the sum (1) has a "right sign" (that is, $-$ in the construction of a majorant or $+$ in the construction of a minorant) and satisfies $\boldsymbol{\alpha}_j \in \boldsymbol{S}_j$ (or $\boldsymbol{U}'_j$ or $\boldsymbol{U}_j$, depending on the value of θ). A direct application of Buchstab's identity gives

$$\sum_{\boldsymbol{\alpha}_j} S\left(\mathcal{A}^q_{p_1\cdots p_j}, p_j\right) = \sum_{\boldsymbol{\alpha}_j} S\left(\mathcal{A}^q_{p_1\cdots p_j}, x^\kappa\right) - \sum_{\substack{\boldsymbol{\alpha}_j \\ \kappa \leqslant \alpha_{j+1} < \min\left(\alpha_j, \frac{1}{2}(1-\alpha_1-\cdots-\alpha_j)\right)}} S\left(\mathcal{A}^q_{p_1\cdots p_j p_{j+1}}, p_{j+1}\right). \tag{2}$$

After applying Buchstab's identity once, sometimes we cannot use it again since $(\alpha_1, \ldots, \alpha_j, \alpha_j) \notin \boldsymbol{S}_{j+1}$ (or $\boldsymbol{U}'_{j+1}$ or $\boldsymbol{U}_{j+1}$). In this case, we cannot give an asymptotic formula for the sum (with a "wrong sign")

$$\sum_{\substack{\boldsymbol{\alpha}_j \\ \kappa \leqslant \alpha_{j+1} < \min\left(\alpha_j, \frac{1}{2}(1-\alpha_1-\cdots-\alpha_j)\right)}} S\left(\mathcal{A}^q_{p_1\cdots p_j p_{j+1}}, x^\kappa\right) \tag{3}$$

after the second Buchstab iteration. The idea of a role-reversal is to apply the second Buchstab iteration on a different variable, usually the largest explicit prime variable (denoted by p_1):

$$(p_1 p_2 \cdots p_{j+1})\beta \Rightarrow (\beta p_2 \cdots p_{j+1}) p_1.$$

Reversing the roles of β and p_1 gives

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{\alpha}_j \\ \kappa \leqslant \alpha_{j+1} < \min\left(\alpha_j, \frac{1}{2}(1-\alpha_1-\cdots-\alpha_j)\right)}} S\left(\mathcal{A}^q_{p_1\cdots p_j p_{j+1}}, p_{j+1}\right) \\
=& \sum_{\substack{\boldsymbol{\alpha}_j \\ \kappa \leqslant \alpha_{j+1} < \min\left(\alpha_j, \frac{1}{2}(1-\alpha_1-\cdots-\alpha_j)\right)}} S\left(\mathcal{A}^q_{\beta p_2\cdots p_j p_{j+1}}, \left(\frac{2x}{\beta p_2 \cdots p_{j+1}}\right)^{\frac{1}{2}}\right).
\end{aligned} \tag{4}$$

Then, using the second Buchstab's identity (actually on the variable p_1 in (3)), we have

$$\begin{aligned}
&\sum_{\boldsymbol{\alpha}_{j+1}} S\left(\mathcal{A}^q_{\beta p_2\cdots p_j p_{j+1}}, \left(\frac{2x}{\beta p_2 \cdots p_{j+1}}\right)^{\frac{1}{2}}\right) \\
=& \sum_{\boldsymbol{\alpha}_{j+1}} S\left(\mathcal{A}^q_{\beta p_2\cdots p_j p_{j+1}}, x^\kappa\right) - \sum_{\substack{\boldsymbol{\alpha}_{j+1} \\ \kappa \leqslant \alpha_{j+2} < \frac{1}{2}\alpha_1}} S\left(\mathcal{A}^q_{\beta p_2\cdots p_j p_{j+1} p_{j+2}}, p_{j+2}\right).
\end{aligned} \tag{5}$$

Note that $\beta \asymp x^{1-\alpha_1-\cdots-\alpha_{j+1}}$. Now we only need the condition $(1-\alpha_1-\cdots-\alpha_j, \alpha_2, \ldots, \alpha_j) \in \boldsymbol{S}_j$ (or $\boldsymbol{U}'_j$ or $\boldsymbol{U}_j$) instead of $(\alpha_1, \ldots, \alpha_j, \alpha_j) \in \boldsymbol{S}_{j+1}$ (or $\boldsymbol{U}'_{j+1}$ or $\boldsymbol{U}_{j+1}$). We can then discard parts of the last sum on the right-hand side of (5) with $(1-\alpha_1-\cdots-\alpha_{j+1}, \alpha_2, \cdots, \alpha_{j+2}) \notin \boldsymbol{G}_{j+2}$. More role-reversals are also possible to apply on other variables (e.g. p_2, p_{j+2}, ...) of this sum.

When $\theta \leqslant \frac{45}{89} - \varepsilon$, $\boldsymbol{\alpha}_j \in \boldsymbol{S}_j$ and $(1-\alpha_1-\cdots-\alpha_j, \alpha_2, \ldots, \alpha_j) \in \boldsymbol{S}_j$, by the definition of $\boldsymbol{S}_j$, we can give asymptotic formulas for the first sums on the right-hand side of (2) and (5) using Lemma 0.1 directly. In [2] the author indeed used Lemma 0.1 to give an asymptotic formula for the first sum on the right-hand side of (5). When $\theta > \frac{45}{89} - \varepsilon$, however, the available results (Lemmas 0.2–0.3) require a descending order between prime variables in the definitions of $\boldsymbol{A}_j$ and $\boldsymbol{A}'_j$. Thus, at least a splitting argument on the (possibly non-prime) variable β is necessary.

Since we have $(\beta, P(p_{j+1})) = 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_{j+1} \geqslant \kappa$, any prime factor of β is $\geqslant x^{\kappa}$, and $\frac{\log \beta}{\log x} = 1 - \alpha_1 - \cdots - \alpha_{j+1} \leqslant 1 - (j+1)\kappa$. Hence, the number of prime factors of β is no more than

$$\lceil \frac{1-(j+1)\kappa}{\kappa} \rceil < \infty. \tag{6}$$

Now, we can split β into a product of finitely many prime variables. Write $\beta = p'_1 p'_2 \cdots p'_l$, $p'_i \asymp x^{\alpha'_i}$ and $\alpha'_1 > \alpha'_2 > \ldots > \alpha'_l > \alpha_{j+1} \geqslant \kappa$. Note that we have $1 - \alpha_1 - \cdots - \alpha_{j+1} = \alpha'_1 + \cdots + \alpha'_l$.

Case 1. $\frac{45}{89} - \varepsilon < \theta \leqslant \frac{17}{32} - \varepsilon$. In order to give an asymptotic formula for the first sum on the right-hand side of (2), we need $\boldsymbol{\alpha}_j \in \boldsymbol{U}'_j$, and thus $\alpha_1 < \frac{3}{7} + \varepsilon$. We also need to give an asymptotic formula for the first sum on the right-hand side of (5). Assume first that $\alpha'_1 < \alpha_1$. Since

$$\alpha_1 > \alpha_2 > \ldots > \alpha_{j+1} \geqslant \kappa \quad \text{and} \quad \alpha_1 > \alpha'_1 > \alpha'_2 > \ldots > \alpha'_l > \alpha_{j+1} \geqslant \kappa,$$

we know that the $j+l$ variables $\alpha_2, \ldots, \alpha_{j+1}, \alpha'_1, \ldots \alpha'_l$ can be arranged into a descending order (denoted by $\boldsymbol{\alpha}'_{j+l}$) with the largest variable (=max(α_2, α'_1)) less than α_1. Hence, we have $\boldsymbol{\alpha}'_{j+l} \in \boldsymbol{A}'_{j+l}$. Now, we only need the condition $(\boldsymbol{\alpha}'_{j+l}, 2\theta - 1 + \varepsilon) \in \boldsymbol{S}_{j+l+1}$, which can be satisfied if we have

$$(\alpha_2, \ldots, \alpha_j, \alpha_{j+1}, 1 - \alpha_1 - \cdots - \alpha_{j+1}, 2\theta - 1 + \varepsilon) \in \boldsymbol{S}_{j+2}.$$

Again, this can be satisfied if we have

$$(\alpha_2, \ldots, \alpha_j, 1 - \alpha_1 - \cdots - \alpha_j, 2\theta - 1 + \varepsilon) \in \boldsymbol{S}_{j+1}.$$

Assume that $\alpha'_1 > \alpha_1$. Then we have $1 - \alpha_1 - \cdots - \alpha_{j+1} = \alpha'_1 + \cdots + \alpha'_l > \alpha'_1 > \alpha_1$. Now, if we have

$$(\alpha_2, \ldots, \alpha_j, \alpha_{j+1}, 1 - \alpha_1 - \cdots - \alpha_{j+1}, 2\theta - 1 + \varepsilon) \in \boldsymbol{S}_{j+2},$$

we must have

$$(\alpha_1, \alpha_2, \ldots, \alpha_j, \alpha_{j+1}, 2\theta - 1 + \varepsilon) \in \boldsymbol{S}_{j+2},$$

and we can give an asymptotic formula for the sum (3). Thus, a role-reversal is not necessary to apply in this case, and we can just perform another direct Buchstab iteration on the last sum on the right-hand side of (2) to get the sums (3) and

$$\sum_{\substack{\boldsymbol{\alpha}_j \\ \kappa \leqslant \alpha_{j+1} < \min\left(\alpha_j, \frac{1}{2}(1-\alpha_1-\cdots-\alpha_j)\right) \\ \kappa \leqslant \alpha_{j+2} < \min\left(\alpha_{j+1}, \frac{1}{2}(1-\alpha_1-\cdots-\alpha_j-\alpha_{j+1})\right)}} S\left(\mathcal{A}^q_{p_1 \cdots p_j p_{j+1} p_{j+2}}, p_{j+2}\right). \tag{7}$$

We can then discard parts of (7) with $\boldsymbol{\alpha}_{j+2} \notin \boldsymbol{G}_{j+2}$.

Case 2. $\frac{17}{32} - \varepsilon < \theta < \frac{11}{20} - \varepsilon$. The discussions are very similar to those in **Case 1**. The only difference is that we need $\boldsymbol{\alpha}_j \in \boldsymbol{U}_j$ instead of $\boldsymbol{\alpha}_j \in \boldsymbol{U}'_j$ (Lemma 0.2 is not available here), so that $\alpha_1 < \tau$.

In other similar cases where $2\theta - 1 + \varepsilon$ is replaced by $2\theta_1 + \theta_2 - 1 + \varepsilon$ or ε (see [2]), and when performing further role-reversals on a sum that a role-reversal has already been performed on it (which means that more almost-prime variables become explicit, see [[1], (34)] for example), the required conditions are similar.

As an application, we get the following better bounds for $C_0^*(\theta)$ using the **first method** in [2] by applying role-reversals in the extended range of θ. Many other bounds in [2] can be improved using a similar approach.

θ	$C_0^*(\theta)$ (Old)	$C_0^*(\theta)$ (New)	θ	$C_0^*(\theta)$ (Old)	$C_0^*(\theta)$ (New)
0.501	**0.8636**	**0.8636**	0.517	0.4455	**0.4936**
0.502	**0.8455**	**0.8455**	0.518	0.4086	**0.4624**
0.503	**0.8298**	**0.8298**	0.519	0.3744	**0.4261**
0.504	**0.8135**	**0.8135**	0.520	0.3363	**0.3890**
0.505	**0.7972**	**0.7972**	0.521	0.2973	**0.3510**
0.506	0.7299	**0.7730**	0.522	0.2546	**0.3113**
0.507	0.7082	**0.7505**	0.523	0.2148	**0.2714**
0.508	0.6894	**0.7334**	0.524	0.1706	**0.2277**
0.509	0.6606	**0.7045**	0.525	0.1218	**0.1785**
0.510	0.6369	**0.6808**	0.526	0.0778	**0.1324**
0.511	0.6128	**0.6551**	0.527	0.0267	**0.0779**
0.512	0.5907	**0.6373**	0.528	0	**0.0256**
0.513	0.5567	**0.6033**	0.529	0	0
0.514	0.5332	**0.5796**	0.530	0	0
0.515	0.5040	**0.5496**	0.531	0	0
0.516	0.4762	**0.5245**			

Table 1: Lower Bounds for $C_0^*(\theta)$ (First Method, $\frac{1}{2} < \theta \leqslant \frac{17}{32} - \varepsilon$)

References

[1] R. Li. On the largest prime factor of integers in short intervals III. *arXiv e-prints*, page arXiv:2601.00910v1, 2026.

[2] R. Li. Primes in arithmetic progressions to large moduli and refinements of Harman's sieve. *arXiv e-prints*, page arXiv:2602.20917v6, 2026.

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