Primes in arithmetic progressions to large moduli

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How many primes are less than x and $\equiv a(\mathrm{mod} q)$?

Dirichlet's Theorem

Let q and a be fixed integers, $q > 0$. If $(q, a) = 1$, then there are infinitely many primes $\equiv a(\mathrm{mod} q)$.

Write

$$\pi(x; q, a) = \sum_{\substack{p \leqslant x \\ p \equiv a (\bmod q)}} 1.$$

When $q > x$, then $\pi(x; q, a) \leqslant 1$. We want to show that for sufficiently large x,

$$\pi(x; q, a) \approx \frac{\pi(x)}{\varphi(q)}$$

with q as large as possible.

Siegel–Walfisz Theorem

If $q \leqslant (\log x)^A$ and $(q, a) = 1$, then we have

$$\left|\pi(x; q, a) - \frac{\pi(x)}{\varphi(q)}\right| \ll \frac{x}{(\log x)^A}.$$

Under GRH: $q \leqslant x^{\frac{1}{2}}(\log x)^{-B}$.
Montgomery's conjecture: $q \leqslant x^{1-\varepsilon}$.

In many applications, we only need a result for almost all moduli q.

Bombieri–Vinogradov Theorem (Bombieri (1965), Vinogradov (1965))

We have

$$\sum_{q \leqslant x^{\frac{1}{2}} (\log x)^{-B}} \max_{(a,q)=1} \left| \pi(x; q, a) - \frac{\pi(x)}{\varphi(q)} \right| \ll \frac{x}{(\log x)^A}.$$

On average over q, this is as good as GRH.
Elliott–Halberstam conjecture: $q \leqslant x^{1-\varepsilon}$.

Beyond $x^{1/2}$

Conjecture (Bombieri–Vinogradov with moduli beyond $x^{1/2}$)

For some $\delta > 0$, we have

$$\sum_{q \leqslant x^{\frac{1}{2}+\delta}} \max_{(a,q)=1} \left| \pi(x;q,a) - \frac{\pi(x)}{\varphi(q)} \right| \ll \frac{x}{(\log x)^A}.$$

Theorem (Bombieri et al. (1987), Bombieri et al. (1989))

Fix $a \in \mathbb{Z} \backslash \{0\}$ and let $\delta \geqslant 0$. For $Q = x^{\frac{1}{2}+\delta}$, we have

$$\sum_{\substack{q \sim Q \\ (q,a)=1}} \left| \pi(x;q,a) - \frac{\pi(x)}{\varphi(q)} \right| \ll \delta^2 \frac{x}{\log x} + \frac{x(\log\log x)^B}{(\log x)^2}.$$

Theorem (Maynard (2025b))

Let $\delta > 0$. Suppose that Q_1 and Q_2 satisfy

$$Q_1 \leqslant x^{\frac{1}{10}-3\delta}(\log x)^{-C}, \quad Q_2 \leqslant x^{\frac{2}{5}+4\delta}(\log x)^{C}.$$

Then we have

$$\sum_{\substack{q_1 \sim Q_1 \\ q_2 \sim Q_2}} \max_{(a,q_1q_2)=1} \left| \pi(x; q_1q_2, a) - \frac{\pi(x)}{\varphi(q_1q_2)} \right| \ll \delta \frac{x}{\log x} + \frac{x(\log\log x)^2}{(\log x)^2}.$$

Beyond $x^{1/2}$

Theorem (Maynard (2025a))

Fix $a \in \mathbb{Z}\backslash\{0\}$. Suppose that Q_1 and Q_2 satisfy

$$Q_1^2 Q_2 < x^{1-\varepsilon}, \quad Q_1^7 Q_2^{12} < x^{4-\varepsilon}, \quad Q_1^{19} Q_2^{20} < x^{10-\varepsilon}.$$

Then we have

$$\sum_{\substack{q_1 \leqslant Q_1 \\ q_2 \leqslant Q_2 \\ (q_1 q_2, a)=1}} \left| \pi(x; q_1 q_2, a) - \frac{\pi(x)}{\varphi(q_1 q_2)} \right| \ll \frac{x}{(\log x)^A}.$$

The above condition yields $Q_1 Q_2 < x^{\frac{11}{21}}$.

Theorem (Maynard (2025b))

Let $0 < \delta < 0.001$ and $Q_1Q_2Q_3 = x^{\frac{1}{2}+\delta}$. Suppose that Q_1, Q_2 and Q_3 satisfy

$$x^{40\delta} < Q_2 < x^{\frac{1}{20}-7\delta}, \quad x^{\frac{1}{10}+12\delta}Q_2^{-1} < Q_3 < x^{\frac{1}{10}-4\delta}Q_2^{-\frac{3}{5}}.$$

Then we have

$$\sum_{\substack{q_1 \leqslant Q_1 \\ q_2 \leqslant Q_2}} \max_{(b,q_1q_2)=1} \sum_{q_3 \leqslant Q_3} \max_{\substack{(a,q_1q_2q_3)=1 \\ a \equiv b (\bmod q_1q_2)}} \left| \pi(x; q_1q_2q_3, a) - \frac{\pi(x)}{\varphi(q_1q_2q_3)} \right| \ll \frac{x}{(\log x)^A}.$$

The problem of bounding

$$\sum_{q \leqslant Q} \left| \pi(x; q, a) - \frac{\pi(x)}{\varphi(q)} \right|$$

can be seen as a problem equivalent to bounding

$$\sum_{q \leqslant Q} \lambda_q \left(\pi(x; q, a) - \frac{\pi(x)}{\varphi(q)} \right),$$

for an arbitrary divisor-bounded weight λ_q. Naturally, we can try to make the weight λ_q more "flexible" to extend the range of Q beyond $x^{\frac{1}{2}}$.

Bilinear estimates

One way is to “split” the weight λ_q to $\lambda_{1,q_1} * \lambda_{2,q_2}$ supported on $[1, Q_1]$ and $[1, Q_2]$ respectively and prove results of the form

$$\sum_{\substack{q_1 \leqslant Q_1 \\ q_2 \leqslant Q_2 \\ (q_1 q_2, a) = 1}} \lambda_{1,q_1} \lambda_{2,q_2} \left(\pi(x; q_1 q_2, a) - \frac{\pi(x)}{\varphi(q_1 q_2)} \right) \ll \frac{x}{(\log x)^A} \tag{Bilinear}$$

for some $Q_1 Q_2 \geqslant x^{\frac{1}{2}}$.

Fix $a \in \mathbb{Z}\backslash\{0\}$. Then (Bilinear) holds if Q_1 and Q_2 satisfy:

Theorem (Bombieri et al. (1986))

$$Q_1 < x^{\frac{1}{3}}, \quad Q_2 < x^{\frac{1}{5}}, \quad Q_1^5 Q_2^2 < x^2, \quad Q_1 Q_2 < x^{\frac{29}{56}}.$$

Theorem (Fouvry (1987))

$$Q_1 Q_2^3 < x, \quad Q_1 Q_2 < x^{\frac{29}{56}}, \quad Q_1 < \max\left(x^{\frac{1}{2}} Q_2^{-1}, x^{\frac{2}{5}} Q_2^{-\frac{2}{5}}\right);$$

$$Q_1 Q_2^3 < x, \quad Q_1 Q_2 < x^{\frac{29}{56}}, \quad Q_1^4 Q_2 < x^{\frac{403}{266}}, \quad Q_1^{\frac{7}{4}} Q_2 < x^{\frac{403}{532}}.$$

Theorem (Baker and Harman (1998))

$$Q_1 < x^{\frac{1}{3}}, \quad Q_2 < x^{\frac{1}{5}}, \quad Q_1 Q_2 < x^{\frac{29}{56}}.$$

Theorem (Maynard (2025a))

$$Q_1^2 Q_2 < x^{1-\varepsilon}, \quad Q_1^7 Q_2^{12} < x^{4-\varepsilon}, \quad Q_1^{19} Q_2^{20} < x^{10-\varepsilon}.$$

A quadrilinear estimate

Theorem (Lichtman (2022))

Fix $a \in \mathbb{Z} \backslash \{0\}$. Suppose that Q_1, Q_2 and Q_3 satisfy

$$Q_1 Q_2 < x^{\frac{1}{2}+\varepsilon}, \quad Q_1 Q_3^2 < x^{\frac{1}{2}-2\varepsilon}, \quad Q_3^2 < Q_2 < x^{\frac{1}{32}-\varepsilon}.$$

Then we have

$$\sum_{\substack{q_1 \leqslant Q_1 \\ q_2 \leqslant Q_2 \\ q_3 \leqslant Q_3 \\ q_4 \leqslant Q_3 \\ (q_1 q_2 q_3 q_4, a)=1}} \lambda_{1,q_1} \lambda_{2,q_2} \lambda_{3,q_3} \lambda_{4,q_4} \left(\pi(x; q_1 q_2 q_3 q_4, a) - \frac{\pi(x)}{\varphi(q_1 q_2 q_3 q_4)} \right) \ll \frac{x}{(\log x)^A}.$$

The above condition yields $Q_1 Q_2 Q_3^2 < x^{\frac{17}{32}}$.

Theorem (L. (2026))

Fix $a \in \mathbb{Z}\backslash\{0\}$. Then (Bilinear) holds if Q_1 and Q_2 satisfy

$$Q_1^2 Q_2 < x^{1-\varepsilon}, \quad Q_1^7 Q_2^{12} < x^{4-\varepsilon}.$$

The above condition yields $Q_1 Q_2 < x^{\frac{9}{17}}$.

A new trilinear estimate

Theorem (L. (2026))

Fix $a \in \mathbb{Z}\backslash\{0\}$. Let $Q_i = x^{\theta_i}$. Suppose that $(\theta_1, \theta_2, \theta_3) \in \boldsymbol{\mathcal{W}}$. Then we have

$$\sum_{\substack{q_1 \leqslant Q_1 \\ q_2 \leqslant Q_2 \\ q_3 \leqslant Q_3 \\ (q_1 q_2 q_3, a) = 1}} \lambda_{1,q_1} \lambda_{2,q_2} \lambda_{3,q_3} \left(\pi(x; q_1 q_2 q_3, a) - \frac{\pi(x)}{\varphi(q_1 q_2 q_3)} \right) \ll \frac{x}{(\log x)^A}.$$

Moreover, we have

$$\left(\frac{15}{32} + \varepsilon, \frac{3}{64} - 2\varepsilon, \frac{1}{64} - \varepsilon \right) \in \boldsymbol{\mathcal{W}}.$$

This implies [Lichtman (2022), Theorem 1.4].

Instead of bounding

$$\sum_{\substack{q\leqslant Q\\(q,a)=1}}\left|\pi(x;q,a)-\frac{\pi(x)}{\varphi(q)}\right|,$$

one can try to bound

$$\sum_{\substack{q\leqslant Q\\(q,a)=1}}\left|\sum_{\substack{n\leqslant x\\n\equiv a(\bmod q)}}\rho_j(n)-\frac{1}{\varphi(q)}\sum_{\substack{n\leqslant x\\(n,q)=1}}\rho_j(n)\right|$$

for some functions $\rho_j(n)$ $(j\in\{0,1\})$ satisfy

$$\rho_0(n)\leqslant 1_p(n)\leqslant\rho_1(n).$$

Moreover, we call $\rho_0(n)$ a minorant for $1_p(n)$ and $\rho_1(n)$ a majorant for $1_p(n)$.

We want the minorants are always positive and the majorants are not too large. That is, we have

$$\sum_{n\leqslant x}\rho_0(n)\geqslant(1+o(1))\frac{C_0(\theta)x}{\log x}\quad\text{and}\quad\sum_{n\leqslant x}\rho_1(n)\leqslant(1+o(1))\frac{C_1(\theta)x}{\log x}$$

for two functions $C_0(\theta)$ and $C_1(\theta)$ satisfy $0<C_0(\theta)\leqslant 1$ and $C_1(\theta)\geqslant 1$, where $\theta=\frac{\log Q}{\log x}$. Moreover, we can try to show that for almost all $q\sim Q$,

$$\frac{C_0^*(\theta)x}{\varphi(q)\log x}\leqslant\pi(x;q,a)\leqslant\frac{C_1^*(\theta)x}{\varphi(q)\log x}$$

for two functions $C_0^*(\theta)$ and $C_1^*(\theta)$ satisfy $0<C_0^*(\theta)\leqslant 1$ and $C_1^*(\theta)\geqslant 1$.

From here, we always suppose that x is sufficiently large, $\varepsilon > 0$ is sufficiently small, $A, B > 0$ (may depend on other variables), and $Q, Q_i, M_i \geqslant 1$. We shall ignore the presence of ε in many places below (mainly in the definitions of regions $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{E}$, $\boldsymbol{F}$ and their subregions) for clarity. The letters p and β, with or without subscript, are reserved for primes and almost-primes respectively. We use $P^{+}(n)$ and $P^{-}(n)$ to denote the largest and smallest prime factor of n.

Put

$$P(z)=\prod_{p<z}p,\quad \psi(n,z)=1_{(n,P(z))=1},\quad \Psi(n,z)=1_{P^+(n)<z}.$$

Let $\mathcal{C}$ denote a finite set of positive integers and put

$$\mathcal{C}_d=\{n:nd\in\mathcal{C}\},\quad S(\mathcal{C},z)=\sum_{n\in\mathcal{C}}\psi(n,z)=\sum_{\substack{n\in\mathcal{C}\\(n,P(z))=1}}1.$$

Put

$$\mathcal{A}^q=\{n:n\sim x,\ n\equiv a(\mathrm{mod} q)\}\quad\text{and}\quad\mathcal{B}^q=\{n:n\sim x,\ (n,q)=1\}.$$

By the definitions of the sieved sets $\mathcal{A}^q$, $\mathcal{B}^q$ and the sieve function $S(\mathcal{C},z)$, using Prime Number Theorem, we have

$$\pi(2x;q,a)-\pi(x;q,a)=\sum_{p\in\mathcal{A}^q}1=S\left(\mathcal{A}^q,(2x)^{\frac{1}{2}}\right)\quad\text{and}\quad S\left(\mathcal{B}^q,(2x)^{\frac{1}{2}}\right)=(1+o(1))\frac{x}{\log x}.$$

Our aim is to show that the sparser set $\mathcal{A}^q$ contains the expected proportion of primes compared to the larger set $\mathcal{B}^q$, which requires us to decompose $S\left(\mathcal{A}^q,(2x)^{\frac{1}{2}}\right)$.

In order to decompose $S\left(\mathcal{A}^q,(2x)^{\frac{1}{2}}\right)$ and bound $C_0(\theta)$, $C_1(\theta)$, $C_0^*(\theta)$ and $C_1^*(\theta)$, we need the following two types of asymptotic formulas.
Type-C:

$$\sum_{\substack{q\sim Q\\(q,a)=1}}\left|S\left(\mathcal{A}^q,z\right)-\frac{1}{\varphi(q)}S\left(\mathcal{B}^q,z\right)\right|\ll\frac{x}{(\log x)^A}. \tag{C}$$

Type-C^*: For almost all $q\sim Q$,

$$S\left(\mathcal{A}^q,z\right)=(1+o(1))\frac{1}{\varphi(q)}S\left(\mathcal{B}^q,z\right). \tag{C^*}$$

Note that an asymptotic formula of the form (C) implies an asymptotic formula of the form (C^*).

There are two conditions that we may want the coefficients to satisfy. We shall use a divisor-bounded coefficient sequence λ_l as an example.

(**Condition A**: Siegel–Walfisz condition) For any $f \geqslant 1$, $k \geqslant 1$, $b \neq 0$ and $(k, b) = 1$, we have

$$\sum_{\substack{l \sim L \\ l \equiv b \pmod k \\ (l,f)=1}} \lambda_l = \frac{1}{\varphi(k)} \sum_{\substack{l \sim L \\ (l,fk)=1}} \lambda_l + O\left(\frac{L(d(f))^B}{(\log L)^A} \right).$$

(**Condition B**: No small prime factors) We have $\lambda_l = 0$ whenever l has a prime factor smaller than $\exp\left(\log x (\log\log x)^{-3}\right)$.

For example, if we produce parallel decompositions of $S\left(\mathcal{A}^q, (2x)^{\frac{1}{2}}\right)$ and $S\left(\mathcal{B}^q, (2x)^{\frac{1}{2}}\right)$ as

$$S\left(\mathcal{A}^q, (2x)^{\frac{1}{2}}\right) = \Sigma_1 - \Sigma_2, \qquad S\left(\mathcal{B}^q, (2x)^{\frac{1}{2}}\right) = \Sigma_1' - \Sigma_2'$$

with $\Sigma_1, \Sigma_2, \Sigma_1', \Sigma_2' \geqslant 0$, and we know that Σ_1 has an asymptotic formula of the form (C), we can use the size of Σ_1 to give an upper bound for $C_1(\theta)$. To calculate the value, we can add the corresponding "loss" Σ_2' on $S\left(\mathcal{B}^q, (2x)^{\frac{1}{2}}\right)$ (corresponds to 1).
In the lower bound case, we can only discard positive terms and subtract the corresponding "loss" from $S\left(\mathcal{B}^q, (2x)^{\frac{1}{2}}\right)$.

Buchstab's identity is the equation

$$S(\mathcal{C}, z) = S(\mathcal{C}, w) - \sum_{w \leqslant p < z} S(\mathcal{C}_p, p),$$

where $2 \leqslant w < z$.
Let $\omega(u)$ denote the Buchstab function determined by the following differential-difference equation

$$\begin{cases} \omega(u) = \frac{1}{u}, & 1 \leqslant u \leqslant 2, \\ (u\omega(u))' = \omega(u-1), & u \geqslant 2. \end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\omega(u) \geqslant \omega_0(u) = \begin{cases} \frac{1}{u}, & 1 \leqslant u < 2, \\ \frac{1+\log(u-1)}{u}, & 2 \leqslant u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t} dt \geqslant 0.5607, & 3 \leqslant u < 4, \\ 0.5612, & u \geqslant 4, \end{cases}$$

$$\omega(u) \leqslant \omega_1(u) = \begin{cases} \frac{1}{u}, & 1 \leqslant u < 2, \\ \frac{1+\log(u-1)}{u}, & 2 \leqslant u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t} dt \leqslant 0.5644, & 3 \leqslant u < 4, \\ 0.5617, & u \geqslant 4. \end{cases}$$

Sometimes we also need this simple equality in our numerical calculations:

$$\int_{1-\theta}^{\frac{1}{2}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt = \int_{1-\theta}^{\frac{1}{2}} \frac{1}{t(1-t)} dt = \log\left(\frac{\theta}{1-\theta}\right) \quad \text{for} \quad \frac{1}{2} < \theta < \frac{2}{3}.$$

A "Type-I$_j$" sum refers to a sum of type

$$\sum_{m_0, m_1, \ldots, m_j} a_{0,m_0},$$

and a "Type-II$_j$" sum refers to a sum of type

$$\sum_{m_1, \ldots, m_j} a_{1,m_1} \cdots a_{j,m_j},$$

where a_{i,m_i} $(0 \leqslant i \leqslant j)$ are divisor-bounded complex sequences. For the sake of simplicity, we often write "Type-I$_1$" as "Type-I" and "Type-II$_2$" as "Type-II".

Type-II information: Asymptotic formula of the form (C) for Type-II sums. Let $M_1M_2 \asymp x$. Suppose that a_{2,m_2} satisfies **Condition A**.

$$\sum_{\substack{q\sim Q\\(q,a)=1}}\left|\sum_{\substack{m_1\sim M_1\\m_2\sim M_2\\m_1m_2\equiv a(\bmod q)}} a_{1,m_1}a_{2,m_2}-\frac{1}{\varphi(q)}\sum_{\substack{m_1\sim M_1\\m_2\sim M_2\\(m_1m_2,q)=1}} a_{1,m_1}a_{2,m_2}\right| \ll \frac{x}{(\log x)^A}.$$

(Fouvry (1984))

$$M_2 \geqslant x^{\varepsilon}, \quad Q^2x^{-1+\varepsilon} \leqslant M_2 \leqslant Q^{-\frac{4}{3}}x^{\frac{5}{6}-\varepsilon}.$$

(Wright (2026))

$$\exp\left((\log x)^{\varepsilon}\right) \leqslant M_2 \leqslant Q^{-\frac{33}{28}}x^{\frac{17}{28}-\varepsilon}.$$

Type-II$_3$ information: Asymptotic formula of the form (C) for Type-II$_3$ sums. Let $M_1M_2M_3 \asymp x$. Suppose that a_{1,m_1}, a_{2,m_2} and a_{3,m_3} satisfy **Condition B**, and a_{2,m_2} also satisfies **Condition A**.

$$\sum_{\substack{q\sim Q\\(q,a)=1}}\left|\sum_{\substack{m_1\sim M_1\\m_2\sim M_2\\m_3\sim M_3\\m_1m_2m_3\equiv a(\bmod q)}} a_{1,m_1}a_{2,m_2}a_{3,m_3}-\frac{1}{\varphi(q)}\sum_{\substack{m_1\sim M_1\\m_2\sim M_2\\m_3\sim M_3\\(m_1m_2m_3,q)=1}} a_{1,m_1}a_{2,m_2}a_{3,m_3}\right| \ll \frac{x}{(\log x)^A}.$$

(Bombieri et al. (1987))

$$\min(M_1,M_2,M_3)>x^{\varepsilon},\quad Qx^{\varepsilon}<M_1M_2,\quad M_1^2M_2^3<Qx^{1-\varepsilon},\quad M_1^5M_2^2<x^{2-\varepsilon},\quad M_1^4M_2^3<x^{2-\varepsilon}.$$

(Baker and Harman (1996))

$$\min(M_1,M_2,M_3)>x^{\varepsilon},\quad Qx^{\varepsilon}<M_1M_2,\quad M_1M_2^2Q^2<x^{2-3\varepsilon},\quad M_1^5M_2^2<x^{2-3\varepsilon}.$$

(Maynard (2025a))

$$\min(M_1,M_2,M_3)>x^{\varepsilon},\quad Q<x^{0.7-\varepsilon},\quad Qx^{\varepsilon}<M_1M_2,\quad M_2<Q^{-1}x^{1-2\varepsilon},$$
$$M_1M_2<Q^{-\frac{1}{7}}x^{\frac{153}{224}-10\varepsilon},\quad M_1^4M_2<Q^{-1}x^{\frac{57}{32}-10\varepsilon}.$$

Arithmetic information

Type-I/II information: Asymptotic formula of the form (C) for Type-I/II sums. Let $M_1M_2M_3 \asymp x$ and $z \ll \exp\left(\log x(\log\log x)^{-1}\right)$.

$$\sum_{\substack{q\sim Q\\(q,a)=1}}\left|\sum_{\substack{m_1\sim M_1\\m_2\in\mathbf{M}\\m_3\sim M_3\\m_1m_2m_3\equiv a(\bmod q)}} a_{1,m_1}a_{2,m_2}a_{3,m_3} - \frac{1}{\varphi(q)}\sum_{\substack{m_1\sim M_1\\m_2\in\mathbf{M}\\m_3\sim M_3\\(m_1m_2m_3,q)=1}} a_{1,m_1}a_{2,m_2}a_{3,m_3}\right| \ll \frac{x}{(\log x)^A}.$$

(Bombieri et al. (1987))

$$M_3Q < x^{1-\varepsilon}, \quad M_1^4M_3Q < x^{2-\varepsilon}, \quad M_1^2M_3Q^2 < x^{2-\varepsilon},$$

and

$$a_{2,m_2} = 1_{m_2\in\mathbf{M}} \qquad \text{or} \qquad a_{2,m_2} = 1_{\substack{m_2\in\mathbf{M}\\(m_2,P(z))=1}}$$

for some interval $\mathbf{M} \subseteq [M_2, 2M_2]$.

Arithmetic information

New Type-II information: Let $\mathbf{M}_i$ $(1 \leqslant i \leqslant r)$ be intervals such that $\mathbf{M}_i \subseteq [M_i, 2M_i]$. Let $\prod_{1\leqslant i\leqslant r} M_i \asymp x$. Suppose that a_{j,m_j} $(1 \leqslant j \leqslant r)$ satisfy **Conditions A and B**.

$$\sum_{\substack{q\sim Q\\(q,a)=1}} \left| \sum_{\substack{m_i\in\mathbf{M}_i\\1\leqslant i\leqslant r}} \left(\prod_{1\leqslant j\leqslant r} a_{j,m_j} \right) \left(\mathbf{1}_{m_1\cdots m_r\equiv a (\bmod q)} - \frac{\mathbf{1}_{(m_1\cdots m_r,q)=1}}{\varphi(q)} \right) \right| \ll \frac{x}{(\log x)^A}.$$

(Maynard (2025a))

$$Q \leqslant x^{\frac{127}{224}-\varepsilon}, \quad x^{\frac{3}{7}+\varepsilon} < \prod_{j\in\mathcal{J}} M_j < Q^{-1}x^{1-\varepsilon}$$

for some set $\mathcal{J} \subseteq \{1,\ldots,r\}$, and

$$a_{j,m_j} = \mathbf{1}_{(m_j,P(z))=1}, \qquad z = \exp\left(\log x(\log\log x)^{-3}\right)$$

for all $m_j > x^{\frac{1}{15}}$.

Let $\theta, \theta_i \in (0,\ 1)$, $Q = x^\theta$, $p_i \asymp x^{\alpha_i}$ and $\boldsymbol{\alpha}_n = (\alpha_1, \ldots, \alpha_n)$. Write

$$\kappa = \kappa(\theta) = \begin{cases} \frac{5-8\theta}{6} - \varepsilon, & \theta \leqslant \frac{17}{32} - \varepsilon, \\ \frac{5-8\theta}{12} - 3\varepsilon, & \frac{17}{32} - \varepsilon < \theta \leqslant \frac{7}{13} - \varepsilon, \\ \frac{3-5\theta}{7} - 2\varepsilon, & \frac{7}{13} - \varepsilon < \theta \leqslant \frac{4}{7} - \varepsilon, \end{cases}$$

$$\kappa' = \kappa'(\theta) = \begin{cases} \frac{11-20\theta}{6} - 2\varepsilon, & \frac{7}{13} - \varepsilon < \theta \leqslant \frac{11}{20} - \varepsilon, \\ \kappa, & \text{otherwise.} \end{cases}$$

$$\tau = \tau(\theta) = \begin{cases} \frac{3(1-\theta)}{5} - \varepsilon, & \theta \leqslant \frac{11}{21}, \\ \frac{2}{7} - \varepsilon, & \frac{11}{21} < \theta \leqslant \frac{6}{11} - \varepsilon, \\ \frac{5-6\theta}{7} - \varepsilon, & \frac{6}{11} - \varepsilon < \theta, \end{cases}$$

$$\tau' = \tau'(\theta) = \begin{cases} \frac{5-6\theta}{7}, & \frac{7}{13} - \varepsilon < \theta \leqslant \frac{11}{20} - \varepsilon, \\ \tau, & \text{otherwise.} \end{cases}$$

The terms *partition* and *exactly partition* are defined in [Harman (2007), Page 162].

Let $\frac{1}{2} \leqslant \theta \leqslant \frac{4}{7} - \varepsilon$. Define

$$\boldsymbol{g}_1 = \boldsymbol{g}_1(\theta) = \left\{(s,t) : 2\theta - 1 + \varepsilon \leqslant s \leqslant \frac{5-8\theta}{6} - \varepsilon\right\},$$

$$\boldsymbol{g}_2 = \boldsymbol{g}_2(\theta) = \left\{(s,t) : s \leqslant \frac{17-33\theta}{28} - \varepsilon\right\},$$

$$\boldsymbol{g}_3 = \boldsymbol{g}_3(\theta) = \{(s,t) : s+t \geqslant \theta + \varepsilon,\ 2s+3t \leqslant 1+\theta-\varepsilon,\ 5s+2t \leqslant 2-\varepsilon,\ 4s+3t \leqslant 2-\varepsilon\},$$

$$\boldsymbol{g}_4 = \boldsymbol{g}_4(\theta) = \{(s,t) : s+t \geqslant \theta + \varepsilon,\ s+2t \leqslant 2-2\theta-\varepsilon,\ 5s+2t \leqslant 2-\varepsilon\},$$

$$\boldsymbol{g}_5 = \boldsymbol{g}_5(\theta) = \left\{(s,t) : s+t \geqslant \theta+\varepsilon,\ t \leqslant 1-\theta-\varepsilon,\ s+t \leqslant \frac{153}{224} - \frac{1}{7}\theta - \varepsilon,\ 4s+t \leqslant \frac{57}{32} - \theta - \varepsilon\right\},$$

$$\boldsymbol{g}_6 = \boldsymbol{g}_6(\theta) = \left\{(s,t) : \theta \leqslant \frac{127}{224} - \varepsilon,\ \frac{3}{7} + \varepsilon \leqslant s \leqslant 1-\theta-\varepsilon \text{ or } \theta+\varepsilon \leqslant s \leqslant \frac{4}{7} - \varepsilon\right\},$$

$$\boldsymbol{\mathcal{G}}_j = \boldsymbol{\mathcal{G}}_j(\theta) = \big\{\boldsymbol{\alpha}_j : \min \boldsymbol{\alpha}_j \geqslant (\log\log x)^{-3},\ \boldsymbol{\alpha}_j \text{ partitions into } \boldsymbol{g}_1 \cup \boldsymbol{g}_2 \cup \boldsymbol{g}_3 \cup \boldsymbol{g}_4 \cup \boldsymbol{g}_5$$
$$\text{or } \min \boldsymbol{\alpha}_j \geqslant 0.001 + 10\varepsilon,\ \boldsymbol{\alpha}_j \text{ partitions into } \boldsymbol{g}_6\big\}.$$

Then

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{\mathcal{G}}_j} S\left(\mathcal{A}^q_{p_1\cdots p_j}, p_j\right),$$

or more general Type-II sums, has an asymptotic formula of the form (C).

Let $\frac{1}{2} \leqslant \theta \leqslant \frac{4}{7} - \varepsilon$. Define

$$\boldsymbol{g}_7 = \boldsymbol{g}_7(\theta) = \left\{(s,t) : \frac{3}{7} + \varepsilon \leqslant s \leqslant 1 - \theta - \varepsilon,\ 0 \leqslant t < \frac{1-s}{2}\right\},$$
$$\boldsymbol{U}_j^* = \boldsymbol{U}_j^*(\theta) = \{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \text{ partitions exactly into } \boldsymbol{g}_7\}.$$

Let $\xi(\boldsymbol{\alpha_j})$ be a continuous function with $\varepsilon^2 \leqslant \xi(\boldsymbol{\alpha_j}) \leqslant \frac{1}{2}$. Then

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j^*} S\left(\mathcal{A}^q_{p_1 \cdots p_j}, x^{\xi(\boldsymbol{\alpha_j})}\right)$$

has an asymptotic formula of the form (C).

Let $\frac{1}{2} \leqslant \theta \leqslant \frac{17}{32} - \varepsilon$ and $M < x^{1-\varepsilon} Q^{-1}$. Then, for a divisor-bounded sequence a_m,

$$\sum_{m \sim M} a_m S\left(\mathcal{A}_m^q, x^{\kappa}\right)$$

has an asymptotic formula of the form (C).
Let $\frac{1}{2} \leqslant \theta < \frac{11}{20} - \varepsilon$ and $M < x^{1-\varepsilon} Q^{-1}$. Then, for a divisor-bounded sequence a_m,

$$\sum_{m \sim M} a_m S\left(\mathcal{A}_m^q, x^{\frac{11-20\theta}{6} - \varepsilon}\right)$$

has an asymptotic formula of the form (C).

Let $\frac{1}{2} \leqslant \theta \leqslant \frac{4}{7} - \varepsilon$. Define

$$
\begin{aligned}
\boldsymbol{S} &= \boldsymbol{S}(\theta) = \{(s,t) : s \leqslant 1-\theta-\varepsilon,\ s+2t \leqslant 2-2\theta-\varepsilon,\ s+4t \leqslant 2-\theta-\varepsilon\},\\
\boldsymbol{S}_j &= \boldsymbol{S}_j(\theta) = \{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \text{ partitions exactly into } \boldsymbol{S}\},\\
\boldsymbol{A}_j &= \boldsymbol{A}_j(\theta) = \left\{\boldsymbol{\alpha}_j : (\log\log x)^{-3} \leqslant \alpha_j < \cdots < \alpha_1 < \tau,\ \alpha_1 + \cdots + \alpha_j \leqslant 1\right\},\\
\boldsymbol{T}^* &= \boldsymbol{T}^*(\theta) = \left\{(s,t) : 0 \leqslant s \leqslant \frac{8\theta-2}{7},\ 0 \leqslant t \leqslant \frac{5-6\theta}{7}\right\},\\
\boldsymbol{T}_j^* &= \boldsymbol{T}_j^*(\theta) = \{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \text{ partitions exactly into } \boldsymbol{T}^*\},\\
\boldsymbol{U}_j' &= \boldsymbol{U}_j'(\theta) = \{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \in \boldsymbol{A}_j,\ (\alpha_1, \ldots, \alpha_j, 2\theta-1+\varepsilon) \in \boldsymbol{S}_{j+1}\},\\
\boldsymbol{U}_j &= \boldsymbol{U}_j(\theta) = \begin{cases} \boldsymbol{U}_j'(\theta), & \theta < \frac{7}{13},\\ \left\{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \in \boldsymbol{A}_j,\ \boldsymbol{\alpha}_j \in \boldsymbol{T}_j^*\right\}, & \theta \geqslant \frac{7}{13}. \end{cases}
\end{aligned}
$$

Let $\frac{1}{2} \leqslant \theta \leqslant \frac{4}{7} - \varepsilon$. Then

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}_j} S\left(\mathcal{A}^q_{p_1 \cdots p_j}, x^{\kappa}\right)$$

has an asymptotic formula of the form (C). Moreover, the condition $\alpha_1 < \tau$ in the definition of $\boldsymbol{A}_j$ can be removed when $\theta \leqslant \frac{17}{32} - \varepsilon$.

Let $\frac{1}{2} \leqslant \theta < \frac{11}{20} - \varepsilon$. Then

$$\sum_{\boldsymbol{\alpha}_j \in \boldsymbol{U}'_j} S\left(\mathcal{A}^q_{p_1 \cdots p_j}, x^{\kappa'}\right)$$

has an asymptotic formula of the form (C).

Let $\frac{1}{2} \leqslant \theta \leqslant \frac{45}{89} - \varepsilon$, $m_1 \sim M_1$ and $m_2 \sim M_2$. Then, for divisor-bounded sequences a_{1,m_1} and a_{2,m_2},

$$\sum_{\left(\frac{\log M_1}{\log x}, \frac{\log M_2}{\log x}\right) \in \boldsymbol{S}} a_{1,m_1} a_{2,m_2} S\left(\mathcal{A}^q_{m_1 m_2}, x^{\kappa}\right)$$

has an asymptotic formula of the form (C).

Define

$$\begin{aligned}
\boldsymbol{G}_j &= \boldsymbol{G}_j(\theta) = \left\{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \in \boldsymbol{\mathcal{G}}_j\right\}, \\
U_2 &= U_2(\theta) = \left\{\boldsymbol{\alpha}_2 : \kappa \leqslant \alpha_1 < \frac{3}{7} + \varepsilon,\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)\right\}, \\
A &= A(\theta) = \left\{\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2 \in U_2,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2,\ \alpha_1 + \alpha_2 < \theta + \varepsilon\right\}, \\
B &= B(\theta) = \left\{\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2 \in U_2,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \cup A,\ \alpha_1 + 4\alpha_2 < 3 - 3\theta - \varepsilon\right\}, \\
C &= C(\theta) = \left\{\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2 \in U_2,\ \boldsymbol{\alpha}_2 \notin \boldsymbol{G}_2 \cup A \cup B\right\}.
\end{aligned}$$

Let $\frac{1}{2} \leqslant \theta \leqslant \frac{11}{21} - \varepsilon$. Then

$$\sum_{\boldsymbol{\alpha}_2 \in A \cup B} S\left(\mathcal{A}^q_{p_1 p_2}, x^{\kappa}\right)$$

has an asymptotic formula of the form (C).

Let $\frac{1}{2} \leqslant \theta \leqslant \frac{17}{32} - \varepsilon$. Then

$$\sum_{\boldsymbol{\alpha}_2 \in A} S\left(\mathcal{A}^q_{p_1 p_2}, x^{\kappa}\right)$$

has an asymptotic formula of the form (C).

A two-dimensional Harman's sieve. Put

$$\mathcal{E}^q = \left\{ m_1 m_2 : x^{1-\theta-\varepsilon} \leqslant n_1 \leqslant x^{\frac{1}{2}},\ m_1 m_2 \sim x,\ m_1 m_2 \equiv a (\mathrm{mod}\, q) \right\}.$$

(Baker and Harman (1996)) Let $\frac{1}{2} \leqslant \theta < \frac{17}{32}$. Then

$$S\left(\mathcal{E}^q, x^\kappa\right) = \sum_{\substack{m_1 m_2 \in \mathcal{A} \\ x^{1-\theta-\varepsilon} \leqslant m_1 \leqslant x^{\frac{1}{2}} \\ (m_1 m_2, P(x^\kappa)) = 1}} 1$$

has an asymptotic formula of the form (C^*).

Corollary 1: Let $\frac{1}{2} \leqslant \theta < \frac{17}{32}$. Then we have, for almost all $q \sim x^{\theta}$,

$$-\sum_{1-\theta-\varepsilon<\alpha_1<\frac{1}{2}} S\left(\mathcal{A}_{p_1}^{q}, p_1\right) \leqslant (1+o(1)) \frac{1}{\varphi(q)} \left(-\sum_{1-\theta-\varepsilon<\alpha_1<\frac{1}{2}} S\left(\mathcal{B}_{p_1}^{q}, p_1\right) \right.$$

$$\left. + \sum_{\substack{\kappa\leqslant\alpha_3<\alpha_1<\frac{\theta+\varepsilon}{2} \\ 1-\theta-\varepsilon<\alpha_1+v\leqslant\theta+\varepsilon \\ (\alpha_1,v)\notin \boldsymbol{G}_2 \\ \alpha_3<\min\left(v,\frac{1}{2}(1-\alpha_1-v)\right) \\ (\alpha_1,v,\alpha_3)\notin \boldsymbol{G}_3}} S\left(\mathcal{B}_{p_1 m p_3}^{q}, p_3\right) \right),$$

where $m = x^{v}$ and $(m, P(p_1)) = 1$.

Corollary 2: Let $\frac{1}{2} \leqslant \theta < \frac{17}{32}$. Then we have, for almost all $q \sim x^{\theta}$,

$$-\sum_{1-\theta-\varepsilon<\alpha_1<\frac{1}{2}} S\left(\mathcal{A}_{p_1}^{q}, p_1\right) \geqslant (1+o(1)) \frac{1}{\varphi(q)} \left(-\sum_{1-\theta-\varepsilon<\alpha_1<\frac{1}{2}} S\left(\mathcal{B}_{p_1}^{q}, p_1\right) \right.$$
$$\left. - \sum_{\substack{\kappa \leqslant \alpha_1 < v \\ 1-\theta-\varepsilon<\alpha_1+v\leqslant\theta+\varepsilon \\ (\alpha_1, v) \notin \boldsymbol{G}_2}} S\left(\mathcal{B}_{p_1 m}^{q}, p_1\right) \right),$$

where $m = x^{v}$ and $(m, P(p_1)) = 1$.

A three-dimensional Harman's sieve. Define

$$\boldsymbol{R} = \boldsymbol{R}(\theta) = \{\boldsymbol{\alpha}_2 : \alpha_2 \leqslant \alpha_1,\ \alpha_1 + 2\alpha_2 \leqslant 1,\ \alpha_1 + 4\alpha_2 \geqslant 3 - 3\theta - \varepsilon,\ 3\alpha_2 \geqslant 2\alpha_1, \\ \max\left(\tau,\ \frac{31\theta - 15}{3} + 5\varepsilon\right) \leqslant \alpha_1 \leqslant \min\left(\frac{3}{7} + \varepsilon,\ 4 - 7\theta - 3\varepsilon\right)\right\},$$

$$\boldsymbol{R}_0 = \boldsymbol{R}_0(\theta) = \{\boldsymbol{\alpha}_2 : \alpha_1 + 2\alpha_2 \leqslant 1,\ \alpha_1 + 4\alpha_2 \geqslant 3 - 3\theta - \varepsilon,\ 3\alpha_2 \geqslant 2\alpha_1 - 3\varepsilon, \\ \max\left(\frac{19\theta - 7}{7},\ \frac{50\theta - 19}{17}\right) + 24\varepsilon \leqslant \alpha_1 \leqslant \frac{3}{7} + \varepsilon\right\}.$$

Put

$$\mathcal{H}_1^q = \left\{m_1 m_2 m_3 : \left(\frac{\log m_1}{\log x}, \frac{\log m_2}{\log x}\right) \in \boldsymbol{R},\ m_1 m_2 m_3 \sim x,\ m_1 m_2 m_3 \equiv a (\bmod q)\right\},$$

$$\mathcal{H}_2^q = \left\{m_1 m_2 m_3 : \left(\frac{\log m_1}{\log x}, \frac{\log m_2}{\log x}\right) \in \boldsymbol{R}_0,\ m_1 m_2 m_3 \sim x,\ m_1 m_2 m_3 \equiv a (\bmod q)\right\}.$$

(Baker and Harman (1996)) Let $\frac{1}{2} \leqslant \theta < \frac{16}{31} - \varepsilon$. Then

$$S\left(\mathcal{H}_1^q, x^\kappa\right) = \sum_{\substack{m_1 m_2 m_3 \in \mathcal{A} \\ \left(\frac{\log m_1}{\log x}, \frac{\log m_2}{\log x}\right) \in \boldsymbol{R} \\ (m_1 m_2 m_3, P(x^\kappa)) = 1}} 1$$

has an asymptotic formula of the form (C^*).
(Baker and Harman (1998)) Let $\frac{25}{49} - \varepsilon \leqslant \theta \leqslant \frac{92}{175} - \varepsilon$. Assume that q is prime. Then

$$S\left(\mathcal{H}_2^q, x^\kappa\right) = \sum_{\substack{m_1 m_2 m_3 \in \mathcal{A} \\ \left(\frac{\log m_1}{\log x}, \frac{\log m_2}{\log x}\right) \in \boldsymbol{R}_0 \\ (m_1 m_2 m_3, P(x^\kappa)) = 1}} 1$$

has an asymptotic formula of the form (C^*).

Define

$$\begin{aligned}
\boldsymbol{D}_1 = \boldsymbol{D}_1(\theta) = \{\boldsymbol{\alpha}_3 : \alpha_1 \geqslant \kappa,\ \alpha_2 \geqslant \kappa,\ \alpha_3 \geqslant \kappa,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\\
(\alpha_1, \alpha_2 + \alpha_3) \in \boldsymbol{R} \text{ with } \alpha_2 \geqslant \alpha_3 \text{ or } (\alpha_1 + \alpha_2, \alpha_3) \in \boldsymbol{R} \text{ with } \alpha_1 \geqslant \alpha_2\},\\
\boldsymbol{D}_2 = \boldsymbol{D}_2(\theta) = \{\boldsymbol{\alpha}_4 : \alpha_1 \geqslant \kappa,\ \alpha_2 \geqslant \kappa,\ \alpha_3 \geqslant \kappa,\ \alpha_4 \geqslant \kappa,\\
\boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ (\alpha_1 + \alpha_2, \alpha_3 + \alpha_4) \in \boldsymbol{R}\},\\
\boldsymbol{D}_3 = \boldsymbol{D}_3(\theta) = \{\boldsymbol{\alpha}_3 : \alpha_1 \geqslant \kappa,\ \alpha_2 \geqslant \kappa,\ \alpha_3 \geqslant \kappa,\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3,\\
(\alpha_1, \alpha_2 + \alpha_3) \in \boldsymbol{R}_0 \text{ with } \alpha_2 \geqslant \alpha_3 \text{ or } (\alpha_1 + \alpha_2, \alpha_3) \in \boldsymbol{R}_0 \text{ with } \alpha_1 \geqslant \alpha_2\},\\
\boldsymbol{D}_4 = \boldsymbol{D}_4(\theta) = \{\boldsymbol{\alpha}_4 : \alpha_1 \geqslant \kappa,\ \alpha_2 \geqslant \kappa,\ \alpha_3 \geqslant \kappa,\ \alpha_4 \geqslant \kappa,\\
\boldsymbol{\alpha}_4 \notin \boldsymbol{G}_4,\ (\alpha_1 + \alpha_2, \alpha_3 + \alpha_4) \in \boldsymbol{R}_0\}.
\end{aligned}$$

Corollary 1: Let $\frac{1}{2} \leqslant \theta < \frac{16}{31} - \varepsilon$. Then we have, for almost all $q \sim x^{\theta}$,

$$\sum_{\boldsymbol{\alpha}_2 \in \boldsymbol{R}} S\left(\mathcal{A}^{q}_{p_1 p_2}, x^{\kappa}\right) \leqslant (1+o(1)) \frac{1}{\varphi(q)} \left(\sum_{\boldsymbol{\alpha}_2 \in \boldsymbol{R}} S\left(\mathcal{B}^{q}_{p_1 p_2}, x^{\kappa}\right) + \frac{x}{\log x}(I_1 + I_2) \right),$$

where

$$I_1 = \frac{1}{\kappa} \int_{(t_1,t_2,t_3) \in \boldsymbol{D}_1} \frac{\omega\left(\frac{1-t_1-t_2-t_3}{\kappa}\right)}{t_1 t_2 t_3} dt_3 dt_2 dt_1,$$

$$I_2 = \frac{1}{\kappa} \int_{(t_1,t_2,t_3,t_4) \in \boldsymbol{D}_2} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{\kappa}\right)}{t_1 t_2 t_3 t_4} dt_4 dt_3 dt_2 dt_1.$$

Corollary 2: Let $\frac{25}{49}-\varepsilon \leqslant \theta \leqslant \frac{92}{175}-\varepsilon$. Then we have, for almost all prime $q \sim x^{\theta}$,

$$\sum_{\boldsymbol{\alpha}_2 \in \boldsymbol{R}_0} S\left(\mathcal{A}_{p_1 p_2}^{q}, x^{\kappa}\right) \leqslant (1+o(1)) \frac{1}{\varphi(q)}\left(\sum_{\boldsymbol{\alpha}_2 \in \boldsymbol{R}_0} S\left(\mathcal{B}_{p_1 p_2}^{q}, x^{\kappa}\right)+\frac{x}{\log x}(I_3+I_4)\right),$$

where

$$I_3=\frac{1}{\kappa} \int_{(t_1, t_2, t_3) \in \boldsymbol{D}_3} \frac{\omega\left(\frac{1-t_1-t_2-t_3}{\kappa}\right)}{t_1 t_2 t_3} d t_3 d t_2 d t_1,$$

$$I_4=\frac{1}{\kappa} \int_{(t_1, t_2, t_3, t_4) \in \boldsymbol{D}_4} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{\kappa}\right)}{t_1 t_2 t_3 t_4} d t_4 d t_3 d t_2 d t_1.$$

We give upper bounds for $C_1(\theta)$ and $C_1^*(\theta)$.
Known results:
(1). $C_1(\theta) = C_1^*(\theta) = 1$ for all $\theta \leqslant 0.5 - \varepsilon$;
(2). $C_1^*(\theta) \leqslant 1 + \varepsilon$ for $\theta = 0.5$;
(3). $C_1^*(\theta)$ is monotonic increasing for $0.5 \leqslant \theta \leqslant 0.6$.
Now we split the range $\theta \in \left[\frac{1}{2},\ 1\right)$ to several subranges and use different methods to treat them. Note that when $\theta \in \left[\frac{1}{2},\ \frac{127}{224} - \varepsilon\right]$, we can remove the condition $t < \frac{1}{2}(1-s)$ when applying the Type-II range $\left[\frac{3}{7} + \varepsilon,\ 1 - \theta - \varepsilon\right]$.
Note that we can only discard negative terms when proving upper bounds.

Case 1. $\frac{1}{2} \leqslant \theta \leqslant \frac{45}{89} - \varepsilon$. Type-II range:

$$\left((\log x)^{\varepsilon-1},\ \frac{1}{6}(5-8\theta)-\varepsilon\right].$$

Using Buchstab's identity, we have

$$\begin{aligned} S\left(\mathcal{A}^q,(2x)^{\frac{1}{2}}\right) &= S\left(\mathcal{A}^q, x^{\kappa}\right) - \sum_{\kappa \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^q_{p_1}, p_1\right) \\ &= S\left(\mathcal{A}^q, x^{\kappa}\right) - \sum_{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon} S\left(\mathcal{A}^q_{p_1}, p_1\right) \\ &\quad - \sum_{\frac{3}{7}+\varepsilon \leqslant \alpha_1 \leqslant 1-\theta-\varepsilon} S\left(\mathcal{A}^q_{p_1}, p_1\right) - \sum_{1-\theta-\varepsilon < \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^q_{p_1}, p_1\right). \end{aligned}$$

The first and the third terms have asymptotic formulas of the form (C).

Case $C_1(\theta)$: We discard the (negative) fourth term.

$$\sum_{1-\theta-\varepsilon<\alpha_1<\frac{1}{2}} S\left(\mathcal{A}_{p_1}^q, p_1\right) \Rightarrow \text{Loss of } \int_{1-\theta}^{\frac{1}{2}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt + O(\varepsilon).$$

The second term:

$$\sum_{\kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon} S\left(\mathcal{A}_{p_1}^q, p_1\right) = \sum_{\kappa\leqslant\alpha_1<\tau} S\left(\mathcal{A}_{p_1}^q, p_1\right) + \sum_{\tau\leqslant\alpha_1<\frac{3}{7}+\varepsilon} S\left(\mathcal{A}_{p_1}^q, p_1\right).$$

We discard the (negative) second term above.

$$\sum_{\tau\leqslant\alpha_1<\frac{3}{7}+\varepsilon} S\left(\mathcal{A}_{p_1}^q, p_1\right) \Rightarrow \text{Loss of } \int_{\tau}^{\frac{3}{7}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt + O(\varepsilon).$$

The first term: Since $\theta \leqslant \frac{11}{21} - \varepsilon$, we have $\boldsymbol{\alpha}_2 \in \boldsymbol{G}_2 \cup A \cup B$ for any $\kappa \leqslant \alpha_2 < \alpha_1 < \tau$. Using Buchstab's identity, we have

$$\begin{aligned}
&\sum_{\kappa \leqslant \alpha_1 < \tau} S\left(\mathcal{A}_{p_1}^{q}, p_1\right) \\
=& \sum_{\kappa \leqslant \alpha_1 < \tau} S\left(\mathcal{A}_{p_1}^{q}, x^{\kappa}\right) - \sum_{\substack{\kappa \leqslant \alpha_1 < \tau \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in \boldsymbol{G}_2}} S\left(\mathcal{A}_{p_1 p_2}^{q}, p_2\right) \\
&- \sum_{\substack{\kappa \leqslant \alpha_1 < \tau \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B}} S\left(\mathcal{A}_{p_1 p_2}^{q}, p_2\right)
\end{aligned}$$

$$
\begin{aligned}
= & \sum_{\kappa \leqslant \alpha_1 < \tau} S\left(\mathcal{A}_{p_1}^q, x^\kappa\right) - \sum_{\substack{\kappa \leqslant \alpha_1 < \tau \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in \boldsymbol{G}_2}} S\left(\mathcal{A}_{p_1 p_2}^q, p_2\right) \\
& - \sum_{\substack{\kappa \leqslant \alpha_1 < \tau \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B}} S\left(\mathcal{A}_{p_1 p_2}^q, x^\kappa\right) + \sum_{\substack{\kappa \leqslant \alpha_1 < \tau \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \in \boldsymbol{G}_3}} S\left(\mathcal{A}_{p_1 p_2 p_3}^q, p_3\right) \\
& + \sum_{\substack{\kappa \leqslant \alpha_1 < \tau \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3}} S\left(\mathcal{A}_{p_1 p_2 p_3}^q, p_3\right).
\end{aligned}
$$

The first, the second, the third, and the fourth terms have asymptotic formulas of the form (C).

The (negative) fifth term:

$$\sum_{\substack{\kappa\leqslant\alpha_1<\tau \\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2\in A\cup B \\ \kappa\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3\notin \boldsymbol{G}_3}} S\left(\mathcal{A}^q_{p_1p_2p_3}, p_3\right).$$

Parts with $(\alpha_1, \alpha_2, \alpha_3, \alpha_3) \notin \boldsymbol{S}_4$: Discard them.
Parts with $(\alpha_1, \alpha_2, \alpha_3, \alpha_3) \in \boldsymbol{S}_4$: Apply Buchstab's identity twice in a similar way to get a (negative) five-dimensional sum.
This process can be used repeatedly until we get a $\lfloor\frac{1}{\kappa}\rfloor$-dimensional sum.

We can also use the following two useful devices.
Reversed Buchstab's identity. Idea:

$$S\left(\mathcal{A}^q_{p_1\cdots p_j}, p_j\right) = \sum_{\substack{p_1\cdots p_j\beta\in\mathcal{A}^q \\ (\beta,P(p_j))=1}} 1 = \sum_{\substack{p_1\cdots p_j p_{j+1}\in\mathcal{A}^q \\ p_{j+1}>p_j}} 1 + \sum_{\substack{p_1\cdots p_j\beta\in\mathcal{A}^q \\ \Omega(\beta)\geqslant 2 \\ (\beta,P(p_j))=1}} 1.$$

We can decompose β in the last sum and use the new prime factors to get asymptotic formulas of the form (C).

An example: When $\alpha_3 < \frac{1}{2}(1 - \alpha_1 - \alpha_2 - \alpha_3)$, we have

$$\sum_{\boldsymbol{\alpha}_3} S\left(\mathcal{A}^q_{p_1p_2p_3}, p_3\right) = \sum_{\boldsymbol{\alpha}_3} S\left(\mathcal{A}^q_{p_1p_2p_3}, \left(\frac{2x}{p_1p_2p_3}\right)^{\frac{1}{2}}\right) + \sum_{\substack{\boldsymbol{\alpha}_3 \\ \alpha_3 < \alpha_4 < \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)}} S\left(\mathcal{A}^q_{p_1p_2p_3p_4}, p_4\right).$$

We can give asymptotic formulas of the form (C) for parts of the last sum with $\boldsymbol{\alpha}_4 \in \boldsymbol{G}_4$. We can use this device again for the remaining parts of the last sum if $\alpha_4 < \frac{1}{2}(1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4)$.
Buchstab's identity: When $\alpha_3 < \frac{1}{2}(1 - \alpha_1 - \alpha_2 - \alpha_3)$, we have

$$\sum_{\boldsymbol{\alpha}_3} S\left(\mathcal{A}^q_{p_1p_2p_3}, \left(\frac{2x}{p_1p_2p_3}\right)^{\frac{1}{2}}\right) = \sum_{\boldsymbol{\alpha}_3} S\left(\mathcal{A}^q_{p_1p_2p_3}, p_3\right) - \sum_{\substack{\boldsymbol{\alpha}_3 \\ \alpha_3 < \alpha_4 < \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)}} S\left(\mathcal{A}^q_{p_1p_2p_3p_4}, p_4\right).$$

(Variable) Role-reversal.
Since only negative terms can be discarded in the upper bound case, we need to use Buchstab's identity twice if we use it directly. Assume that $\boldsymbol{\alpha}_j \in \boldsymbol{S}_j$.

$$\sum_{\boldsymbol{\alpha}_j} S\left(\mathcal{A}^q_{p_1\cdots p_j}, p_j\right) = \sum_{\boldsymbol{\alpha}_j} S\left(\mathcal{A}^q_{p_1\cdots p_j}, x^\kappa\right) - \sum_{\substack{\boldsymbol{\alpha}_j \\ \kappa\leqslant\alpha_{j+1}<\min\left(\alpha_j, \frac{1}{2}(1-\alpha_1-\cdots-\alpha_j)\right)}} S\left(\mathcal{A}^q_{p_1\cdots p_j p_{j+1}}, p_{j+1}\right).$$

After applying Buchstab's identity once, sometimes we cannot use it again since $(\alpha_1, \ldots, \alpha_j, \alpha_j) \notin \boldsymbol{S}_{j+1}$. In this case, we cannot give an asymptotic formula for (positive)

$$\sum_{\substack{\boldsymbol{\alpha}_j \\ \kappa\leqslant\alpha_{j+1}<\min\left(\alpha_j, \frac{1}{2}(1-\alpha_1-\cdots-\alpha_j)\right)}} S\left(\mathcal{A}^q_{p_1\cdots p_j p_{j+1}}, x^\kappa\right)$$

after the second Buchstab iteration.

Idea: Applying the second Buchstab iteration on a different variable, usually the largest explicit prime variable (denoted by p_1).

$$(p_1 p_2 \cdots p_{j+1})\beta \Rightarrow (\beta p_2 \cdots p_{j+1}) p_1$$

We have

$$\sum_{\substack{\boldsymbol{\alpha}_j \\ \kappa \leqslant \alpha_{j+1} < \min\left(\alpha_j, \frac{1}{2}(1-\alpha_1-\cdots-\alpha_j)\right)}} S\left(\mathcal{A}^q_{p_1 \cdots p_j p_{j+1}}, p_{j+1}\right)$$

$$= \sum_{\substack{\boldsymbol{\alpha}_j \\ \kappa \leqslant \alpha_{j+1} < \min\left(\alpha_j, \frac{1}{2}(1-\alpha_1-\cdots-\alpha_j)\right)}} S\left(\mathcal{A}^q_{\beta p_2 \cdots p_j p_{j+1}}, \left(\frac{2x}{\beta p_2 \cdots p_{j+1}}\right)^{\frac{1}{2}}\right).$$

Then, using Buchstab's identity, we have

$$\sum_{\boldsymbol{\alpha}_{j+1}} S\left(\mathcal{A}^{q}_{\beta p_2 \cdots p_j p_{j+1}}, \left(\frac{2x}{\beta p_2 \cdots p_{j+1}}\right)^{\frac{1}{2}}\right)$$
$$= \sum_{\boldsymbol{\alpha}_{j+1}} S\left(\mathcal{A}^{q}_{\beta p_2 \cdots p_j p_{j+1}}, x^{\kappa}\right) - \sum_{\substack{\boldsymbol{\alpha}_{j+1} \\ \kappa \leqslant \alpha_{j+2} < \frac{1}{2}\alpha_1}} S\left(\mathcal{A}^{q}_{\beta p_2 \cdots p_j p_{j+1} p_{j+2}}, p_{j+2}\right).$$

Note that $\beta \asymp x^{1-\alpha_1-\cdots-\alpha_{j+1}}$. Now we only need the condition $(1-\alpha_1-\cdots-\alpha_j, \alpha_2, \ldots, \alpha_j) \in \boldsymbol{S}_j$ instead of $(\alpha_1, \ldots, \alpha_j, \alpha_j) \in \boldsymbol{S}_{j+1}$. We can discard parts of the (negative) last sum with $(1-\alpha_1-\cdots-\alpha_{j+1}, \alpha_2, \cdots, \alpha_{j+2}) \notin \boldsymbol{G}_{j+2}$.
Role-reversals are very useful in the lower bound case, which we will discuss later. Due to some reasons, we shall only use role-reversals when $\frac{1}{2} \leqslant \theta \leqslant \frac{45}{89} - \varepsilon$.

Final decompositions, 1-1

The differences between role-reversals in Harman's sieve and the switching principle of Chen and Iwaniec: In some problems, such as the Goldbach problem ($2n = p + p'$) and the twin prime problem ($p + 2 = p'$), we have more than one prime variable to study and decompose. Role-reversals focus on the size of factors of elements in one sieved set, while the switching principle allows us to change the decomposed element and construct a new sieved set to work with.

An example: Diophantine approximation with Goldbach numbers. Harman (2020) proved that for any irrational α and arbitrary real β, there are infinitely many solutions to

$$\|\alpha n + \beta\| < n^{-\frac{5}{6}}, \qquad n = p + p', \qquad p,\ p' > 2.$$

Final decompositions, 1-1

A role-reversal:

$$\begin{aligned} n-p' = p \Rightarrow n-p' &= (p_1p_2\cdots p_j)\beta_1 \\ \Rightarrow n-p' &= (\beta_1p_2\cdots p_j)p_1 \\ \Rightarrow n-p' &= (\beta_1p_2\cdots p_j)p_{j+1}\cdots p_k\beta_2. \end{aligned}$$

The switching principle:

$$\begin{aligned} n-p' = p \Rightarrow n-p' &= (p_1p_2\cdots p_j)\beta_1 \\ \Rightarrow n-(p_1p_2\cdots p_j)\beta_1 &= p' \\ \Rightarrow n-(p_1p_2\cdots p_j)\beta_1 &= (p_1'p_2'\cdots p_k')\beta_1'. \end{aligned}$$

By combining these two devices (maybe the first time to do this in the history of sieves), Harman (2020) obtained a lower bound constant 0.16. In an unpublished note, we improved this to 0.2. Role-reversals are also possible to use (with a splitting argument needed) in the proof of famous results like Chen's Theorem $(1+2)$. We hope someone can accomplish this work.

After decompositions, we can obtain an upper bound for $C_1(\theta)$ of the following form:

$$C_1(\theta) \leqslant 1 + \int_{1-\theta}^{\frac{1}{2}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt + \int_{\tau}^{\frac{3}{7}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt + \text{3D Loss} - \text{4D Saving} + \text{5D Loss} + O(\varepsilon).$$

Case $C_1^*(\theta)$: Since $\theta < \frac{17}{32}$, we use a two-dimensional Harman's sieve to deal with the fourth term.

$$\sum_{1-\theta-\varepsilon<\alpha_1<\frac{1}{2}} S\left(\mathcal{A}_{p_1}^q, p_1\right) \Rightarrow \text{Loss of } \int_{(t_1,t_2,t_3)} \frac{\omega\left(\frac{t_2}{t_1}\right)\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1^2 t_3^2} dt_3 dt_2 dt_1 + O(\varepsilon).$$

The second term:

$$\sum_{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon} S\left(\mathcal{A}_{p_1}^{q}, p_1\right).$$

Using Buchstab's identity, we have

$$\begin{aligned}
&\sum_{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon} S\left(\mathcal{A}_{p_1}^{q}, p_1\right) \\
=& \sum_{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon} S\left(\mathcal{A}_{p_1}^{q}, x^{\kappa}\right) - \sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in \boldsymbol{G}_2}} S\left(\mathcal{A}_{p_1 p_2}^{q}, p_2\right) \\
&- \sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B}} S\left(\mathcal{A}_{p_1 p_2}^{q}, p_2\right) - \sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in C}} S\left(\mathcal{A}_{p_1 p_2}^{q}, p_2\right)
\end{aligned}$$

$$
\begin{aligned}
= & \sum_{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon} S\left(\mathcal{A}_{p_1}^q, x^\kappa\right) - \sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in \boldsymbol{G}_2}} S\left(\mathcal{A}_{p_1 p_2}^q, p_2\right) \\
& - \sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B}} S\left(\mathcal{A}_{p_1 p_2}^q, x^\kappa\right) - \sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in C}} S\left(\mathcal{A}_{p_1 p_2}^q, x^\kappa\right) \\
& + \sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B \cup C \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \in \boldsymbol{G}_3}} S\left(\mathcal{A}_{p_1 p_2 p_3}^q, p_3\right) + \sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B \cup C \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3}} S\left(\mathcal{A}_{p_1 p_2 p_3}^q, p_3\right).
\end{aligned}
$$

The first, the second, the third, and the fifth terms have asymptotic formulas of the form (C^*).

Since $\theta \leqslant \frac{25}{49} - \varepsilon$, we know that $\boldsymbol{\alpha}_2 \in C$ implies $\boldsymbol{\alpha}_2 \in \boldsymbol{R}$. We use a three-dimensional Harman's sieve to deal with the fourth term.

$$\sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7} + \varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in C}} S\left(\mathcal{A}^q_{p_1 p_2}, x^\kappa\right) \Rightarrow \text{Loss of } I_1 + I_2 + O(\varepsilon).$$

The sixth term:

$$\sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7} + \varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B \cup C \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right) \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3}} S\left(\mathcal{A}^q_{p_1 p_2 p_3}, p_3\right).$$

The decompositions on this sum are similar to the corresponding decompositions in Case $C_1(\theta)$.

After decompositions, we can obtain an upper bound for $C_1^*(\theta)$ of the following form:

$$C_1^*(\theta) \leqslant 1 + \int_{(t_1, t_2, t_3)} \frac{\omega\left(\frac{t_2}{t_1}\right)\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1^2 t_3^2} dt_3 dt_2 dt_1 + I_1 + I_2$$
$$+ \text{3D Loss} - \text{4D Saving} + \text{5D Loss} + O(\varepsilon).$$

Case 2. $\frac{45}{89} - \varepsilon < \theta \leqslant \frac{11}{21} - \varepsilon$. Type-II range:

$$\left[2\theta - 1 + \varepsilon,\ \frac{1}{6}(5 - 8\theta) - \varepsilon\right].$$

Case $C_1(\theta)$: The decompositions are very similar to the decompositions in Case 1. We only need to replace the condition $(\alpha_1, \alpha_2, \alpha_3, \alpha_3) \in \boldsymbol{S}_4$ and similar conditions involving $\boldsymbol{S}_j$ with $(\alpha_1, \alpha_2, \alpha_3, \alpha_3) \in \boldsymbol{U}_4$ and corresponding ones with $\boldsymbol{U}_j$. Also, we do not use role-reversals here.

Case $C_1^*(\theta)$: For $\theta \leqslant \frac{25}{49} - \varepsilon$, the decompositions are very similar to the decompositions in Case 1. For $\frac{25}{49} - \varepsilon < \theta \leqslant \frac{11}{21} - \varepsilon$, we only need to make some modifications on the decompositions of the sum

$$\sum_{\kappa \leqslant \alpha_1 < \frac{3}{7} + \varepsilon} S\left(\mathcal{A}_{p_1}^{q}, p_1\right).$$

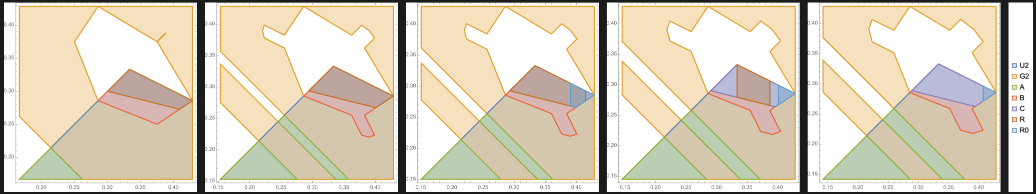
For parts of this sum with

$$\max\left(\tau,\ \frac{31\theta - 15}{3} + 5\varepsilon\right) \leqslant \alpha_1 \leqslant \min\left(\frac{3}{7} + \varepsilon,\ 4 - 7\theta - 3\varepsilon\right),$$

we can perform the corresponding decompositions in Case 1 if $\theta < \frac{16}{31} - \varepsilon$. When q is prime, we can also perform similar decompositions on parts of this sum with

$$\max\left(\frac{19\theta - 7}{7},\ \frac{50\theta - 19}{17}\right) + 24\varepsilon \leqslant \alpha_1 \leqslant \frac{3}{7} + \varepsilon.$$

We discard the remaining parts of this sum.



Plots of U_2 $(\theta = \frac{1}{2},\ \frac{25}{49} - \varepsilon,\ \frac{21}{41} - \varepsilon,\ \frac{16}{31} - 2\varepsilon,\ \frac{14}{27} - \varepsilon)$

Case 3. $\frac{11}{21} - \varepsilon < \theta \leqslant \frac{17}{32} - \varepsilon$. Type-II range:

$$\left[2\theta - 1 + \varepsilon,\ \frac{1}{6}(5 - 8\theta) - \varepsilon\right].$$

Case $C_1(\theta)$: The decompositions are very similar to the decompositions in Case 2.
Case $C_1^*(\theta)$: (Baker and Harman (1996)) We have

$$C_1^*(\theta) \leqslant 1.725 + 13.125(\theta - 0.523).$$

Case 4. $\frac{17}{32} - \varepsilon < \theta \leqslant \frac{4}{7} - \varepsilon$.
From here, all high-dimensional sieve results mentioned before are unavailable. The decompositions are the same in the cases $C_1(\theta)$ and $C_1^*(\theta)$. When $\theta > \frac{127}{224} - \varepsilon$, we need to add the condition $t < \frac{1}{2}(1-s)$ when applying the Type-II range $\left[\frac{3}{7} + \varepsilon,\ 1 - \theta - \varepsilon\right]$.

$$C_1(\theta),\ C_1^*(\theta) \leqslant 1 + \int_{1-\theta}^{\frac{1}{2}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt + \int_{\tau}^{\frac{3}{7}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt + \text{3D Loss} - \text{4D Saving} + \text{5D Loss} + O(\varepsilon).$$

We shall only use the above upper bound for $C_1^*(\theta)$ when $\theta \leqslant 0.565$. When $\theta > 0.565$, we have better upper bounds for $C_1^*(\theta)$, and we shall discuss them in the next case.

Case 5. $\frac{4}{7} - \varepsilon < \theta < 1$.
From here, almost all of the arithmetic information becomes unavailable, and we can only use the standard upper bound sieve to construct the majorant $\rho_1(n)$ in the case $C_1(\theta)$. Following Fouvry and Radziwiłł (2022), we pick the sieve weights $\lambda_{d,z}$ as in Selberg's upper bound sieve:

$$\sum_{\substack{d|n \\ d\leqslant z}} \lambda_{d,z} = \left(\sum_{\substack{d|n \\ d\leqslant \sqrt{z}}} \mu(d) \left(1 - \frac{\log d}{\log \sqrt{z}} \right) \right)^2$$

and $\lambda_{d,z} = 0$ for $d > z$. It is clear that we have

$$1_p(n) \leqslant \sum_{d|n} \lambda_{d,z}$$

for $n > \sqrt{z}$ and

$$\sum_{n\sim x} \sum_{d|n} \lambda_{d,z} = 2(1 + o(1)) \frac{x}{\log z}.$$

Let $\rho_1(n) = \sum_{d|n} \lambda_{d,z}$.

By our choice of $\rho_1(n)$, a result in Wright (2026) and the generalized Bombieri–Vinogradov Theorem, we have the following bounds for $C_1(\theta)$:

$$C_1(\theta) \leqslant \begin{cases} 4+\varepsilon, & \frac{4}{7}-\varepsilon < \theta \leqslant \frac{529}{630}+\varepsilon, \\ \frac{178}{45}+\varepsilon, & \frac{529}{630}+\varepsilon < \theta \leqslant \frac{177}{178}, \\ \frac{66}{28\theta-11}+\varepsilon, & \frac{177}{178} < \theta < 1. \end{cases}$$

Case $C_1^*(\theta)$: We need to consider the range $0.565 < \theta < 1$. For $\frac{177}{178} < \theta < 1$ we can use the above upper bounds for $C_1(\theta)$ to bound $C_1^*(\theta)$; for $0.565 < \theta \leqslant \frac{11}{14}$ we can use the existing results of Baker and Harman (1996) and Fouvry (1985) based on the Rosser–Iwaniec sieve with a bilinear error term:

$$C_1^*(\theta) \leqslant \begin{cases} \frac{14}{12-13\theta} - \log\left(\frac{4(1-\theta)}{3\theta}\right) + \varepsilon, & 0.565 < \theta < \frac{4}{7}, \\ \frac{14}{12-13\theta} + \varepsilon, & \frac{4}{7} \leqslant \theta < 0.6, \\ \frac{8}{3-\theta} + \varepsilon, & 0.6 \leqslant \theta < \frac{5}{7}, \\ \frac{6}{1+\theta} + \varepsilon, & \frac{5}{7} \leqslant \theta < \frac{3}{4}, \\ \frac{12}{5-2\theta} + \varepsilon, & \frac{3}{4} \leqslant \theta < \frac{11}{14}. \end{cases}$$

For $\frac{11}{14} \leqslant \theta \leqslant \frac{177}{178}$, we improve the result of Fouvry (1985).

Theorem (L. (2026))

For $\frac{11}{14} \leqslant \theta \leqslant \frac{177}{178}$, we have

$$C_1^*(\theta) \leqslant \max\left(\frac{16}{3+2\theta}, \frac{4}{3-2\theta}\right) + \varepsilon.$$

Combining those bounds we obtain

$$C_1^*(\theta) \leqslant \begin{cases} \frac{14}{12-13\theta} - \log\left(\frac{4(1-\theta)}{3\theta}\right) + \varepsilon, & 0.565 < \theta < \frac{4}{7}, \\ \frac{14}{12-13\theta} + \varepsilon, & \frac{4}{7} \leqslant \theta < 0.6, \\ \frac{8}{3-\theta} + \varepsilon, & 0.6 \leqslant \theta < \frac{5}{7}, \\ \frac{6}{1+\theta} + \varepsilon, & \frac{5}{7} \leqslant \theta < \frac{3}{4}, \\ \frac{12}{5-2\theta} + \varepsilon, & \frac{3}{4} \leqslant \theta < \frac{11}{14}, \\ \frac{16}{3+2\theta} + \varepsilon, & \frac{11}{14} \leqslant \theta < \frac{9}{10}, \\ \frac{4}{3-2\theta} + \varepsilon, & \frac{9}{10} \leqslant \theta \leqslant \frac{177}{178}, \\ \frac{66}{28\theta-11} + \varepsilon, & \frac{177}{178} < \theta < 1. \end{cases}$$

θ	$C_1(\theta)$	θ	$C_1(\theta)$	θ	$C_1(\theta)$
0.500	1.5823	0.524	1.7805	0.548	2.3725
0.501	1.5900	0.525	1.7894	0.549	2.3738
0.502	1.5977	0.526	1.7966	0.550	2.3996
0.503	1.6055	0.527	1.8041	0.551	2.4318
0.504	1.6132	0.528	1.8133	0.552	2.4393
0.505	1.6214	0.529	1.8239	0.553	2.4404
0.506	1.6292	0.530	1.8338	0.554	2.4426
0.507	1.6368	0.531	1.8451	0.555	2.4540
0.508	1.6446	0.532	1.8619	0.556	2.4776
0.509	1.6532	0.533	1.8658	0.557	2.4987
0.510	1.6614	0.534	1.8780	0.558	2.5120
0.511	1.6692	0.535	1.9028	0.559	2.5327
0.512	1.6776	0.536	1.9200	0.560	2.5345
0.513	1.6862	0.537	1.9277	0.561	2.6895
0.514	1.6944	0.538	1.9466	0.562	2.7144
0.515	1.7031	0.539	1.9568	0.563	2.8054
0.516	1.7104	0.540	1.9993	0.564	2.8788
0.517	1.7193	0.541	2.0582	0.565	2.9548
0.518	1.7283	0.542	2.0793	0.566	3.0791
0.519	1.7342	0.543	2.1223	0.567	3.3065
0.520	1.7394	0.544	2.1604	0.568	3.4768
0.521	1.7493	0.545	2.2179	0.569	3.6696
0.522	1.7611	0.546	2.3122	0.570	3.7733
0.523	1.7710	0.547	2.3310	0.571	3.8666

Upper Bounds for $C_1(\theta)$ $(0.5 \leqslant \theta \leqslant \frac{4}{7} - \varepsilon)$

θ	$C_1^*(\theta)$	θ	$C_1^*(\theta)$	θ	$C_1^*(\theta)$
0.500	$1+\varepsilon$	0.522	1.7320	0.544	2.1604
0.501	1.0001	0.523	1.7382	0.545	2.2179
0.502	1.0003	0.524	1.7382	0.546	2.3122
0.503	1.0013	0.525	1.7500	0.547	2.3310
0.504	1.0019	0.526	1.7644	0.548	2.3725
0.505	1.0030	0.527	1.7775	0.549	2.3738
0.506	1.0043	0.528	1.7907	0.550	2.3996
0.507	1.0071	0.529	1.8038	0.551	2.4318
0.508	1.0091	0.530	1.8169	0.552	2.4393
0.509	1.0111	0.531	1.8300	0.553	2.4404
0.510	1.0150	0.532	1.8619	0.554	2.4426
0.511	1.0466	0.533	1.8658	0.555	2.4540
0.512	1.0830	0.534	1.8780	0.556	2.4776
0.513	1.1593	0.535	1.9028	0.557	2.4987
0.514	1.2525	0.536	1.9200	0.558	2.5120
0.515	1.3466	0.537	1.9277	0.559	2.5327
0.516	1.4353	0.538	1.9466	0.560	2.5345
0.517	1.6781	0.539	1.9568	0.561	2.6895
0.518	1.6876	0.540	1.9993	0.562	2.7144
0.519	1.6947	0.541	2.0582	0.563	2.8054
0.520	1.7028	0.542	2.0793	0.564	2.8788
0.521	1.7160	0.543	2.1223	0.565	2.9548

Upper Bounds for $C_1^*(\theta)$ $(0.5 \leqslant \theta \leqslant 0.565)$

θ	$C_1^*(\theta)$	θ	$C_1^*(\theta)$	θ	$C_1^*(\theta)$
0.500	$1+\varepsilon$	0.522	1.6873	0.544	2.1604
0.501	1.0001	0.523	1.7138	0.545	2.2179
0.502	1.0003	0.524	1.7382	0.546	2.3122
0.503	1.0013	0.525	1.7500	0.547	2.3310
0.504	1.0019	0.526	1.7644	0.548	2.3725
0.505	1.0030	0.527	1.7775	0.549	2.3738
0.506	1.0043	0.528	1.7907	0.550	2.3996
0.507	1.0071	0.529	1.8038	0.551	2.4318
0.508	1.0091	0.530	1.8169	0.552	2.4393
0.509	1.0111	0.531	1.8300	0.553	2.4404
0.510	1.0150	0.532	1.8619	0.554	2.4426
0.511	1.0237	0.533	1.8658	0.555	2.4540
0.512	1.0315	0.534	1.8780	0.556	2.4776
0.513	1.0789	0.535	1.9028	0.557	2.4987
0.514	1.1431	0.536	1.9200	0.558	2.5120
0.515	1.2199	0.537	1.9277	0.559	2.5327
0.516	1.3199	0.538	1.9466	0.560	2.5345
0.517	1.5740	0.539	1.9568	0.561	2.6895
0.518	1.5948	0.540	1.9993	0.562	2.7144
0.519	1.6136	0.541	2.0582	0.563	2.8054
0.520	1.6320	0.542	2.0793	0.564	2.8788
0.521	1.6591	0.543	2.1223	0.565	2.9548

Upper Bounds for $C_1^*(\theta)$ ($0.5 \leqslant \theta \leqslant 0.565$), Prime Moduli

Combining various bounds for $C_1^*(\theta)$, we can get the estimate

$$\int_{0.5}^{0.679} C_1^*(\theta)d\theta < 0.5 - \varepsilon$$

under the assumption that q is prime. This implies

Theorem (L. (2025))

There are infinitely many primes p such that

$$P^+(p+a) > p^{0.679}.$$

However, we are still unable to show that

$$\int_{0.5}^{0.68} C_1^*(\theta)d\theta < 0.5 - \varepsilon.$$

We give lower bounds for $C_0(\theta)$ and $C_0^*(\theta)$.
Known results:
(1). $C_0(\theta) = C_0^*(\theta) = 1$ for all $\theta \leqslant 0.5 - \varepsilon$;
(2). $C_0^*(\theta) \geqslant 1 - \varepsilon$ for $\theta = 0.5$;
(3). $C_0^*(\theta)$ is monotonic decreasing for $0.5 \leqslant \theta < \frac{17}{32}$.
Now we focus on the range $\theta \in \left[\frac{1}{2},\ \frac{17}{32} - \varepsilon\right]$.
Case $C_0(\theta)$: Using Buchstab's identity, we have

$$\begin{aligned} S\left(\mathcal{A}^q, (2x)^{\frac{1}{2}}\right) &= S\left(\mathcal{A}^q, x^\kappa\right) - \sum_{\kappa \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^q_{p_1}, p_1\right) \\ &= S\left(\mathcal{A}^q, x^\kappa\right) - \sum_{\kappa \leqslant \alpha_1 \leqslant 1-\theta-\varepsilon} S\left(\mathcal{A}^q_{p_1}, p_1\right) - \sum_{1-\theta-\varepsilon < \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^q_{p_1}, p_1\right). \end{aligned}$$

However, we cannot give an asymptotic formula of the form (C) for the last sum above. Hence we cannot get any nontrivial lower bound for $C_0(\theta)$ for $\theta \geqslant \frac{1}{2}$.

We shall use two different methods to give lower bounds for $C_0^*(\theta)$.
First Method: Harman's sieve with Buchstab iterations.
Case 1. $\frac{1}{2} \leqslant \theta \leqslant \frac{45}{89} - \varepsilon$. Type-II range:

$$\left((\log x)^{\varepsilon-1},\ \frac{1}{6}(5-8\theta)-\varepsilon\right].$$

Using Buchstab's identity, we have

$$\begin{aligned} S\left(\mathcal{A}^q,(2x)^{\frac{1}{2}}\right) &= S\left(\mathcal{A}^q,x^\kappa\right) - \sum_{\kappa\leqslant\alpha_1<\frac{1}{2}} S\left(\mathcal{A}^q_{p_1},p_1\right) \\ &= S\left(\mathcal{A}^q,x^\kappa\right) - \sum_{\kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon} S\left(\mathcal{A}^q_{p_1},p_1\right) \\ &\quad - \sum_{\frac{3}{7}+\varepsilon\leqslant\alpha_1\leqslant1-\theta-\varepsilon} S\left(\mathcal{A}^q_{p_1},p_1\right) - \sum_{1-\theta-\varepsilon<\alpha_1<\frac{1}{2}} S\left(\mathcal{A}^q_{p_1},p_1\right) \end{aligned}$$

Final decompositions, 1-2

$$
\begin{aligned}
&= S\left(\mathcal{A}^q, x^\kappa\right) - \sum_{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon} S\left(\mathcal{A}^q_{p_1}, x^\kappa\right) - \sum_{\frac{3}{7}+\varepsilon \leqslant \alpha_1 \leqslant 1-\theta-\varepsilon} S\left(\mathcal{A}^q_{p_1}, p_1\right) \\
&\quad - \sum_{1-\theta-\varepsilon < \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^q_{p_1}, p_1\right) + \sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right)}} S\left(\mathcal{A}^q_{p_1 p_2}, p_2\right).
\end{aligned}
$$

The first, the second and the third terms have asymptotic formulas of the form (C^*). Since $\theta < \frac{17}{32}$, we use a two-dimensional Harman's sieve to deal with the fourth term.

$$
\sum_{1-\theta-\varepsilon < \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^q_{p_1}, p_1\right) \Rightarrow \text{Loss of} \int_{(t_1, t_2)} \frac{\omega\left(\frac{t_2}{t_1}\right) \omega\left(\frac{1-t_1-t_2}{t_1}\right)}{t_1^3} dt_2 dt_1 + O(\varepsilon).
$$

The fifth term:

$$\begin{aligned}
&\sum_{\substack{\kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)}} S\left(\mathcal{A}^q_{p_1p_2},p_2\right)\\
=&\sum_{\substack{\kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in\boldsymbol{G}_2}} S\left(\mathcal{A}^q_{p_1p_2},p_2\right)+\sum_{\substack{\kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in A\cup B}} S\left(\mathcal{A}^q_{p_1p_2},p_2\right)\\
&+\sum_{\substack{\kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in C}} S\left(\mathcal{A}^q_{p_1p_2},p_2\right).
\end{aligned}$$

The first term above has an asymptotic formula of the form (C^*). We discard the (positive) third term above.

The remaining second term:

$$\sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B}} S\left(\mathcal{A}^q_{p_1 p_2}, p_2\right).$$

Parts with $(\alpha_1, \alpha_2, \alpha_2) \in \boldsymbol{S}_3$: Apply Buchstab's identity twice to get a (positive) four-dimensional sum.
Remaining parts with $(\alpha_1, \alpha_2) \in \boldsymbol{S}_2$ and $(1-\alpha_1-\alpha_2, \alpha_2) \in \boldsymbol{S}_2$: Apply Buchstab's identity twice with a role-reversal to get a (positive) four-dimensional sum. Role-reversals are very important in giving better lower bounds for $C_0^*(\theta)$ in this case.
Remaining parts: Use the reversed Buchstab's identity to gain some savings.
This process can be used repeatedly until we get a $\lfloor \frac{1}{\kappa} \rfloor$-dimensional sum.

Case 2. $\frac{45}{89} - \varepsilon < \theta \leqslant \frac{17}{32} - \varepsilon$.
The decompositions are very similar to the decompositions in Case 1. We only need to replace the condition $(\alpha_1, \alpha_2, \alpha_2) \in \boldsymbol{S}_3$ and similar conditions involving $\boldsymbol{S}_j$ with $(\alpha_1, \alpha_2, \alpha_2) \in \boldsymbol{U}_3$ and corresponding ones with $\boldsymbol{U}_j$. Also, we do not use role-reversals here.
Since we can remove the condition $\alpha_1 < \tau$ in the definition of $\boldsymbol{A}_j$ when $\theta \leqslant \frac{17}{32} - \varepsilon$, we can decompose the sum

$$\sum_{\substack{\tau \leqslant \alpha_1 < \frac{3}{7} + \varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B}} S\left(\mathcal{A}^q_{p_1 p_2}, p_2\right).$$

This helps us obtain better lower bounds for $C_0^*(\theta)$.

After decompositions, we can obtain a lower bound for $C_0^*(\theta)$ of the following form:

$$C_0^*(\theta) \geqslant 1 - \int_{(t_1,t_2)} \frac{\omega\left(\frac{t_2}{t_1}\right)\omega\left(\frac{1-t_1-t_2}{t_1}\right)}{t_1^3} dt_2 dt_1 - \text{2D Loss} + \text{3D Saving} - \text{4D Loss} + O(\varepsilon).$$

Second Method: Mikawa's modified sieve.

Idea: Introduce a generalized sieve function that involves a sum over Möbius function $\mu(d)$, "split" this sieve function in two different ways to get two sums of primes together with "other sums" that count almost primes, and use a combinatorial way to measure the contribution from "other sums".

Mikawa used his modified sieve in Mikawa (2001) and Mikawa (2000) to study primes and Goldbach numbers in arithmetic progressions.

Preliminary lemmas:
(Mikawa (2001)) For square-free n, we have

$$\sum_{\substack{d\mid n\\ d<\sqrt{n}}} \mu(d) = \begin{cases} 0, & \mu(n)=1, \\ 1, & n=p, \\ 0, & p \mid n \text{ with } \sqrt{n}<p<n, \\ -2, & n=p_1p_2p_3 \text{ with } p_3<p_2<p_1<\sqrt{n}, \\ -20, & \Omega(n)=7 \text{ with } P^-(n)>n^{\frac{1}{8}}. \end{cases}$$

(Mikawa (2001)) For square-free n, we have

$$\sum_{\substack{d|n \\ \mu^2(d)=1 \\ \Omega(d)=3 \\ \sqrt{n}<d<\sqrt{nP^-(d)}}} 2 = \begin{cases} 2, & n = p_1p_2p_3p_4(p_4 < p_3 < p_2 < p_1) \text{ with } p_1 < p_2p_3p_4 \text{ and } p_2p_3 < p_1, \\ 0, & \Omega(n) \leqslant 3, \\ 0, & n = p_1p_2p_3p_4(p_4 < p_3 < p_2 < p_1) \text{ with } p_1 > p_2p_3p_4 \text{ or } p_2p_3 > p_1, \\ \leqslant 20, & \Omega(n) = 6, \\ 0, & \Omega(n) = 7 \text{ with } P^-(n) > n^{\frac{1}{8}}. \end{cases}$$

(Mikawa (2001)) For $n = p_1p_2p_3p_4p_5$ with $p_5 < p_4 < p_3 < p_2 < p_1$, we have

$$0 \leqslant \sum_{\substack{d|n \\ \mu^2(d)=1 \\ \Omega(d)=3 \\ \sqrt{n}<d<\sqrt{nP^-(d)}}} 2 - \sum_{\substack{d|n \\ d<\sqrt{n}}} \mu(d) \leqslant \begin{cases} 2, & p_2p_3 < p_1p_5, \\ 0, & \text{otherwise.} \end{cases}$$

Write $y = x(\log x)^{-4}$ and put

$$\mathcal{A}' = \{n : x - y < n \leqslant x,\ n \equiv a(\mathrm{mod} q)\} \quad \text{and} \quad \mathcal{B}' = \{n : x - y < n \leqslant x\}.$$

New asymptotic formulas of the form (C^*):
(Mikawa (2001)) Let $\frac{1}{2} \leqslant \theta \leqslant \frac{17}{32} - \varepsilon$. Then, for almost all $q \sim Q$, we have

$$\sum_{n\in\mathcal{A}'} \left(\sum_{\substack{d|n \\ d<\sqrt{x}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) = \frac{1}{\varphi(q)} \sum_{\substack{n\in\mathcal{B}' \\ (n,q)=1}} \left(\sum_{\substack{d|n \\ d<\sqrt{x}}} \mu(d) \right) \psi\left(n, x^{\kappa}\right) + O\left(\frac{y}{q(\log x)^3} \right)$$

and

$$\sum_{n\in\mathcal{A}'} \left(\sum_{\substack{d|n \\ \mu^2(d)=1 \\ \Omega(d)=3 \\ \sqrt{x}<d<\sqrt{xP^-(d)}}} 2 \right) \psi\left(n, x^{\kappa}\right) = \frac{1}{\varphi(q)} \sum_{\substack{n\in\mathcal{B}' \\ (n,q)=1}} \left(\sum_{\substack{d|n \\ \mu^2(d)=1 \\ \Omega(d)=3 \\ \sqrt{x}<d<\sqrt{xP^-(d)}}} 2 \right) \psi\left(n, x^{\kappa}\right) + O\left(\frac{y}{q(\log x)^3} \right).$$

After decompositions, we obtain

$$\sum_{p\in\mathcal{A}'} 1 \geqslant \frac{1}{\varphi(q)}\sum_{p\in\mathcal{B}'} 1 + \left(\sum_{\substack{p_1p_2p_3\in\mathcal{A}' \\ x^\kappa<p_3<p_2<p_1<p_2p_3}} 2 - \frac{1}{\varphi(q)} \sum_{\substack{p_1p_2p_3\in\mathcal{B}' \\ x^\kappa<p_3<p_2<p_1<p_2p_3}} 2 \right)$$

$$+ \left(\sum_{\substack{p_1\cdots p_7\in\mathcal{A}' \\ x^\kappa<p_7<\cdots<p_1}} 20 - \frac{1}{\varphi(q)} \sum_{\substack{p_1\cdots p_7\in\mathcal{B}' \\ x^\kappa<p_7<\cdots<p_1}} 20 \right)$$

$$- \frac{1}{\varphi(q)} \left(\sum_{\substack{p_1p_2p_3p_4\in\mathcal{B}' \\ x^\kappa<p_4<p_3<p_2<p_1 \\ p_1<p_2p_3p_4 \\ p_2p_3<p_1}} 2 + \sum_{\substack{p_1\cdots p_5\in\mathcal{B}' \\ x^\kappa<p_5<\cdots<p_1 \\ p_2p_3<p_1p_5}} 2 + \sum_{\substack{p_1\cdots p_6\in\mathcal{B}' \\ x^\kappa<p_6<\cdots<p_1}} 20 \right) + O\left(\frac{y}{q(\log x)^3}\right)$$

for almost all $q \sim Q$.

We can estimate the right-hand side by the asymptotic formulas given earlier (with small modifications) and Prime Number Theorem in short intervals. Summing over $\asymp (\log x)^5$ subintervals, we can obtain a lower bound for $C_0^*(\theta)$.

θ	$C_0^*(\theta)$ (Method 1)	$C_0^*(\theta)$ (Method 2)	θ	$C_0^*(\theta)$ (Method 1)	$C_0^*(\theta)$ (Method 2)
0.501	**0.8636**	0.8025	0.517	0.4455	**0.6033**
0.502	**0.8455**	0.8003	0.518	0.4086	**0.5851**
0.503	**0.8298**	0.7981	0.519	0.3744	**0.5689**
0.504	**0.8135**	0.7859	0.520	0.3363	**0.5487**
0.505	**0.7972**	0.7699	0.521	0.2973	**0.5323**
0.506	0.7299	**0.7579**	0.522	0.2546	**0.5139**
0.507	0.7082	**0.7439**	0.523	0.2148	**0.4919**
0.508	0.6894	**0.7339**	0.524	0.1706	**0.4699**
0.509	0.6606	**0.7199**	0.525	0.1218	**0.4499**
0.510	0.6369	**0.7079**	0.526	0.0778	**0.4299**
0.511	0.6128	**0.6919**	0.527	0.0267	**0.4079**
0.512	0.5907	**0.6838**	0.528	0	**0.3839**
0.513	0.5567	**0.6658**	0.529	0	**0.3579**
0.514	0.5332	**0.6517**	0.530	0	**0.3339**
0.515	0.5040	**0.6355**	0.531	0	**0.3099**
0.516	0.4762	**0.6215**			

Lower Bounds for $C_0^*(\theta)$ ($\frac{1}{2} < \theta \leqslant \frac{17}{32} - \varepsilon$)

Now we focus on the first 2-factored case, where the moduli $q = q_1 q_2$ with $q_1 \sim Q_1 = x^{\theta_1}$ and $q_2 \sim Q_2 = x^{\theta_2}$. We want to show that

$$\sum_{\substack{q_1 \leqslant Q_1 \\ q_2 \leqslant Q_2 \\ (q_1 q_2, a)=1}} \left| \sum_{\substack{n \leqslant x \\ n \equiv a (\bmod q_1 q_2)}} \rho_j(n) - \frac{1}{\varphi(q_1 q_2)} \sum_{\substack{n \leqslant x \\ (n, q_1 q_2)=1}} \rho_j(n) \right| \ll \frac{x}{(\log x)^A}$$

for some functions $\rho_j(n)$ $(j \in \{0, 1\})$ satisfy

$$\rho_0(n) \leqslant 1_p(n) \leqslant \rho_1(n).$$

We want the minorants are always positive and the majorants are not too large. That is, we have

$$\sum_{n\leqslant x}\rho_0(n)\geqslant(1+o(1))\frac{C_0(\theta_1,\theta_2)x}{\log x}\quad\text{and}\quad\sum_{n\leqslant x}\rho_1(n)\leqslant(1+o(1))\frac{C_1(\theta_1,\theta_2)x}{\log x}$$

for two functions $C_0(\theta_1,\theta_2)$ and $C_1(\theta_1,\theta_2)$ satisfy $0<C_0(\theta_1,\theta_2)\leqslant 1$ and $C_1(\theta_1,\theta_2)\geqslant 1$. Moreover, we can try to show that for almost all $q\sim Q$ such that $q_1\mid q$ and $q_1\sim Q_1$,

$$\frac{C_0^*(\theta_1,\theta_2)x}{\varphi(q)\log x}\leqslant\pi(x;q,a)\leqslant\frac{C_1^*(\theta_1,\theta_2)x}{\varphi(q)\log x}$$

for two functions $C_0^*(\theta_1,\theta_2)$ and $C_1^*(\theta_1,\theta_2)$ satisfy $0<C_0(\theta_1,\theta_2)\leqslant C_0^*(\theta_1,\theta_2)\leqslant 1$ and $1\leqslant C_1^*(\theta_1,\theta_2)\leqslant C_1(\theta_1,\theta_2)$.

By the definitions of the sieved sets $\mathcal{A}^q$, $\mathcal{B}^q$ and the sieve function $S(\mathcal{C}, z)$, using Prime Number Theorem, we have

$$\pi(2x; q_1q_2, a) - \pi(x; q_1q_2, a) = \sum_{p \in \mathcal{A}^{q_1q_2}} 1 = S\left(\mathcal{A}^{q_1q_2}, (2x)^{\frac{1}{2}}\right)$$

and

$$S\left(\mathcal{B}^{q_1q_2}, (2x)^{\frac{1}{2}}\right) = (1 + o(1))\frac{x}{\log x}.$$

Our aim is again to show that the sparser set $\mathcal{A}^{q_1q_2}$ contains the expected proportion of primes compared to the larger set $\mathcal{B}^{q_1q_2}$, which requires us to decompose $S\left(\mathcal{A}^{q_1q_2}, (2x)^{\frac{1}{2}}\right)$.

In order to decompose $S\left(\mathcal{A}^{q_1q_2},(2x)^{\frac{1}{2}}\right)$ and bound $C_0(\theta_1,\theta_2)$, $C_1(\theta_1,\theta_2)$, $C_0^*(\theta_1,\theta_2)$ and $C_1^*(\theta_1,\theta_2)$, we need the following two types of asymptotic formulas.
Type-C_2:

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\left|S\left(\mathcal{A}^{q_1q_2},z\right)-\frac{1}{\varphi(q_1q_2)}S\left(\mathcal{B}^{q_1q_2},z\right)\right|\ll\frac{x}{(\log x)^A}. \tag{C_2}$$

Type-C_2^*: For almost all $q\sim Q$ such that $q_1\mid q$ and $q_1\sim Q_1$, we have

$$S\left(\mathcal{A}^{q_1q_2},z\right)=(1+o(1))\frac{1}{\varphi(q_1q_2)}S\left(\mathcal{B}^{q_1q_2},z\right). \tag{C_2^*}$$

Note that an asymptotic formula of the form (C_2) implies an asymptotic formula of the form (C_2^*). The asymptotic formulas of the forms (C) and (C^*) also imply asymptotic formulas of the forms (C_2) and (C_2^*) respectively.

Type-II information: Asymptotic formula of the form (C_2) for Type-II sums. Let $M_1M_2 \asymp x$. Suppose that a_{2,m_2} satisfies **Conditions A and B**.

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}}\left|\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_1m_2\equiv a \pmod{q_1q_2}}} a_{1,m_1}a_{2,m_2} - \frac{1}{\varphi(q_1q_2)}\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ (m_1m_2,q_1q_2)=1}} a_{1,m_1}a_{2,m_2}\right| \ll \frac{x}{(\log x)^A}.$$

(Maynard (2025a))

$$Q_2x^{\varepsilon} < M_2,\quad Q_1^2Q_2^3M_2^6 < x^{2-15\varepsilon},\quad Q_1^3Q_2^3M_2^3 < x^{2-15\varepsilon},\quad Q_1^4Q_2^3M_2^3 < x^{\frac{5}{2}-15\varepsilon},\quad Q_1^2Q_2^2 < M_2x^{1-4\varepsilon};$$

$$Q_1Q_2^2 < M_2x^{1-7\varepsilon},\quad Q_1^8Q_2^7M_2^6 < x^{4-13\varepsilon};$$

$$Q_1^7Q_2^{12} < x^{4-19\varepsilon},\quad Q_1x^{\varepsilon} < M_2 < Q_1^{-1}x^{1-6\varepsilon}.$$

Arithmetic information

Type-I$_3$ information: Asymptotic formula of the form (C_2) for Type-I$_3$ sums. Let $M_1M_2 \asymp x$ and $z = \exp\left(\log x(\log\log x)^{-3}\right)$. Let $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ be intervals such that $\mathbf{M}_i \subseteq [M_i, 2M_i]$.

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}} \left| \sum_{\substack{m_0\sim M_0\\ m_1\in\mathbf{M}_1\\ m_2\in\mathbf{M}_2\\ m_3\in\mathbf{M}_3\\ (m_1m_2m_3,P(z))=1\\ m_0m_1m_2m_3\equiv a(\bmod q_1q_2)}} a_{m_0} - \frac{1}{\varphi(q_1q_2)} \sum_{\substack{m_0\sim M_0\\ m_1\in\mathbf{M}_1\\ m_2\in\mathbf{M}_2\\ m_3\in\mathbf{M}_3\\ (m_1m_2m_3,P(z))=1\\ (m_0m_1m_2m_3,q_1q_2)=1}} a_{m_0} \right| \ll \frac{x}{(\log x)^A}.$$

(Maynard (2025a))

$$x^\varepsilon \leqslant M_3 \leqslant M_2 \leqslant M_1 \leqslant x^{\frac{3}{7}+\varepsilon}, \quad x^\varepsilon \leqslant M_0, \quad Q_1^7Q_2^9 < x^4, \quad Q_1^9Q_2 < x^{\frac{32}{7}}, \quad Q_1^{\frac{15}{8}}Q_2^{\frac{15}{8}}M_0 < x^{1-20\varepsilon}.$$

(Lichtman (2022))

$$x^\varepsilon \leqslant M_3 \leqslant M_2 \leqslant M_1 \leqslant x^{\frac{3}{7}+\varepsilon}, \quad M_0 = x^\varepsilon, \quad Q_1^3Q_2^2 < x^{\frac{11}{7}-30\varepsilon}, \quad Q_1^{11}Q_2^{12} < x^{6-30\varepsilon}, \quad Q_1Q_2 < x^{\frac{8}{15}-30\varepsilon}.$$

Write $\theta = \theta_1 + \theta_2$ and assume that $\theta_1 \geqslant \theta_2$. Define

$$\begin{aligned}
\boldsymbol{U} &= \{(\theta_1, \theta_2) : 0 \leqslant \theta_2 \leqslant \theta_1 < 1,\ \theta_1 + \theta_2 < 1\}, \\
\boldsymbol{I} &= \Big\{(\theta_1, \theta_2) : (\theta_1, \theta_2) \in \boldsymbol{U};\ \theta_1 + \theta_2 \leqslant \frac{1}{2} - \varepsilon \\
&\qquad \text{or } 2\theta_1 + \theta_2 \leqslant 1 - \varepsilon,\ 7\theta_1 + 12\theta_2 \leqslant 4 - \varepsilon,\ 19\theta_1 + 20\theta_2 \leqslant 10 - \varepsilon\}, \\
\boldsymbol{T}_1 &= \left\{(\theta_1, \theta_2) : (\theta_1, \theta_2) \in \boldsymbol{U};\ 7\theta_1 + 9\theta_2 < 4,\ 9\theta_1 + \theta_2 < \frac{32}{7}\right\}, \\
\boldsymbol{T}_2 &= \left\{(\theta_1, \theta_2) : (\theta_1, \theta_2) \in \boldsymbol{U} \backslash \boldsymbol{T}_1;\ 3\theta_1 + 2\theta_2 < \frac{11}{7} - \varepsilon,\ 11\theta_1 + 12\theta_2 < 6 - \varepsilon,\ \theta_1 + \theta_2 < \frac{8}{15} - \varepsilon\right\}, \\
\boldsymbol{T} &= \{(\theta_1, \theta_2) : (\theta_1, \theta_2) \in \boldsymbol{U};\ (\theta_1, \theta_2) \in \boldsymbol{T}_1 \cup \boldsymbol{T}_2\},
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{A} = \Bigg\{(\theta_1,\theta_2) : (\theta_1,\theta_2) \in \boldsymbol{U}\backslash\boldsymbol{I};\ & \frac{5}{14} < \theta_1 \leqslant \frac{2}{5},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{9}(2-2\theta_1) \\
& \text{or } \frac{2}{5} < \theta_1 \leqslant \frac{4}{9},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{6}(2-3\theta_1) \\
& \text{or } \frac{4}{9} < \theta_1 \leqslant \frac{1}{2},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{9}(5-9\theta_1) \\
& \text{or } \frac{1}{2} < \theta_1 < \frac{11}{20},\ 0 < \theta_2 < \frac{1}{18}(11-20\theta_1)\Bigg\}, \\
\boldsymbol{B} = \Bigg\{(\theta_1,\theta_2) : (\theta_1,\theta_2) \in \boldsymbol{U}\backslash\boldsymbol{I};\ & \frac{1}{4} < \theta_1 \leqslant \frac{10}{33},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \theta_1 \\
& \text{or } \frac{10}{33} < \theta_1 \leqslant \frac{1}{2},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{19}(10-14\theta_1) \\
& \text{or } \frac{1}{2} < \theta_1 < \frac{5}{7},\ 0 < \theta_2 < \frac{1}{19}(10-14\theta_1)\Bigg\}, \\
\boldsymbol{C} = \Bigg\{(\theta_1,\theta_2) : (\theta_1,\theta_2) \in \boldsymbol{U}\backslash\boldsymbol{I};\ & \frac{2}{5} < \theta_1 < \frac{1}{2},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{12}(4-7\theta_1)\Bigg\}.
\end{aligned}
$$

Let $j \geqslant 4$, $P_1 P_2 \cdots P_j \asymp x$ and $P_1 \geqslant P_2 \geqslant \cdots \geqslant P_j \geqslant x^{\frac{1}{7}+10\varepsilon}$. Suppose that

$$2\theta_1 + \theta_2 < 1 - 100\varepsilon \quad \text{and} \quad 7\theta_1 + 12\theta_2 < 4 - 100\varepsilon.$$

Then

$$\sum_{\substack{p_1, \ldots, p_j \\ p_i \sim P_i,\ 1 \leqslant i \leqslant j \\ p_1 \cdots p_j \equiv a (\bmod q_1 q_2)}} 1$$

has an asymptotic formula of the form (C_2).

A new three-dimensional Harman's sieve. Put

$$\mathcal{F}^q = \left\{ m_1 m_2 m_3 : x^{\varepsilon} \leqslant m_3 \leqslant m_2 \leqslant m_1 \leqslant x^{\frac{3}{7}+\varepsilon},\ m_1 m_2 m_3 \sim x,\ m_1 m_2 m_3 \equiv a(\bmod q) \right\}.$$

We can give an asymptotic formula of the form (C_2) for

$$S\left(\mathcal{F}^{q_1 q_2}, x^{\kappa}\right)$$

with $\kappa = \frac{5-8\theta}{6} - \varepsilon$ or $\frac{2-3\theta}{3} - \varepsilon$ or some other values using the Type-II information and the Type-I_3 information above, under several conditions of θ_1, θ_2 and θ.

When the available Type-II range starts from $[2\theta - 1 + \varepsilon,\ \cdots)$, we need $2\theta - 1 + \varepsilon < 1 - \frac{15}{8}\theta - 20\varepsilon$, or $\theta < \frac{16}{31} - 10\varepsilon$. Since $3(2\theta - 1 + \varepsilon) < \kappa$ when $\theta < \frac{16}{31} - 10\varepsilon$, we can give an asymptotic formula of the form (C_2) for $S\left(\mathcal{F}^{q_1 q_2}, x^{\kappa}\right)$ if $(\theta_1, \theta_2) \in \boldsymbol{T}_1$. When the available Type-II range starts from $[2\theta_1 + \theta_2 - 1 + \varepsilon,\ \cdots)$, we need the condition $44\theta_1 + 26\theta_2 < 23$ and the details are similar. In these two cases only Maynard's Type-I_3 information is available.

When the available Type-II range starts from $[\varepsilon,\ \cdots)$, we can use both Maynard's and Lichtman's Type-I_3 information since we can now take $M_0 = x^{\varepsilon}$. For all $\theta < \frac{17}{32}$, the Type-II range $[\varepsilon,\ \kappa]$ is sufficient to give an asymptotic formula of the form (C_2) for $S\left(\mathcal{F}^{q_1 q_2}, x^{\kappa}\right)$ if $(\theta_1, \theta_2) \in \boldsymbol{T}$.

We give upper bounds for $C_1(\theta_1,\theta_2)$ and $C_1^*(\theta_1,\theta_2)$.
Known results:
(1). $C_1(\theta_1,\theta_2)=C_1(\theta_2,\theta_1)$, $C_1^*(\theta_1,\theta_2)=C_1^*(\theta_2,\theta_1)$;
(2). $C_1(\theta_1,\theta_2)=C_1^*(\theta_1,\theta_2)=1$ for all θ_1,θ_2 satisfy $\theta_1+\theta_2\leqslant 0.5-\varepsilon$;
(3). $C_1(\theta_1,\theta_2)=C_1^*(\theta_1,\theta_2)=1$ for all θ_1,θ_2 satisfy $2\theta_1+\theta_2\leqslant 1-\varepsilon$, $7\theta_1+12\theta_2\leqslant 4-\varepsilon$ and $19\theta_1+20\theta_2\leqslant 10-\varepsilon$;
(4). $C_1(\theta_1,\theta_2)\leqslant C_1(\theta_1+\theta_2)$, $C_1^*(\theta_1,\theta_2)\leqslant C_1^*(\theta_1+\theta_2)$ for $0.5\leqslant\theta_1+\theta_2\leqslant 1$;
(5). $C_1^*(\theta_1,\theta_2)\leqslant C_1(\theta_1,\theta_2)\leqslant 1+\varepsilon$ for all θ_1,θ_2 satisfy $\theta_1+\theta_2=0.5$;
(6). $C_1^*(\theta_1,\theta_2)\leqslant C_1(\theta_1,\theta_2)\leqslant 1+\varepsilon$ for all θ_1,θ_2 satisfy $\theta_1\leqslant 0.5$ and $\theta_2=\min\left(1-2\theta_1,\frac{4-7\theta_1}{12},\frac{10-19\theta_1}{20}\right)$.
We divide the region $\boldsymbol{U}$ into subregions based on the available Type-II ranges in each subregion. The decompositions are similar to which in Final decompositions 1-1 (with more available arithmetic information), and we can use the new three-dimensional Harman's sieve in Case $C_1(\theta_1,\theta_2)$. In Final decompositions 1-1, we can only use the three-dimensional Harman's sieve in Case $C_1^*(\theta)$.

For those subregions with a Type-II range

$$[2\theta_1 + \theta_2 - 1 + \varepsilon,\ \cdots) \quad \text{or} \quad [\varepsilon,\ \cdots)\,,$$

sometimes we can replace the condition $(\alpha_1, \ldots, \alpha_j, 2\theta - 1 + \varepsilon) \in \boldsymbol{S}_{j+1}$ in the definition of $\boldsymbol{U}_j$ by $(\alpha_1, \ldots, \alpha_j, 2\theta_1 + \theta_2 - 1 + \varepsilon) \in \boldsymbol{S}_{j+1}$ or $(\alpha_1, \ldots, \alpha_j, \varepsilon) \in \boldsymbol{S}_{j+1}$.

A sketch proof of the main theorem in Maynard (2025a).
If we want to use the asymptotic formula for numbers with $\geqslant 4$ prime factors, then we need to have

$$2\theta_1 + \theta_2 \leqslant 1 - 100\varepsilon \quad \text{and} \quad 7\theta_1 + 12\theta_2 \leqslant 4 - 100\varepsilon.$$

When $\theta \geqslant \frac{1}{2}$, $7\theta_1 + 12\theta_2 \leqslant 4 - 100\varepsilon$ yields $(\theta_1, \theta_2) \in \boldsymbol{C}$, and we can use the Type-II range $(\theta_1 + \varepsilon,\ 1 - \theta_1 - \varepsilon)$. Note that $2\theta_1 + \theta_2 \leqslant 1 - 100\varepsilon$ means that $\theta_1 + \varepsilon \leqslant 1 - \theta - \varepsilon$, the new Type-II range $(\theta_1 + \varepsilon,\ 1 - \theta_1 - \varepsilon)$ covers the whole interval $\left(1 - \theta - \varepsilon,\ \frac{1}{2}\right)$. If we want to use the new three-dimensional Harman's sieve with $\theta \geqslant \frac{16}{31}$, then our Type-II range cannot be $[2\theta - 1 + \varepsilon, \cdots)$, which means that we need our Type-II range starts from ε. The requirement of that is

$$\frac{1}{6}(4 - 7\theta_1 - 8\theta_2) - \varepsilon \geqslant 2\theta - 1 + \varepsilon, \quad \text{or} \quad 19\theta_1 + 20\theta_2 \leqslant 10 - 100\varepsilon.$$

Now $\theta \leqslant \frac{11}{21} - 11\varepsilon$ is satisfied automatically, and we have $\frac{2-3\theta}{3} - \varepsilon \geqslant \frac{1}{7} + 10\varepsilon$. The available Type-II range is

$$\left[\varepsilon,\ \frac{2 - 3\theta}{3} - \varepsilon\right].$$

Final decompositions, 2-1

Write $\kappa = \frac{2-3\theta}{3} - \varepsilon$ here. Using Buchstab's identity, we have

$$
\begin{aligned}
S\left(\mathcal{A}^{q_1 q_2}, (2x)^{\frac{1}{2}}\right) = S\left(\mathcal{A}^{q_1 q_2}, x^{\kappa}\right) &- \sum_{\frac{3}{7}+\varepsilon \leqslant \alpha_1 \leqslant 1-\theta-\varepsilon} S\left(\mathcal{A}^{q_1 q_2}_{p_1}, p_1\right) - \sum_{1-\theta-\varepsilon < \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^{q_1 q_2}_{p_1}, p_1\right) \\
&- \sum_{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon} S\left(\mathcal{A}^{q_1 q_2}_{p_1}, x^{\kappa}\right) + \sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in \boldsymbol{G}_2}} S\left(\mathcal{A}^{q_1 q_2}_{p_1 p_2}, p_2\right) \\
&+ \sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B}} S\left(\mathcal{A}^{q_1 q_2}_{p_1 p_2}, x^{\kappa}\right) + \sum_{\substack{m_1 m_2 m_3 \in \mathcal{A}^{q_1 q_2} \\ \kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in C \\ (m_1 m_2 m_3, P(x^{\kappa}))=1}} 1 \\
&- \sum_{\substack{\kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in A \cup B \cup C \\ \kappa \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1}{2}(1-\alpha_1-\alpha_2)\right)}} S\left(\mathcal{A}^{q_1 q_2}_{p_1 p_2 p_3}, p_3\right) - \sum_{\substack{m_1 m_2 m_3 \in \mathcal{A}^{q_1 q_2} \\ \kappa \leqslant \alpha_1 < \frac{3}{7}+\varepsilon \\ \kappa \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2 \in C \\ (m_1 m_2 m_3, P(x^{\kappa}))=1 \\ \Omega(m_1 m_2) \geqslant 3}} 1.
\end{aligned}
$$

We can give asymptotic formulas of the form (C_2) for all terms above.

$\theta_2\backslash\theta_1$	0.26	0.27	0.28	0.29	0.30	0.31	0.32	0.33	0.34	0.35	0.36	0.37	0.38	0.39
0.25	1.0486	1.7394	1.8338	1.9993	2.3996	—	—	—	—	—	—	—	—	—
0.24	$1+\varepsilon$	1.0486	1.7394	1.8338	1.9993	2.3996	—	—	—	—	—	—	—	—
0.23	1	$1+\varepsilon$	1.0486	1.7394	1.8338	1.9993	2.3996	—	—	—	—	—	—	—
0.22	1	1	$1+\varepsilon$	1.0486	1.7394	1.8338	1.9993	2.3996	—	—	—	—	—	—
0.21	1	1	1	$1+\varepsilon$	1.0486	1.7394	1.8338	1.9993	2.3996	—	—	—	—	—
0.20	1	1	1	1	$1+\varepsilon$	1.0486	1.7394	1.8338	1.9993	2.3996	—	—	—	—
0.19	1	1	1	1	1	$1+\varepsilon$	1.0486	1.7394	1.8338	1.9993	2.3996	—	—	—
0.18	1	1	1	1	1	1	$1+\varepsilon$	1.0486	1.7394	1.8338	1.9993	2.3996	—	—
0.17	1	1	1	1	1	1	1	$1+\varepsilon$	1.0486	1.7394	1.8338	1.9993	2.3996	—
0.16	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0486	1.7394	1.8338	1.9993	2.3996
0.15	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0486	1.7394	1.8338	1.9993
0.14	1	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0486	1.7394	1.8338
0.13	1	1	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0486	1.7394
0.12	1	1	1	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0486

$\theta_2\backslash\theta_1$	0.40	0.41	0.42	0.43	0.44	0.45	0.46	0.47	0.48	0.49	0.50	0.51	0.52	0.53	0.54
0.15	2.3996	—	—	—	—	—	—	—	—	—	—	—	—	—	—
0.14	1.9993	2.3996	—	—	—	—	—	—	—	—	—	—	—	—	—
0.13	1.8250	1.9993	2.3996	—	—	—	—	—	—	—	—	—	—	—	—
0.12	1.7394	1.8182	1.9851	2.3996	—	—	—	—	—	—	—	—	—	—	—
0.11	1.0486	1.1067	1.8182	1.9486	2.3852	—	—	—	—	—	—	—	—	—	—
0.10	$1+\varepsilon$	1.0486	1.1111	1.8182	1.9486	2.3511	—	—	—	—	—	—	—	—	—
0.09	1	1	1.0486	1.1111	1.8182	1.9486	2.3511	—	—	—	—	—	—	—	—
0.08	1	1	1	1	1.1099	1.8182	1.9486	2.3511	—	—	—	—	—	—	—
0.07	1	1	1	1	1	1	1.8182	1.9486	2.3511	—	—	—	—	—	—
0.06	1	1	1	1	1	1	1	1.8182	1.9486	2.3511	—	—	—	—	—
0.05	1	1	1	1	1	1	1	1	1.7215	1.9486	2.3511	—	—	—	—
0.04	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.7676	1.9486	2.3549	—	—	—
0.03	1	1	1	1	1	1	1	1	1	1.0503	1.8182	1.9615	2.3648	—	—
0.02	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.1099	1.8212	1.9765	2.3928	—
0.01	1	1	1	1	1	1	1	1	1	1	1.0482	1.7415	1.8271	1.9893	2.3935

Upper Bounds for $C_1(\theta_1, \theta_2)$ $(0.5 < \theta \leqslant 0.55)$

$\theta_2\backslash\theta_1$	0.26	0.27	0.28	0.29	0.30	0.31	0.32	0.33	0.34	0.35	0.36	0.37	0.38	0.39
0.25	1.0150	1.7028	1.8169	1.9993	2.3996	—	—	—	—	—	—	—	—	—
0.24	$1+\varepsilon$	1.0150	1.7028	1.8169	1.9993	2.3996	—	—	—	—	—	—	—	—
0.23	1	$1+\varepsilon$	1.0150	1.7028	1.8169	1.9993	2.3996	—	—	—	—	—	—	—
0.22	1	1	$1+\varepsilon$	1.0150	1.7028	1.8169	1.9993	2.3996	—	—	—	—	—	—
0.21	1	1	1	$1+\varepsilon$	1.0150	1.7028	1.8169	1.9993	2.3996	—	—	—	—	—
0.20	1	1	1	1	$1+\varepsilon$	1.0150	1.7028	1.8169	1.9993	2.3996	—	—	—	—
0.19	1	1	1	1	1	$1+\varepsilon$	1.0150	1.7028	1.8169	1.9993	2.3996	—	—	—
0.18	1	1	1	1	1	1	$1+\varepsilon$	1.0150	1.7028	1.8169	1.9993	2.3996	—	—
0.17	1	1	1	1	1	1	1	$1+\varepsilon$	1.0150	1.7028	1.8169	1.9993	2.3996	—
0.16	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0150	1.7028	1.8169	1.9993	2.3996
0.15	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0150	1.7028	1.8169	1.9993
0.14	1	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0150	1.7028	1.8169
0.13	1	1	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0150	1.7028
0.12	1	1	1	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0150

$\theta_2\backslash\theta_1$	0.40	0.41	0.42	0.43	0.44	0.45	0.46	0.47	0.48	0.49	0.50	0.51	0.52	0.53	0.54
0.15	2.3996	—	—	—	—	—	—	—	—	—	—	—	—	—	—
0.14	1.9993	2.3996	—	—	—	—	—	—	—	—	—	—	—	—	—
0.13	1.8169	1.9993	2.3996	—	—	—	—	—	—	—	—	—	—	—	—
0.12	1.7028	1.8120	1.9851	2.3996	—	—	—	—	—	—	—	—	—	—	—
0.11	1.0150	1.0678	1.8120	1.9486	2.3852	—	—	—	—	—	—	—	—	—	—
0.10	$1+\varepsilon$	1.0150	1.0687	1.8120	1.9486	2.3511	—	—	—	—	—	—	—	—	—
0.09	1	1	1.0150	1.0670	1.8120	1.9486	2.3511	—	—	—	—	—	—	—	—
0.08	1	1	1	1	1.0636	1.8120	1.9486	2.3511	—	—	—	—	—	—	—
0.07	1	1	1	1	1	1	1.8120	1.9486	2.3511	—	—	—	—	—	—
0.06	1	1	1	1	1	1	1	1.8120	1.9486	2.3511	—	—	—	—	—
0.05	1	1	1	1	1	1	1	1	1.6992	1.9486	2.3511	—	—	—	—
0.04	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.7406	1.9486	2.3549	—	—	—
0.03	1	1	1	1	1	1	1	1	1	1.0187	1.8120	1.9615	2.3648	—	—
0.02	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0636	1.8169	1.9765	2.3928	—
0.01	1	1	1	1	1	1	1	1	1	1	1.0150	1.6991	1.8169	1.9893	2.3935

Upper Bounds for $C_1^*(\theta_1, \theta_2)$ $(0.5 < \theta \leqslant 0.55)$

We give lower bounds for $C_0(\theta_1, \theta_2)$ and $C_0^*(\theta_1, \theta_2)$.
Known results:
(1). $C_0(\theta_1, \theta_2) = C_0(\theta_2, \theta_1)$, $C_0^*(\theta_1, \theta_2) = C_0^*(\theta_2, \theta_1)$;
(2). $C_0(\theta_1, \theta_2) = C_0^*(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $\theta_1 + \theta_2 \leqslant 0.5 - \varepsilon$;
(3). $C_0(\theta_1, \theta_2) = C_0^*(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $2\theta_1 + \theta_2 \leqslant 1 - \varepsilon$, $7\theta_1 + 12\theta_2 \leqslant 4 - \varepsilon$ and $19\theta_1 + 20\theta_2 \leqslant 10 - \varepsilon$;
(4). $C_0(\theta_1, \theta_2) \geqslant C_0(\theta_1 + \theta_2)$, $C_0^*(\theta_1, \theta_2) \geqslant C_0^*(\theta_1 + \theta_2)$ for $0.5 \leqslant \theta_1 + \theta_2 \leqslant 1$;
(5). $C_0^*(\theta_1, \theta_2) \geqslant 1 - \varepsilon$ for all θ_1, θ_2 satisfy $\theta_1 + \theta_2 = 0.5$.
Case $C_0(\theta_1, \theta_2)$: by the earlier discussions, we know that we need to give an asymptotic formula of the form (C_2) for the sum

$$\sum_{1-\theta-\varepsilon<\alpha_1<\frac{1}{2}} S\left(\mathcal{A}_{p_1}^{q_1 q_2}, p_1\right).$$

By the earlier discussions, we need

$$2\theta_1 + \theta_2 \leqslant 1 - \varepsilon \quad \text{and} \quad 7\theta_1 + 12\theta_2 \leqslant 4 - \varepsilon.$$

If $19\theta_1 + 20\theta_2 \leqslant 10 - \varepsilon$, then we have $C_0(\theta_1, \theta_2) = 1$ by the main theorem in Maynard (2025a).

Case $C_0^*(\theta_1, \theta_2)$: We still use two different methods to give lower bounds for $C_0^*(\theta_1, \theta_2)$.
First Method: Harman's sieve with Buchstab iterations.
The decompositions are similar to which in Final decompositions 1-2 (with more available arithmetic information).
Second Method: Mikawa's modified sieve.
The decompositions are similar to which in Final decompositions 1-3 (with more available arithmetic information).
When applying Mikawa's modified sieve, we need the Type-II range $[r_0,\ r_1]$ satisfies $r_1 > 2r_0$. In the case $q \sim Q$ in Final decompositions 1-3, the Type-II range is fixed on $\left[2\theta - 1 + \varepsilon,\ \frac{5-8\theta}{6} - \varepsilon\right]$ (when $\theta < \frac{45}{89}$, the Type-II range $\left((\log x)^{\varepsilon - 1},\ 2\theta - 1 + \varepsilon\right)$ would not bring useful improvements here), and that is why this method is not applicable when $\theta \geqslant \frac{17}{32}$.
In the first 2-factored moduli case, we can enlarge r_1 to values like $\frac{2-3\theta}{3} - \varepsilon$ and $\frac{5-8\theta_1-6\theta_2}{6} - \varepsilon$ for some special (θ_1, θ_2). For example, when κ is replaced by $\frac{2-3\theta}{3} - \varepsilon$, then we only need

$$\frac{2-3\theta}{3} - \varepsilon > 2(2\theta - 1 + \varepsilon), \quad \text{or} \quad \theta < \frac{8}{15} - 9\varepsilon$$

for the Type-II requirements. Can we extend the "Mikawa applicable range" of θ to $\geqslant \frac{17}{32}$?

NO

The real problem is not on the Type-II requirements but on the Type-I requirements.
When applying Mikawa's modified sieve, we also need the following conditions to apply the Type-I/II information:

$$\frac{1}{2} - (2\theta - 1) < 1 - \theta, \quad \frac{1}{2} + 15(2\theta - 1) < 2 - \theta \quad \text{and} \quad \frac{1}{2} + 7(2\theta - 1) < 2 - 2\theta.$$

The last condition above is also equivalent to $\theta < \frac{17}{32}$.
Another idea is to use the Type-II_3 information to cover the region that the Type-I/II information cannot cover; Type-II_3 information actually covers the remaining region for $\theta < \frac{17}{30}$. However, the coefficients in the corresponding Type-I sum are convolutions involving the Ψ function, which means that they do not satisfy **Condition B** (No small prime factors), and the Type-II_3 information is not applicable here.

0.13	−0.12 **0.3477**	—	—	—	—	—	—	—	—	—	—	—	—
0.12	0.3363 **0.5487**	0.0055 **0.3625**	—	—	—	—	—	—	—	—	—	—	—
0.11	0.6369 **0.7079**	0.3506 **0.5519**	0.0055 **0.3625**	—	—	—	—	—	—	—	—	—	—
0.10	$1-\varepsilon$	0.6369 **0.7079**	0.3616 **0.5551**	0.0055 **0.3625**	—	—	—	—	—	—	—	—	—
0.09	1	1	0.6369 **0.7079**	0.3786 **0.5569**	0.0055 **0.3625**	—	—	—	—	—	—	—	—
0.08	1	1	1	1	0.3828 **0.5633**	0.0055 **0.3625**	—	—	—	—	—	—	—
0.07	1	1	1	1	1	1	0.0055 **0.3625**	—	—	—	—	—	—
0.06	1	1	1	1	1	1	1	0.0055 **0.3625**	—	—	—	—	—
0.05	1	1	1	1	1	1	1	1	0.5212 **0.5237**	—	—	—	—
0.04	1	1	1	1	1	1	1	1	**0.7961** 0.7051	0.2650 **0.4419**	—	—	—
0.03	1	1	1	1	1	1	1	1	1	0.5903 **0.6339**	0.0055 **0.3625**	—	—
0.02	1	1	1	1	1	1	1	1	1	**0.8186** 0.7741	0.3828 **0.5634**	−0.04 **0.3509**	—
0.01	1	1	1	1	1	1	1	1	1	1	0.6530 **0.7097**	0.3616 **0.5552**	−0.09 **0.3449**
$\theta_2\backslash\theta_1$	**0.40**	**0.41**	**0.42**	**0.43**	**0.44**	**0.45**	**0.46**	**0.47**	**0.48**	**0.49**	**0.50**	**0.51**	**0.52**

Lower Bounds for $C_0^*(\theta_1,\theta_2)$ $(0.5 < \theta \leqslant 0.53)$

Now we focus on the second 2-factored case (the bilinear case), where the absolute values in the first case are replaced by divisor-bounded coefficients. We want to show that

$$\sum_{\substack{q_1 \leqslant Q_1 \\ q_2 \leqslant Q_2 \\ (q_1 q_2, a)=1}} \lambda_{1,q_1} \lambda_{2,q_2} \left(\sum_{\substack{n \leqslant x \\ n \equiv a \pmod{q_1 q_2}}} \rho_j(n) - \frac{1}{\varphi(q_1 q_2)} \sum_{\substack{n \leqslant x \\ (n, q_1 q_2)=1}} \rho_j(n) \right) \ll \frac{x}{(\log x)^A}$$

for some functions $\rho_j(n)$ $(j \in \{0,1\})$ satisfy

$$\rho_0(n) \leqslant 1_p(n) \leqslant \rho_1(n).$$

We want the minorants are always positive and the majorants are not too large. That is, we have

$$\sum_{n \leqslant x} \rho_0(n) \geqslant (1+o(1)) \frac{C_0'(\theta_1, \theta_2) x}{\log x} \quad \text{and} \quad \sum_{n \leqslant x} \rho_1(n) \leqslant (1+o(1)) \frac{C_1'(\theta_1, \theta_2) x}{\log x}$$

for two functions $C_0'(\theta_1, \theta_2)$ and $C_1'(\theta_1, \theta_2)$ satisfy $0 < C_0'(\theta_1, \theta_2) \leqslant 1$ and $C_1'(\theta_1, \theta_2) \geqslant 1$.

Now we need the following type of asymptotic formulas.

$$\sum_{\substack{q_1 \sim Q_1 \\ q_2 \sim Q_2 \\ (q_1 q_2, a)=1}} \lambda_{1,q_1} \lambda_{2,q_2} \left(\sum_{\substack{n \sim x \\ n \equiv a (\bmod q_1 q_2)}} f(n) - \frac{1}{\varphi(q_1 q_2)} \sum_{\substack{n \sim x \\ (n, q_1 q_2)=1}} f(n) \right) \ll \frac{x}{(\log x)^A}. \qquad (**)$$

Note that asymptotic formulas of the forms (C) and (C_2) also imply asymptotic formulas of the form $(**)$.

Type-II information: Asymptotic formula of the form $(**)$ for Type-II sums. Let $M_1M_2 \asymp x$. Suppose that a_{2,m_2} satisfies **Conditions A and B**.

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}} \lambda_{1,q_1}\lambda_{2,q_2}\left(\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_1m_2\equiv a (\bmod q_1q_2)}} a_{1,m_1}a_{2,m_2}-\frac{1}{\varphi(q_1q_2)}\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ (m_1m_2,q_1q_2)=1}} a_{1,m_1}a_{2,m_2}\right)\ll\frac{x}{(\log x)^A}.$$

(Fouvry (1987))

$$Q_1Q_2^2 \leqslant M_2x^{1-\varepsilon},\ Q_1^8Q_2^7M_2^6 \leqslant x^{4-\varepsilon};$$
$$Q_1Q_2^2 \leqslant x^{1-\varepsilon},\ Q_1^8Q_2^7M_2^5 \leqslant x^{4-\varepsilon};$$
$$Q_1Q_2^2 \leqslant M_2x^{1-\varepsilon},\ Q_1^5Q_2^2M_2^6 \leqslant x^{2-\varepsilon},\ Q_1^4Q_2^3M_2^3 \leqslant x^{2-\varepsilon},\ Q_1^9Q_2^8M_2^6 \leqslant x^{5-\varepsilon};$$
$$Q_1Q_2^2 \leqslant x^{1-\varepsilon},\ Q_1^5Q_2^2M_2^5 \leqslant x^{2-\varepsilon},\ Q_1^8Q_2^6M_2^5 \leqslant x^{4-\varepsilon};$$
$$Q_1Q_2^2M_2 \leqslant x^{1-\varepsilon},\ Q_1^5Q_2^2M_2^4 \leqslant x^{2-\varepsilon},\ Q_1^8Q_2^6M_2^5 \leqslant x^{4-\varepsilon}.$$

(Bombieri et al. (1986))

$$Q_2x^{\varepsilon} \ll M_2 \ll x^{-\varepsilon}\min\left(x^{\frac{1}{2}}Q_1^{-\frac{1}{2}}, x^2Q_1^{-5}Q_2^{-1}, xQ_1^{-2}Q_2^{-\frac{1}{2}}\right);$$
$$Q_2x^{\varepsilon} \ll M_2 \ll x^{-\varepsilon}\min\left(x^{\frac{1}{2}}Q_1^{-1}Q_2^{\frac{1}{2}}, x^{\frac{2}{5}}Q_1^{-\frac{2}{5}}, x^{\frac{1}{2}}Q_1^{-\frac{3}{4}}\right).$$

(Iwaniec and Pomykala (1993))

$$Q_2x^{\varepsilon} < M_2,\ Q_1^2Q_2 < M_2x^{1-\varepsilon},\ Q_1^8Q_2^4M_2^6 < x^{5-\varepsilon}.$$

Arithmetic information

Type-I information: Asymptotic formula of the form $(**)$ for Type-I sums. Let $M_1M_2 \asymp x$ and $z \ll \exp\big(\log x(\log\log x)^{-1}\big)$.

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ (q_1q_2,a)=1}} \lambda_{1,q_1}\lambda_{2,q_2}\left(\sum_{\substack{m_1\in\mathbf{M}\\ m_2\sim M_2\\ m_1m_2\equiv a \pmod{q_1q_2}}} a_{1,m_1}a_{2,m_2} - \frac{1}{\varphi(q_1q_2)}\sum_{\substack{m_1\in\mathbf{M}\\ m_2\sim M_2\\ (m_1m_2,q_1q_2)=1}} a_{1,m_1}a_{2,m_2}\right) \ll \frac{x}{(\log x)^A}.$$

(Bombieri et al. (1986))

$$Q_1M_2 < x^{1-\varepsilon}, \quad Q_1Q_2^4M_2 < x^{2-\varepsilon}, \quad Q_1Q_2^2M_2^2 < x^{2-2\varepsilon}, \quad Q_1^3Q_2^4M_2 < x^{3-\varepsilon},$$

and

$$a_{1,m_1} = 1_{m_1\in\mathbf{M}} \qquad \text{or} \qquad a_{1,m_1} = 1_{\substack{m_1\in\mathbf{M}\\ (m_1,P(z))=1}}$$

for some interval $\mathbf{M} \subseteq [M_1, 2M_1]$.

Write $\theta = \theta_1 + \theta_2$ and assume that $\theta_1 \geqslant \theta_2$. We can enlarge the region $\boldsymbol{S}$ defined before using our new Type-I information. Define

$$\begin{aligned}
\boldsymbol{S} = \; & \{(s,t) : s \leqslant 1-\theta-\varepsilon,\; s+2t \leqslant 2-2\theta-\varepsilon,\; s+4t \leqslant 2-\theta-\varepsilon \\
& \qquad \text{or } s+t < \min\left(1-\theta_1, 2-\theta_1-4\theta_2, 1-\frac{1}{2}\theta_1-\theta_2, 3-3\theta_1-4\theta_2\right)-\varepsilon\Big\},\\
\boldsymbol{J} = \; & \Big\{(\theta_1,\theta_2) : (\theta_1,\theta_2) \in \boldsymbol{U};\; \theta_1+\theta_2 \leqslant \frac{1}{2}-\varepsilon \\
& \qquad \text{or } 2\theta_1+\theta_2 \leqslant 1-\varepsilon,\; 7\theta_1+12\theta_2 \leqslant 4-\varepsilon,\; 19\theta_1+20\theta_2 \leqslant 10-\varepsilon \\
& \qquad \text{or } \theta_1 < \frac{1}{3},\; \theta_2 < \frac{1}{5},\; \theta_1+\theta_2 < \frac{29}{56} \\
& \qquad \text{or } \theta_1+3\theta_2 < 1,\; \theta_1+\theta_2 < \frac{29}{56},\; \theta_2 < \max\left(\frac{1-2\theta_1}{2}, \frac{2-2\theta_1}{5}\right) \\
& \qquad \text{or } \theta_1+3\theta_2 < 1,\; \theta_1+\theta_2 < \frac{29}{56},\; 4\theta_1+\theta_2 < \frac{403}{266},\; \frac{7}{4}\theta_1+\theta_2 < \frac{403}{532}\Big\},
\end{aligned}$$

Asymptotic formulas

$$\boldsymbol{E} = \Big\{(\theta_1,\theta_2) : (\theta_1,\theta_2) \in \boldsymbol{U}\backslash\boldsymbol{J};\ \frac{1}{4} < \theta_1 \leqslant \frac{2}{7},\ \frac{1}{2}(1-2\theta_1) < \theta_2 \leqslant \theta_1$$

$$\text{or } \frac{2}{7} < \theta_1 \leqslant \frac{2}{5},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{4}(2-3\theta_1)$$

$$\text{or } \frac{2}{5} < \theta_1 < \frac{1}{2},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{15}(11-20\theta_1)$$

$$\text{or } \frac{1}{2} \leqslant \theta_1 < \frac{11}{20},\ 0 < \theta_2 < \frac{1}{15}(11-20\theta_1)\Big\},$$

$$\boldsymbol{F}_1 = \Big\{(\theta_1,\theta_2) : (\theta_1,\theta_2) \in \boldsymbol{U}\backslash\boldsymbol{J};\ \frac{1}{4} < \theta_1 \leqslant \frac{5}{18},\ \frac{1}{2}(1-2\theta_1) < \theta_2 \leqslant \theta_1$$

$$\text{or } \frac{5}{18} < \theta_1 < \frac{1}{2},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{10}(5-8\theta_1)$$

$$\text{or } \frac{1}{2} \leqslant \theta_1 < \frac{11}{20},\ 0 < \theta_2 < \frac{1}{10}(11-20\theta_1)\Big\},$$

$$\boldsymbol{F}_2 = \Big\{(\theta_1,\theta_2) : (\theta_1,\theta_2) \in \boldsymbol{U}\backslash\boldsymbol{J};\ \frac{1}{4} < \theta_1 < \frac{2}{7},\ \frac{1}{2}(1-2\theta_1) < \theta_2 \leqslant \theta_1$$

$$\text{or } \frac{2}{7} \leqslant \theta_1 \leqslant \frac{2}{5},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < \frac{1}{4}(2-3\theta_1)$$

$$\text{or } \frac{2}{5} < \theta_1 < \frac{1}{2},\ \frac{1}{2}(1-2\theta_1) < \theta_2 < 1-2\theta_1\Big\}.$$

A new three-dimensional Harman's sieve, continued. In previous applications of the three-dimensional Harman's sieve, we always assume that $\kappa \geqslant \frac{1}{7} + \varepsilon$. In fact, this device is still applicable in some cases with $\kappa < \frac{1}{7} + \varepsilon$.
Suppose that $\frac{1}{8} < \kappa < \frac{1}{7} + \varepsilon$, and we apply a three-dimensional Harman's sieve on a region R. Since $\alpha_1 < \frac{1}{2} = \frac{4}{8}$ and $\alpha_2 < \frac{1}{3} < \frac{3}{8}$, we have $\Omega(m_1) \leqslant 3$ and $\Omega(m_2) \leqslant 2$, and $\Omega(m_1 m_2)$ can be 3, 4 or 5.
Now we have 3 cases based on different values of $\Omega(m_1 m_2)$:
(1). $\Omega(m_1) = 1,\ \Omega(m_2) = 2$ or $\Omega(m_1) = 2,\ \Omega(m_2) = 1$;
(2). $\Omega(m_1) = 2,\ \Omega(m_2) = 2$ or $\Omega(m_1) = 3,\ \Omega(m_2) = 1$;
(3). $\Omega(m_1) = 3,\ \Omega(m_2) = 2$.

These 3 cases correspond to 3 loss integrals:

$$\frac{1}{\kappa}\int_{\substack{t_1,t_2,t_3\geqslant\kappa\\(t_1,t_2+t_3)\in R,\ t_2\geqslant t_3\\ \text{or } (t_1+t_2,t_3)\in R,\ t_1\geqslant t_2\\(t_1,t_2,t_3)\notin \boldsymbol{G}_3}}\frac{\omega\left(\frac{1-t_1-t_2-t_3}{\kappa}\right)}{t_1t_2t_3}dt_3dt_2dt_1,$$

$$\frac{1}{\kappa}\int_{\substack{t_1,t_2,t_3,t_4\geqslant\kappa\\(t_1+t_2,t_3+t_4)\in R,\ t_1\geqslant t_2,\ t_3\geqslant t_4\\ \text{or } (t_1+t_2+t_3,t_4)\in R,\ t_1\geqslant t_2\geqslant t_3\\(t_1,t_2,t_3,t_4)\notin \boldsymbol{G}_4}}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{\kappa}\right)}{t_1t_2t_3t_4}dt_4dt_3dt_2dt_1,$$

and

$$\frac{1}{\kappa}\int_{\substack{t_1,t_2,t_3,t_4,t_5\geqslant\kappa\\(t_1+t_2+t_3,t_4+t_5)\in R\\ t_1\geqslant t_2\geqslant t_3,\ t_4\geqslant t_5\\(t_1,t_2,t_3,t_4,t_5)\notin \boldsymbol{G}_5}}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{\kappa}\right)}{t_1t_2t_3t_4t_5}dt_5dt_4dt_3dt_2dt_1.$$

When $\frac{11}{21} - \varepsilon < \theta \leqslant \frac{17}{32} - \varepsilon$, the new three-dimensional Harman's sieve should be applied on both B and C if we use it, since we can only perform straightforward decompositions on A where $\alpha_1 + \alpha_2 < \theta$. Since we have $\alpha_2 < \frac{1}{3}$, $\alpha_1 < \frac{3}{7} + \varepsilon$ and $\alpha_1 + \alpha_2 > \frac{4}{7} - \varepsilon$ for $\boldsymbol{\alpha}_2 \in B \cup C$, the new three-dimensional Harman's sieve is available here. Now we want to prove that $\boldsymbol{\alpha}_2 \in \boldsymbol{U}_2$ for $\boldsymbol{\alpha}_2 \in A$. The proof is trivial when $\alpha_1 < \tau = \frac{2}{7}$. When $\frac{2}{7} \leqslant \alpha_1 \leqslant \frac{3}{7} + \varepsilon$, we need to show that $\boldsymbol{\alpha}_2 \in \boldsymbol{U}_2$ or $(\alpha_1, \alpha_2, 2\theta - 1 + \varepsilon) \in \boldsymbol{S}_3$. Assume that $\alpha_1 < 2 - 3\theta - 3\varepsilon$, we get

$$\alpha_1 + 2\theta - 1 + \varepsilon < 1 - \theta - \varepsilon,$$

$$\alpha_1 + 2\theta - 1 + \varepsilon + 2\alpha_2 < 4\theta - \alpha_1 - 1 + \varepsilon \leqslant 4\theta - \frac{9}{7} + \varepsilon < 2 - 2\theta - \varepsilon,$$

and

$$\alpha_1 + 2\theta - 1 + \varepsilon + 4\alpha_2 < 6\theta - 3\alpha_1 - 1 + \varepsilon \leqslant 6\theta - \frac{13}{7} + \varepsilon < 2 - \theta - \varepsilon.$$

Since we have $\alpha_2 > \frac{5-8\theta}{6} - \varepsilon$ and $\theta - \frac{5-8\theta}{6} + 2\varepsilon < 2 - 3\theta - 3\varepsilon$ for all $\theta < \frac{17}{32} - \varepsilon$, we know that the condition $\alpha_1 < 2 - 3\theta - 3\varepsilon$ is satisfied automatically when $\boldsymbol{\alpha}_2 \in A$.
We often decide to use the new three-dimensional Harman's sieve instead of discarding the region $\frac{2}{7} \leqslant \alpha_1 \leqslant \frac{3}{7} + \varepsilon$ since the higher dimensional loss is usually smaller than the one-dimensional loss.

We give upper bounds for $C_1'(\theta_1, \theta_2)$.
Known results:
(1). $C_1'(\theta_1, \theta_2) = C_1'(\theta_2, \theta_1)$;
(2). $C_1'(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $\theta_1 + \theta_2 \leqslant 0.5 - \varepsilon$;
(3). $C_1'(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $2\theta_1 + \theta_2 \leqslant 1 - \varepsilon$, $7\theta_1 + 12\theta_2 \leqslant 4 - \varepsilon$ and $19\theta_1 + 20\theta_2 \leqslant 10 - \varepsilon$;
(4). $C_1'(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $\theta_1 < \frac{1}{3}$, $\theta_2 < \frac{1}{5}$ and $\theta_1 + \theta_2 < \frac{29}{56}$;
(5). $C_1'(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $\theta_1 + 3\theta_2 < 1$, $\theta_1 + \theta_2 < \frac{29}{56}$ and $\theta_2 < \max\left(\frac{1-2\theta_1}{2}, \frac{2-2\theta_1}{5}\right)$;
(6). $C_1'(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $\theta_1 + 3\theta_2 < 1$, $\theta_1 + \theta_2 < \frac{29}{56}$, $4\theta_1 + \theta_2 < \frac{403}{266}$ and $\frac{7}{4}\theta_1 + \theta_2 < \frac{403}{532}$;
(7). $C_1'(\theta_1, \theta_2) \leqslant C_1(\theta_1, \theta_2) \leqslant C_1(\theta_1 + \theta_2)$ for $0.5 \leqslant \theta_1 + \theta_2 \leqslant 1$;
(8). $C_1'(\theta_1, \theta_2) \leqslant 1 + \varepsilon$ for all θ_1, θ_2 satisfy $\theta_1 + \theta_2 = 0.5$;
(9). $C_1'(\theta_1, \theta_2) \leqslant 1 + \varepsilon$ for all θ_1, θ_2 satisfy $\theta_1 \leqslant 0.5$ and $\theta_2 = \min\left(1 - 2\theta_1, \frac{4-7\theta_1}{12}, \frac{10-19\theta_1}{20}\right)$.
We divide the region $\boldsymbol{U}$ into subregions based on the available Type-II ranges in each subregion. The decompositions are similar to which in Final decompositions 1-1, with more available arithmetic information.

Some special cases. Define

$$\mathcal{Z}_{\mathbf{1}} = \left\{ (\theta_1, \theta_2) : \frac{1}{4} < \theta_1 < \frac{5}{17},\ \frac{1}{2}(1 - 2\theta_1) < \theta_2 < \frac{1}{3}(1 - \theta_1) \right.$$
$$\left. \text{or } \frac{5}{17} \leqslant \theta_1 < \frac{3}{10},\ \frac{1}{2}(1 - 2\theta_1) < \theta_2 < 2 - 6\theta_1 \right\}.$$

Theorem (L. (2026))

Let $(\theta_1, \theta_2) \in \mathcal{Z}_{\mathbf{1}}$. Suppose that we have

$$\theta_1 + \theta_2 \leqslant \frac{11}{21} - \varepsilon.$$

Then we have

$$C_1'(\theta_1, \theta_2) \leqslant 1 + \int_{1-\theta}^{\frac{1}{2}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt + O(\varepsilon) = 1 + \log\left(\frac{\theta}{1-\theta}\right) + O(\varepsilon).$$

Proof. The Type-II range for $\mathcal{Z}_1$ is

$$\left[\varepsilon,\ \frac{1}{6}(5-8\theta_1-8\theta_2)-\varepsilon\right]\cup\left[\theta_2+\varepsilon,\ \frac{1}{2}(1-\theta_2)-\varepsilon\right]\cup\left[\frac{3}{7}+\varepsilon,\ 1-\theta-\varepsilon\right].$$

Since we have $(\theta_1,\theta_2)\in\mathcal{Z}_1$ and $\theta_1+\theta_2\leqslant\frac{11}{21}-\varepsilon$, all of the following conditions hold true:

$$\kappa=\frac{5-8\theta}{6}-\varepsilon>\frac{17}{126}-\varepsilon>\frac{1}{8}+\varepsilon,\quad 11\theta_1+12\theta_2<6-\varepsilon,\quad 8\theta_1+11\theta_2<5-8\varepsilon,$$

$$3\theta_1+2\theta_2<\frac{11}{7}-\varepsilon,\quad \theta_1+3\theta_2\leqslant 1,\quad \theta_2<\frac{1-\theta}{2}\leqslant\frac{1}{4}\quad\text{and}\quad\frac{1}{2}(1-\theta_2)>\frac{3}{8}.$$

We can simplify our Type-II range in this case to

$$\left[\varepsilon,\ \frac{17}{126}-\varepsilon\right]\cup\left[\frac{1}{4}+\varepsilon,\ \frac{3}{8}-\varepsilon\right]\cup\left[\frac{3}{7}+\varepsilon,\ 1-\theta-\varepsilon\right].$$

Final decompositions, 3-1

Using Buchstab's identity, we have

$$\psi\left(n,(2x)^{\frac{1}{2}}\right)=\psi\left(n,x^{\kappa}\right)-\sum_{\substack{n=p_1\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon}}\psi\left(\beta,x^{\kappa}\right)+\sum_{\substack{n=p_1p_2\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in\boldsymbol{G}_2}}\psi\left(\beta,p_2\right)$$

$$+\sum_{\substack{n=p_1p_2\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in A\cup B}}\psi\left(\beta,x^{\kappa}\right)-\sum_{\substack{n=p_1p_2p_3\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in A\cup B\cup C\\ \kappa\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)}}\psi\left(\beta,p_3\right)$$

$$+\sum_{\substack{n=m_1m_2m_3\\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in C}}\psi\left(m_1m_2m_3,x^{\kappa}\right)-\sum_{\substack{n=m_1m_2m_3\\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in C\\ \Omega(m_1m_2)\geqslant 3}}\psi\left(m_1m_2m_3,x^{\kappa}\right)$$

$$-\sum_{\substack{n=p_1\beta\\ \frac{3}{7}+\varepsilon\leqslant\alpha_1\leqslant 1-\theta-\varepsilon}}\psi\left(\beta,p_1\right)-\sum_{\substack{n=p_1\beta\\ 1-\theta-\varepsilon<\alpha_1<\frac{1}{2}}}\psi\left(\beta,p_1\right).$$

Now we only need to show that $(**)$ holds for $f(n)=$ sums that count numbers with 4 or more prime factors.

Since $\kappa > \frac{1}{8}$, we have $\Omega(n) \leqslant 7$. Assume that $\Omega(n) \geqslant 4$, and we only need to consider the following 7 cases:
Case 1: $\Omega(n) = 7$. Suppose that $n = p_1 \cdots p_7$ and $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_7$. Now we have $\alpha_6 + \alpha_7 \leqslant \frac{2}{7}$. Since we have $\theta_2 \leqslant \frac{1}{4} < \frac{2}{7}$ and $\frac{1}{2}(1 - \theta_2) > \frac{3}{8} > \frac{2}{7}$, if $\alpha_6 + \alpha_7 \in \left[\theta_2 + \varepsilon,\ \frac{3}{8} - \varepsilon\right]$, then $(**)$ holds for $f(n)$. Otherwise we have $\alpha_6 + \alpha_7 < \theta_2 + \varepsilon$. But since we have $8\theta_1 + 11\theta_2 < 5 - 8\varepsilon$, we get

$$\alpha_6 + \alpha_7 < \theta_2 + \varepsilon < \frac{5 - 8\theta}{3} - 2\varepsilon < \alpha_6 + \alpha_7,$$

making a contradiction. Hence $(**)$ holds for $f(n)$ in **Case 1**.
Case 2: $\Omega(n) = 6$. Suppose that $n = p_1 \cdots p_6$ and $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_6$. Now we have $\alpha_5 + \alpha_6 \leqslant \frac{1}{3}$. Since we have $\theta_2 \leqslant \frac{1}{4} < \frac{1}{3}$ and $\frac{1}{2}(1 - \theta_2) > \frac{3}{8} > \frac{1}{3}$, if $\alpha_5 + \alpha_6 \in \left[\theta_2 + \varepsilon,\ \frac{3}{8} - \varepsilon\right]$, then $(**)$ holds for $f(n)$. Otherwise we have $\alpha_5 + \alpha_6 < \theta_2 + \varepsilon$. But since we have $8\theta_1 + 11\theta_2 < 5 - 8\varepsilon$, we get

$$\alpha_5 + \alpha_6 < \theta_2 + \varepsilon < \frac{5 - 8\theta}{3} - 2\varepsilon < \alpha_5 + \alpha_6,$$

making a contradiction. Hence $(**)$ holds for $f(n)$ in **Case 2**.

Case 3: $\Omega(n) = 4$ **or** 5**, no product of variables lies in** $(1-\theta-\varepsilon,\ \theta+\varepsilon)$**.** Suppose that $n = p_1 \cdots p_k$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_k$, while the remaining part $\alpha_i = \alpha_j$ gives a "loss" of size $O(\varepsilon)$. In this case we can "view" $\left[\frac{3}{7}+\varepsilon,\ \frac{4}{7}-\varepsilon\right]$ as a "fake" Type-II range. Since we have $\theta < \frac{11}{21} < \frac{9}{17}$, the results in [Maynard (2025a), Chapter 9] can be applied here if any of the following conditions holds:
(1). $\alpha_k \geqslant \frac{1}{7}+\varepsilon$;
(2). $\Omega(n) = 5$, $\alpha_3+\alpha_4+\alpha_5 > \frac{4}{7}-\varepsilon$;
(3). $\Omega(n) = 4$, $\alpha_1 \leqslant \frac{3}{7}+\varepsilon$, $\alpha_1+\alpha_4 > \frac{4}{7}-\varepsilon$.
By Condition (1), we can assume that $\frac{1}{8} < \alpha_k < \frac{1}{7}+\varepsilon$. We first consider the subcase $\Omega(n) = 5$. By Condition (2) and the "fake" Type-II range $\left[\frac{3}{7}+\varepsilon,\ \frac{4}{7}-\varepsilon\right]$, we can also assume that $\alpha_3+\alpha_4+\alpha_5 < \frac{3}{7}+\varepsilon$. This means that $\alpha_1+\alpha_2 > \frac{4}{7}-\varepsilon$, and thus $\alpha_1 > \frac{2}{7}-\frac{1}{2}\varepsilon$. Since $\theta_2 < \frac{1}{4} < \frac{2}{7}-\frac{1}{2}\varepsilon < \frac{3}{8} < \frac{1}{2}(1-\theta_2)$, we can assume that $\alpha_1 > \frac{3}{8}-\varepsilon$. Now we have $\frac{3}{7}+\varepsilon < \frac{1}{2}-\varepsilon = \frac{3}{8}-\varepsilon+\frac{1}{8} < \alpha_1+\alpha_5$, and we can assume that $\alpha_1+\alpha_5 > \frac{4}{7}-\varepsilon$. Since $\alpha_5 < \frac{1}{7}+\varepsilon$, we have $\alpha_1 > \frac{3}{7}-2\varepsilon$. If $\alpha_1 \geqslant \frac{3}{7}+\varepsilon$, we can assume that $\alpha_1 > \frac{4}{7}-\varepsilon$. But now we have $1 = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 > \frac{4}{7}-\varepsilon+4\cdot\frac{1}{8} > 1$, making a contradiction. The remaining part is $\frac{3}{7}-2\varepsilon < \alpha_1 < \frac{3}{7}+\varepsilon$, leading to a "loss" of size $O(\varepsilon)$. Hence $(**)$ holds for $f(n)$ with an error $O(\varepsilon)$ in **Case 3** if $\Omega(n) = 5$.
Next, we consider the second subcase $\Omega(n) = 4$. Suppose that $\alpha_1+\alpha_4 \leqslant \frac{4}{7}-\varepsilon$. By Condition (3) and the "fake" Type-II range $\left[\frac{3}{7}+\varepsilon,\ \frac{4}{7}-\varepsilon\right]$, we can assume that $\alpha_1+\alpha_4 < \frac{3}{7}+\varepsilon$. Since $\alpha_4 > \frac{1}{8}$, we have $\alpha_1 < \frac{17}{56}+\varepsilon = \frac{3}{7}+\varepsilon-\frac{1}{8}$. Since $\theta_2 < \frac{1}{4} < \frac{17}{56}+\varepsilon < \frac{3}{8} < \frac{1}{2}(1-\theta_2)$, we can assume that $\alpha_1 \leqslant \frac{1}{4}+\varepsilon$. If $\alpha_1 \leqslant \frac{1}{4}$, then we have $1 = \alpha_1+\alpha_2+\alpha_3+\alpha_4 < 4\cdot\frac{1}{4} = 1$, making a contradiction. The remaining part is $\frac{1}{4} < \alpha_1 \leqslant \frac{1}{4}+\varepsilon$, leading to a "loss" of size $O(\varepsilon)$.
Now, by the "fake" Type-II range $\left[\frac{3}{7}+\varepsilon,\ \frac{4}{7}-\varepsilon\right]$, we only need to consider the subcase $\Omega(n) = 4$ with $\alpha_1 > \frac{4}{7}-\varepsilon$. This means that $\alpha_2+\alpha_3+\alpha_4 < \frac{3}{7}+\varepsilon$. Since $\alpha_2 > \alpha_3 > \alpha_4 > \frac{1}{8}+\varepsilon$, we have $\alpha_3+\alpha_4 > \frac{1}{4}+\varepsilon > \theta_2+\varepsilon$. Thus, we can assume that $\alpha_3+\alpha_4 > \frac{3}{8}-\varepsilon$. But now we have

$$\frac{1}{2} < \frac{3}{2}\left(\frac{3}{8}-\varepsilon\right) < \frac{3}{2}(\alpha_3+\alpha_4) < \alpha_2+\alpha_3+\alpha_4 < \frac{3}{7}+\varepsilon,$$

making a contradiction. Hence $(**)$ holds for $f(n)$ with an error $O(\varepsilon)$ in **Case 3** if $\Omega(n) = 4$.

Case 4: $\Omega(n)=5$**, one variable lies in** $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Suppose that $n=p_1\cdots p_5$ and $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_5$. Now we have $\alpha_1 > 1-\theta-\varepsilon$ and $\alpha_2 \geqslant \cdots \geqslant \alpha_5 \geqslant \frac{5-8\theta}{6}-\varepsilon$. But since $\theta < \frac{11}{21} < \frac{10}{19}-\varepsilon$, we have

$$1-\theta-\varepsilon+4\left(\frac{5-8\theta}{6}-\varepsilon\right)>1,$$

making a contradiction. Hence **Case 4** is empty.

Case 5: $\Omega(n)=5$**, a product of two variables lies in** $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Suppose that $n=p_1\cdots p_5$ and $\alpha_1+\alpha_2 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ (of course, $\alpha_3+\alpha_4+\alpha_5 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ too). Since $\alpha_i+\alpha_j \geqslant \frac{5-8\theta}{3}-2\varepsilon > \theta_2+\varepsilon$ (see **Case 1**), we can assume that $\alpha_i+\alpha_j > \frac{1}{2}(1-\theta_2)-\varepsilon > \frac{3}{8}-\varepsilon$ for all $i,j\in\{1,2,3,4,5\}$ (otherwise $\alpha_i+\alpha_j \in \left[\theta_2+\varepsilon,\ \frac{1}{2}(1-\theta_2)-\varepsilon\right]$, and $(**)$ holds for $f(n)$). But now we have

$$\alpha_3+\alpha_4+\alpha_5=\frac{(\alpha_3+\alpha_4)+(\alpha_3+\alpha_5)+(\alpha_4+\alpha_5)}{2}>\frac{3}{2}\left(\frac{3}{8}-\varepsilon\right)>\frac{9}{16}-2\varepsilon>\frac{11}{21}+2\varepsilon>\theta+\varepsilon,$$

making a contradiction. Hence $(**)$ holds for $f(n)$ in **Case 5**.

Case 6: $\Omega(n)=4$**, a product of two variables lies in** $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Suppose that $n=p_1p_2p_3p_4$ and $\alpha_1+\alpha_2 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ (of course, $\alpha_3+\alpha_4 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ too). Without loss of generality, we further assume that $\alpha_1>\alpha_2$ and $\alpha_3>\alpha_4$, while the remaining part $\alpha_i=\alpha_j$ gives a "loss" of size $O(\varepsilon)$. Now we have $\alpha_2,\alpha_4 < \frac{\theta+\varepsilon}{2}$. Since $\frac{1}{2}<\theta\leqslant\frac{11}{21}$, we have $\frac{\theta+\varepsilon}{2}<\frac{11}{42}+\varepsilon<\frac{3}{8}-\varepsilon<\frac{1}{2}(1-\theta_2)-\varepsilon$. Now we can assume that $\alpha_2,\alpha_4<\theta_2+\varepsilon$ (otherwise α_2 or $\alpha_4 \in \left[\theta_2+\varepsilon,\ \frac{1}{2}(1-\theta_2)-\varepsilon\right]$, and $(**)$ holds for $f(n)$). Since we can assume that $\alpha_2+\alpha_4>\frac{3}{8}-\varepsilon$ (see **Case 5**), if $\alpha_1,\alpha_3>\frac{3}{8}-\varepsilon$, then $1=\alpha_1+\alpha_3+(\alpha_2+\alpha_4)>\frac{9}{8}-3\varepsilon>1$, making a contradiction. Now we assume that $\min(\alpha_1,\alpha_3)\leqslant\frac{3}{8}-\varepsilon<\frac{1}{2}(1-\theta_2)-\varepsilon$. Without loss of generality, we further assume that $\alpha_1>\alpha_3$, while the remaining part $\alpha_1=\alpha_3$ gives a "loss" of size $O(\varepsilon)$. If $\alpha_3\in\left[\theta_2+\varepsilon,\ \frac{1}{2}(1-\theta_2)-\varepsilon\right]$, then $(**)$ holds for $f(n)$. Otherwise we have $\alpha_2,\alpha_3,\alpha_4<\theta_2+\varepsilon$. If $\alpha_3<\theta_2-2\varepsilon$, then we have $\alpha_3+\alpha_4\leqslant 2\theta_2-\varepsilon\leqslant 1-\theta-\varepsilon$, making a contradiction with the assumption $\alpha_3+\alpha_4\in(1-\theta-\varepsilon,\ \theta+\varepsilon)$. The remaining part is $\theta_2-2\varepsilon<\alpha_3\leqslant\theta_2+\varepsilon$, leading to a "loss" of size $O(\varepsilon)$. Hence $(**)$ holds for $f(n)$ with an error $O(\varepsilon)$ in **Case 6**.

Case 7: $\Omega(n)=4$**, one variable lies in** $(1-\theta-\varepsilon,\ \theta+\varepsilon)$**.** Suppose that $n=p_0p_1p_2p_3$ and $\alpha_0 \geqslant \alpha_1 \geqslant \alpha_2 \geqslant \alpha_3$. Now we have $\alpha_0 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ and $\alpha_1+\alpha_2+\alpha_3 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$. We also have $\alpha_1+\alpha_3 \leqslant \frac{1}{2}$ and $\alpha_2 \leqslant \frac{1}{3}$. Note that the sum of $\psi\left(m_1m_2m_3, x^{\kappa}\right)$ only counts numbers with all prime factors smaller than $\max(m_1,m_2,m_3) \leqslant x^{\frac{3}{7}+\varepsilon}$, one can easily find that this type of n will not be counted in the sum of $\psi\left(m_1m_2m_3, x^{\kappa}\right)$ by a simple observation. Hence, this type of n will only be counted in one part of the sum of $\psi\left(\beta, p_3\right)$:

$$\sum_{\substack{n=p_1p_2p_3\beta \\ 1-\theta-\varepsilon<\alpha_1+\alpha_2+\alpha_3<\theta+\varepsilon}} \psi\left(\beta, p_3\right),$$

where β here is a large prime. Using Buchstab's identity, we have

$$\sum_{\substack{n=p_1p_2p_3\beta \\ 1-\theta-\varepsilon<\alpha_1+\alpha_2+\alpha_3<\theta+\varepsilon}} \psi\left(\beta, p_3\right) = \sum_{\substack{n=p_1p_2p_3\beta \\ 1-\theta-\varepsilon<\alpha_1+\alpha_2+\alpha_3<\theta+\varepsilon}} \psi\left(\beta, x^{\kappa}\right) - \sum_{\substack{n=p_1p_2p_3p_4\beta \\ 1-\theta-\varepsilon<\alpha_1+\alpha_2+\alpha_3<\theta+\varepsilon \\ \kappa\leqslant\alpha_4<\min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)}} \psi\left(\beta, p_4\right).$$

We know that $(**)$ holds for the first sum since $(\alpha_1+\alpha_2, \alpha_3) \in A$, and we only need to deal with the second sum that counts numbers with 5 or more prime factors, with a product of 3 variables lies in $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Now we reduce this case to **Cases 1, 2 and 5**. Since the second sum is a positive sum that cannot be dropped directly, we need $(**)$ holds for $f(n)$ in **Cases 1, 2 and 5** without an error $O(\varepsilon)$. By the discussions above, we know that $(**)$ holds for $f(n)$ in **Cases 1, 2 and 5** without an error $O(\varepsilon)$, and thus $(**)$ holds for $f(n)$ in **Case 7**. Combining all the 7 cases above, the proof of Theorem is completed.

When (θ_1, θ_2) lies in the boundary of this region:

$$\theta_1 \in \left[\frac{1}{4}, \frac{2}{7} - \varepsilon\right] \cup \left[\frac{2}{7} + \varepsilon, \frac{3}{10} - \varepsilon\right], \quad \theta_2 = \min\left(\frac{1-\theta_1}{3}, \frac{11-21\theta_1}{21}, 2 - 6\theta_1\right),$$

we can use the same decomposing process (with some small modifications) to prove the same upper bound:

$$C_1'(\theta_1, \theta_2) \leqslant 1 + \int_{1-\theta}^{\frac{1}{2}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt + O(\varepsilon) = 1 + \log\left(\frac{\theta}{1-\theta}\right) + O(\varepsilon).$$

For example, we have

$$C_1'(0.28, 0.24) \leqslant 1 + \log\left(\frac{13}{12}\right) + O(\varepsilon) < 1.0801.$$

Note that we have $(\theta_1, \theta_2) \in \boldsymbol{T}$ except for $\theta_1 \in \left(\frac{2}{7} - \varepsilon, \frac{2}{7} + \varepsilon\right)$.

Define

$$\boldsymbol{\mathcal{Z}_2} = \Big\{(\theta_1, \theta_2) : \frac{1}{4} < \theta_1 < \frac{6}{23},\ \frac{1}{3}(1-\theta_1) \leqslant \theta_2 \leqslant \theta_1$$
$$\text{or } \frac{6}{23} \leqslant \theta_1 < \frac{5}{17},\ \frac{1}{3}(1-\theta_1) \leqslant \theta_2 < \frac{1}{13}(6-10\theta_1)\Big\}.$$

Theorem (L. (2026))

Let $(\theta_1, \theta_2) \in \boldsymbol{\mathcal{Z}_2}$. Suppose that we have

$$\theta_2 \leqslant \frac{1}{4} - \varepsilon \quad \text{and} \quad 17\theta_1 + 26\theta_2 \leqslant 11 - 14\varepsilon.$$

Then we have

$$C_1'(\theta_1, \theta_2) \leqslant 1 + \int_{1-\theta}^{\frac{1}{2}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt + O(\varepsilon) = 1 + \log\left(\frac{\theta}{1-\theta}\right) + O(\varepsilon).$$

Proof. The Type-II range for $\mathcal{Z}_{\mathbf{2}}$ is

$$\left[\varepsilon,\ \frac{1}{6}(5-8\theta_1-8\theta_2)-\varepsilon\right]\cup\left[\theta_2+\varepsilon,\ \frac{1}{2}(2-\theta_1-4\theta_2)-\varepsilon\right]\cup\left[\frac{3}{7}+\varepsilon,\ 1-\theta-\varepsilon\right].$$

Since we have $(\theta_1,\theta_2)\in\mathcal{Z}_{\mathbf{2}}$, $\theta_2\leqslant\frac{1}{4}-\varepsilon$ and $17\theta_1+26\theta_2\leqslant 11-14\varepsilon$, all of the following conditions hold true:

$$\kappa=\frac{5-8\theta}{6}-\varepsilon>\frac{17}{126}-\varepsilon>\frac{1}{8}+\varepsilon,\quad 11\theta_1+12\theta_2<6-\varepsilon,\quad 8\theta_1+11\theta_2<5-8\varepsilon,$$

$$3\theta_1+2\theta_2<\frac{11}{7}-\varepsilon,\quad \frac{1}{2}\leqslant\theta_1+\theta_2\leqslant\frac{11}{21}-\varepsilon,\quad 7\theta_1+16\theta_2<6-8\varepsilon\quad\text{and}\quad\frac{1}{2}(2-\theta_1-4\theta_2)>\frac{9}{25}.$$

We can simplify our Type-II range in this case to

$$\left[\varepsilon,\ \frac{17}{126}-\varepsilon\right]\cup\left[\frac{1}{4}+\varepsilon,\ \frac{9}{25}-\varepsilon\right]\cup\left[\frac{3}{7}+\varepsilon,\ 1-\theta-\varepsilon\right].$$

Final decompositions, 3-1

Using Buchstab's identity, we have

$$\begin{aligned}
\psi\left(n,(2x)^{\frac{1}{2}}\right) = \psi\left(n,x^{\kappa}\right) &- \sum_{\substack{n=p_1\beta \\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon}} \psi\left(\beta,x^{\kappa}\right) + \sum_{\substack{n=p_1p_2\beta \\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon \\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2\in\boldsymbol{G}_2}} \psi\left(\beta,p_2\right) \\
&+ \sum_{\substack{n=p_1p_2\beta \\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon \\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2\in A\cup B}} \psi\left(\beta,x^{\kappa}\right) - \sum_{\substack{n=p_1p_2p_3\beta \\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon \\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2\in A\cup B\cup C \\ \kappa\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)}} \psi\left(\beta,p_3\right) \\
&+ \sum_{\substack{n=m_1m_2m_3 \\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon \\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2\in C}} \psi\left(m_1m_2m_3,x^{\kappa}\right) - \sum_{\substack{n=m_1m_2m_3 \\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon \\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2\in C \\ \Omega(m_1m_2)\geqslant 3}} \psi\left(m_1m_2m_3,x^{\kappa}\right) \\
&- \sum_{\substack{n=p_1\beta \\ \frac{3}{7}+\varepsilon\leqslant\alpha_1\leqslant 1-\theta-\varepsilon}} \psi\left(\beta,p_1\right) - \sum_{\substack{n=p_1\beta \\ 1-\theta-\varepsilon<\alpha_1<\frac{1}{2}}} \psi\left(\beta,p_1\right).
\end{aligned}$$

Now we only need to show that $(**)$ holds for $f(n) =$ sums that count numbers with 4 or more prime factors.

Since $\kappa > \frac{1}{8}$, we have $\Omega(n) \leqslant 7$. Assume that $\Omega(n) \geqslant 4$, and we only need to consider the following 7 cases:
Case 1: $\Omega(n) = 7$. Suppose that $n = p_1 \cdots p_7$ and $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_7$. Now we have $\alpha_6 + \alpha_7 \leqslant \frac{2}{7}$. Since we have $\theta_2 \leqslant \frac{1}{4} < \frac{2}{7}$ and $\frac{1}{2}(2 - \theta_1 - 4\theta_2) > \frac{9}{25} > \frac{2}{7}$, if $\alpha_6 + \alpha_7 \in \left[\theta_2 + \varepsilon,\ \frac{9}{25} - \varepsilon\right]$, then $(**)$ holds for $f(n)$. Otherwise we have $\alpha_6 + \alpha_7 < \theta_2 + \varepsilon$. But since we have $8\theta_1 + 11\theta_2 < 5 - 8\varepsilon$, we get

$$\alpha_6 + \alpha_7 < \theta_2 + \varepsilon < \frac{5 - 8\theta}{3} - 2\varepsilon < \alpha_6 + \alpha_7,$$

making a contradiction. Hence $(**)$ holds for $f(n)$ in **Case 1**.
Case 2: $\Omega(n) = 6$. Suppose that $n = p_1 \cdots p_6$ and $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_6$. Now we have $\alpha_5 + \alpha_6 \leqslant \frac{1}{3}$. Since we have $\theta_2 \leqslant \frac{1}{4} < \frac{1}{3}$ and $\frac{1}{2}(2 - \theta_1 - 4\theta_2) > \frac{9}{25} > \frac{1}{3}$, if $\alpha_5 + \alpha_6 \in \left[\theta_2 + \varepsilon,\ \frac{9}{25} - \varepsilon\right]$, then $(**)$ holds for $f(n)$. Otherwise we have $\alpha_5 + \alpha_6 < \theta_2 + \varepsilon$. But since we have $8\theta_1 + 11\theta_2 < 5 - 8\varepsilon$, we get

$$\alpha_5 + \alpha_6 < \theta_2 + \varepsilon < \frac{5 - 8\theta}{3} - 2\varepsilon < \alpha_5 + \alpha_6,$$

making a contradiction. Hence $(**)$ holds for $f(n)$ in **Case 2**.

Case 3: $\Omega(n) = 4$ **or** 5**, no product of variables lies in** $(1-\theta-\varepsilon,\ \theta+\varepsilon)$**.** Suppose that $n = p_1 \cdots p_k$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_k$, while the remaining part $\alpha_i = \alpha_j$ gives a "loss" of size $O(\varepsilon)$. In this case we can "view" $\left[\frac{3}{7}+\varepsilon,\ \frac{4}{7}-\varepsilon\right]$ as a "fake" Type-II range. Since we have $\theta < \frac{11}{21} < \frac{9}{17}$, the results in [Maynard (2025a), Chapter 9] can be applied here if any of the following conditions holds:
(1). $\alpha_k \geqslant \frac{1}{7}+\varepsilon$;
(2). $\Omega(n) = 5$, $\alpha_3+\alpha_4+\alpha_5 > \frac{4}{7}-\varepsilon$;
(3). $\Omega(n) = 4$, $\alpha_1 \leqslant \frac{3}{7}+\varepsilon$, $\alpha_1+\alpha_4 > \frac{4}{7}-\varepsilon$.
By Condition (1), we can assume that $\frac{1}{8} < \alpha_k < \frac{1}{7}+\varepsilon$. We first consider the subcase $\Omega(n) = 5$. By Condition (2) and the "fake" Type-II range $\left[\frac{3}{7}+\varepsilon,\ \frac{4}{7}-\varepsilon\right]$, we can also assume that $\alpha_3+\alpha_4+\alpha_5 < \frac{3}{7}+\varepsilon$. This means that $\alpha_1+\alpha_2 > \frac{4}{7}-\varepsilon$, and thus $\alpha_1 > \frac{2}{7}-\frac{1}{2}\varepsilon$. Since $\theta_2 < \frac{1}{4} < \frac{2}{7}-\frac{1}{2}\varepsilon < \frac{9}{25} < \frac{1}{2}(2-\theta_1-4\theta_2)$, we can assume that $\alpha_1 > \frac{9}{25}-\varepsilon$. Now we have $\frac{3}{7}+\varepsilon < \frac{97}{200}-\varepsilon = \frac{9}{25}-\varepsilon+\frac{1}{8} < \alpha_1+\alpha_5$, and we can assume that $\alpha_1+\alpha_5 > \frac{4}{7}-\varepsilon$. Since $\alpha_5 < \frac{1}{7}+\varepsilon$, we have $\alpha_1 > \frac{3}{7}-2\varepsilon$. If $\alpha_1 \geqslant \frac{3}{7}+\varepsilon$, we can assume that $\alpha_1 > \frac{4}{7}-\varepsilon$. But now we have $1 = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 > \frac{4}{7}-\varepsilon+4\cdot\frac{1}{8} > 1$, making a contradiction. The remaining part is $\frac{3}{7}-2\varepsilon < \alpha_1 < \frac{3}{7}+\varepsilon$, leading to a "loss" of size $O(\varepsilon)$. Hence $(**)$ holds for $f(n)$ with an error $O(\varepsilon)$ in **Case 3** if $\Omega(n) = 5$.
Next, we consider the second subcase $\Omega(n) = 4$. Suppose that $\alpha_1+\alpha_4 \leqslant \frac{4}{7}-\varepsilon$. By Condition (3) and the "fake" Type-II range $\left[\frac{3}{7}+\varepsilon,\ \frac{4}{7}-\varepsilon\right]$, we can assume that $\alpha_1+\alpha_4 < \frac{3}{7}+\varepsilon$. Since $\alpha_4 > \frac{1}{8}$, we have $\alpha_1 < \frac{17}{56}+\varepsilon = \frac{3}{7}+\varepsilon-\frac{1}{8}$. Since $\theta_2 < \frac{1}{4} < \frac{17}{56}+\varepsilon < \frac{9}{25} < \frac{1}{2}(2-\theta_1-4\theta_2)$, we can assume that $\alpha_1 \leqslant \frac{1}{4}+\varepsilon$. If $\alpha_1 \leqslant \frac{1}{4}$, then we have $1 = \alpha_1+\alpha_2+\alpha_3+\alpha_4 < 4\cdot\frac{1}{4} = 1$, making a contradiction. The remaining part is $\frac{1}{4} < \alpha_1 \leqslant \frac{1}{4}+\varepsilon$, leading to a "loss" of size $O(\varepsilon)$.
Now, by the "fake" Type-II range $\left[\frac{3}{7}+\varepsilon,\ \frac{4}{7}-\varepsilon\right]$, we only need to consider the subcase $\Omega(n) = 4$ with $\alpha_1 > \frac{4}{7}-\varepsilon$. This means that $\alpha_2+\alpha_3+\alpha_4 < \frac{3}{7}+\varepsilon$. Since $\alpha_2 > \alpha_3 > \alpha_4 > \frac{1}{8}+\varepsilon$, we have $\alpha_3+\alpha_4 > \frac{1}{4}+\varepsilon > \theta_2+\varepsilon$. Thus, we can assume that $\alpha_3+\alpha_4 > \frac{9}{25}-\varepsilon$. But now we have

$$\frac{1}{2} < \frac{3}{2}\left(\frac{9}{25}-\varepsilon\right) < \frac{3}{2}(\alpha_3+\alpha_4) < \alpha_2+\alpha_3+\alpha_4 < \frac{3}{7}+\varepsilon,$$

making a contradiction. Hence $(**)$ holds for $f(n)$ with an error $O(\varepsilon)$ in **Case 3** if $\Omega(n) = 4$.

Case 4: $\Omega(n) = 5$, one variable lies in $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Suppose that $n = p_1 \cdots p_5$ and $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_5$. Now we have $\alpha_1 > 1-\theta-\varepsilon$ and $\alpha_2 \geqslant \cdots \geqslant \alpha_5 \geqslant \frac{5-8\theta}{6} - \varepsilon$. But since $\theta < \frac{11}{21} < \frac{10}{19} - \varepsilon$, we have

$$1-\theta-\varepsilon+4\left(\frac{5-8\theta}{6}-\varepsilon\right) > 1,$$

making a contradiction. Hence **Case 4** is empty.

Case 5: $\Omega(n) = 5$, a product of two variables lies in $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Suppose that $n = p_1 \cdots p_5$ and $\alpha_1+\alpha_2 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ (of course, $\alpha_3+\alpha_4+\alpha_5 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ too). Since $\alpha_i+\alpha_j \geqslant \frac{5-8\theta}{3} - 2\varepsilon > \theta_2+\varepsilon$ (see **Case 1**), we can assume that $\alpha_i+\alpha_j > \frac{1}{2}(2-\theta_1-4\theta_2)-\varepsilon$ for all $i,j \in \{1,2,3,4,5\}$ (otherwise $\alpha_i+\alpha_j \in \left[\theta_2+\varepsilon,\ \frac{1}{2}(2-\theta_1-4\theta_2)-\varepsilon\right]$, and $(**)$ holds for $f(n)$). Now we have

$$\alpha_3+\alpha_4+\alpha_5 = \frac{(\alpha_3+\alpha_4)+(\alpha_3+\alpha_5)+(\alpha_4+\alpha_5)}{2} > \frac{3}{4}(2-\theta_1-4\theta_2)-\varepsilon.$$

Since $7\theta_1+16\theta_2 < 6-8\varepsilon$, we have $\frac{3}{4}(2-\theta_1-4\theta_2)-\varepsilon > \theta+\varepsilon$, making a contradiction with $\alpha_3+\alpha_4+\alpha_5 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$. Hence $(**)$ holds for $f(n)$ in **Case 5**.

Case 6: $\Omega(n) = 4$, a product of two variables lies in $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Suppose that $n = p_1p_2p_3p_4$ and $\alpha_1+\alpha_2 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ (of course, $\alpha_3+\alpha_4 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ too). Without loss of generality, we further assume that $\max(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \alpha_1$, while the remaining part $\alpha_i = \alpha_j$ gives a "loss" of size $O(\varepsilon)$. Since $\alpha_1 > \frac{1}{4} \geqslant \theta_2+\varepsilon$, we can assume that $\alpha_1 > \frac{1}{2}(2-\theta_1-4\theta_2)-\varepsilon$ (otherwise $\alpha_1 \in \left[\theta_2+\varepsilon,\ \frac{1}{2}(2-\theta_1-4\theta_2)-\varepsilon\right]$, and $(**)$ holds for $f(n)$). Now we have

$$\alpha_1+\alpha_2 > \frac{2-\theta_1-4\theta_2}{2} + \frac{5-8\theta}{6} - 2\varepsilon = \frac{11-11\theta_1-20\theta_2}{6} - 2\varepsilon.$$

Since $17\theta_1+26\theta_2 < 11-14\varepsilon$, we have $\frac{11-11\theta_1-20\theta_2}{6} - 2\varepsilon > \theta+\varepsilon$, making a contradiction with the assumption $\alpha_1+\alpha_2 \in [1-\theta,\ \theta]$. Hence $(**)$ holds for $f(n)$ with an error $O(\varepsilon)$ in **Case 6**.

Case 7: $\Omega(n)=4$**, one variable lies in** $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Suppose that $n=p_0p_1p_2p_3$ and $\alpha_0 \geqslant \alpha_1 \geqslant \alpha_2 \geqslant \alpha_3$. Now we have $\alpha_0 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ and $\alpha_1+\alpha_2+\alpha_3 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$. We also have $\alpha_1+\alpha_3 \leqslant \frac{1}{2}$ and $\alpha_2 \leqslant \frac{1}{3}$. Note that the sum of $\psi\left(m_1m_2m_3, x^{\kappa}\right)$ only counts numbers with all prime factors smaller than $\max(m_1,m_2,m_3) \leqslant x^{\frac{3}{7}+\varepsilon}$, one can easily find that this type of n will not be counted in the sum of $\psi\left(m_1m_2m_3, x^{\kappa}\right)$ by a simple observation. Hence, this type of n will only be counted in one part of the sum of $\psi\left(\beta, p_3\right)$:

$$\sum_{\substack{n=p_1p_2p_3\beta \\ 1-\theta-\varepsilon<\alpha_1+\alpha_2+\alpha_3<\theta+\varepsilon}} \psi\left(\beta, p_3\right),$$

where β here is a large prime. Using Buchstab's identity, we have

$$\sum_{\substack{n=p_1p_2p_3\beta \\ 1-\theta-\varepsilon<\alpha_1+\alpha_2+\alpha_3<\theta+\varepsilon}} \psi\left(\beta, p_3\right) = \sum_{\substack{n=p_1p_2p_3\beta \\ 1-\theta-\varepsilon<\alpha_1+\alpha_2+\alpha_3<\theta+\varepsilon}} \psi\left(\beta, x^{\kappa}\right) - \sum_{\substack{n=p_1p_2p_3p_4\beta \\ 1-\theta-\varepsilon<\alpha_1+\alpha_2+\alpha_3<\theta+\varepsilon \\ \kappa\leqslant\alpha_4<\min\left(\alpha_3, \frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)}} \psi\left(\beta, p_4\right).$$

We know that $(**)$ holds for the first sum since $(\alpha_1+\alpha_2, \alpha_3) \in A$, and we only need to deal with the second sum that counts numbers with 5 or more prime factors, with a product of 3 variables lies in $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Now we reduce this case to **Cases 1, 2 and 5**. Since the second sum is a positive sum that cannot be dropped directly, we need $(**)$ holds for $f(n)$ in **Cases 1, 2 and 5** without an error $O(\varepsilon)$. By the discussions above, we know that $(**)$ holds for $f(n)$ in **Cases 1, 2 and 5** without an error $O(\varepsilon)$, and thus $(**)$ holds for $f(n)$ in **Case 7**. Combining all the 7 cases above, the proof of Theorem is completed.

When (θ_1, θ_2) lies in the boundary of this region:

$$\frac{1}{4} \leqslant \theta_1 < \frac{7}{25}, \quad \theta_2 = \min\left(\frac{1}{4}, \frac{11 - 17\theta_1}{26}\right),$$

we can use the same decomposing process (with some small modifications) to prove the same upper bound:

$$C_1'(\theta_1, \theta_2) \leqslant 1 + \int_{1-\theta}^{\frac{1}{2}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt + O(\varepsilon) = 1 + \log\left(\frac{\theta}{1-\theta}\right) + O(\varepsilon).$$

For example, we have

$$C_1'(0.26, 0.25) \leqslant 1 + \log\left(\frac{51}{49}\right) + O(\varepsilon) < 1.0401.$$

Define

$$\mathcal{Z}_{\mathbf{3}} = \left\{(\theta_1, \theta_2) : \frac{1}{3} < \theta_1 \leqslant \frac{3}{8},\ \frac{1}{2}(1 - 2\theta_1) \leqslant \theta_2 < \frac{1}{14}(5 - 8\theta_1)\right\}.$$

Theorem (L. (2026))

Let $(\theta_1, \theta_2) \in \mathcal{Z}_{\mathbf{3}}$. Suppose that we have

$$16\theta_1 + 8\theta_2 \leqslant 7 - 14\varepsilon.$$

Then $(**)$ holds for

$$f(n) = 1_p(n) + \sum_{\substack{n = p_1\beta \\ 1-\theta-\varepsilon < \alpha_1 < \frac{1}{2}}} \psi(\beta, p_1),$$

and we have

$$C_1'(\theta_1, \theta_2) \leqslant 1 + \int_{1-\theta}^{\frac{1}{2}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt + O(\varepsilon) = 1 + \log\left(\frac{\theta}{1-\theta}\right) + O(\varepsilon).$$

Proof. The Type-II range for $\mathcal{Z}_3$ is

$$\left[\varepsilon,\ \frac{1}{6}(5-8\theta_1-4\theta_2)-\varepsilon\right]\cup\left[\theta_1+\varepsilon,\ \frac{1}{2}(1-\theta_2)-\varepsilon\right]\cup\left[\frac{3}{7}+\varepsilon,\ 1-\theta-\varepsilon\right].$$

Since we have $(\theta_1,\theta_2)\in\mathcal{Z}_3$ and $16\theta_1+8\theta_2\leqslant 7-14\varepsilon$, all of the following conditions hold true:

$$\kappa=\frac{5-8\theta_1-4\theta_2}{6}-\varepsilon>\frac{1}{4},\quad \frac{1}{2}<\theta<\frac{11}{21}-\varepsilon.$$

Now, we can decompose $\psi\left(n,(2x)^{\frac{1}{2}}\right)$. By previous discussions, we only need to show that $(**)$ holds for $f(n)=$ sums that count numbers with 4 or more prime factors. Since $\kappa>\frac{1}{4}$, any $n\sim x$ with 4 or more prime factors must has at least one factor smaller than x^κ, and we can use our Type-II range $\left[\varepsilon,\ \frac{5-8\theta_1-4\theta_2}{6}-\varepsilon\right]$ to show that $(**)$ holds for $f(n)$ that count numbers with 4 or more prime factors. The proof of Theorem is completed.

For example, we have

$$C_1'(0.36,0.15)\leqslant 1+\log\left(\frac{51}{49}\right)+O(\varepsilon)<1.0401.$$

Final decompositions, 3-1

Define

$$\mathcal{Z}_4 = \Big\{(\theta_1,\theta_2) : \frac{5}{11} < \theta_1 \leqslant \frac{8}{17},\ \frac{1}{20}(10-19\theta_1) \leqslant \theta_2 \leqslant \frac{1}{12}(4-7\theta_1) - \varepsilon$$

$$\text{or } \frac{8}{17} < \theta_1 < \frac{10}{21},\ \frac{1}{20}(10-19\theta_1) \leqslant \theta_2 \leqslant 1 - 2\theta_1 - 2\varepsilon \Big\}.$$

Theorem (L. (2026))

Let $(\theta_1, \theta_2) \in \mathcal{Z}_4$. Then $(**)$ holds for

$$f(n) = 1_p(n),$$

and we have

$$C_1'(\theta_1,\theta_2) = C_0'(\theta_1,\theta_2) = 1.$$

Combining this with the main theorem proved in Maynard (2025a), we get

Theorem (L. (2026))

Fix $a \in \mathbb{Z}\backslash\{0\}$. Then (Bilinear) holds if Q_1 and Q_2 satisfy

$$Q_1^2 Q_2 < x^{1-\varepsilon}, \quad Q_1^7 Q_2^{12} < x^{4-\varepsilon}.$$

Proof. The Type-II range for $\mathcal{Z}_4$ is

$$\left[\varepsilon,\ \frac{1}{6}(5-8\theta_1-4\theta_2)-\varepsilon\right]\cup\left[\theta_1+\varepsilon,\ \frac{1}{2}(1-\theta_2)-\varepsilon\right]\cup\left[\frac{3}{7}+\varepsilon,\ 1-\theta-\varepsilon\right].$$

Since $2\theta_1+\theta_2<1-\varepsilon$, we have

$$\frac{5-8\theta_1-4\theta_2}{6}-\varepsilon=\frac{5-4(2\theta_1+\theta_2)}{6}-\varepsilon>\frac{1}{6}-\varepsilon>\frac{1}{7}.$$

Final decompositions, 3-1

Using Buchstab's identity, we have

$$\begin{aligned}
\psi\left(n,(2x)^{\frac{1}{2}}\right) = \psi\left(n,x^{\kappa}\right) - \sum_{\substack{n=p_1\beta \\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon}} \psi\left(\beta,x^{\kappa}\right) + \sum_{\substack{n=p_1p_2\beta \\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon \\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2\in\boldsymbol{G}_2}} \psi\left(\beta,p_2\right) \\
+ \sum_{\substack{n=p_1p_2\beta \\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon \\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2\in A}} \psi\left(\beta,x^{\kappa}\right) - \sum_{\substack{n=p_1p_2p_3\beta \\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon \\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2\in A\cup B\cup C \\ \kappa\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)}} \psi\left(\beta,p_3\right) \\
+ \sum_{\substack{n=m_1m_2m_3 \\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon \\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2\in B\cup C}} \psi\left(m_1m_2m_3,x^{\kappa}\right) - \sum_{\substack{n=m_1m_2m_3 \\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon \\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right) \\ \boldsymbol{\alpha}_2\in B\cup C \\ \Omega(m_1m_2)\geqslant 3}} \psi\left(m_1m_2m_3,x^{\kappa}\right) \\
- \sum_{\substack{n=p_1\beta \\ \frac{3}{7}+\varepsilon\leqslant\alpha_1\leqslant 1-\theta-\varepsilon}} \psi\left(\beta,p_1\right) - \sum_{\substack{n=p_1\beta \\ 1-\theta-\varepsilon<\alpha_1<\frac{1}{2}}} \psi\left(\beta,p_1\right).
\end{aligned}$$

We can give asymptotic formulas of the form $(**)$ for all terms above. The proof of Theorem is completed.

Now we focus on the region

$$\left\{(\theta_1, \theta_2) : \theta_1 \leqslant \frac{1}{3} - \varepsilon,\ \theta_2 \leqslant \frac{1}{5},\ \frac{1}{2} \leqslant \theta \leqslant \frac{11}{21} - \varepsilon\right\}.$$

Baker and Harman (1998) proved that $(**)$ holds for $f(n) = 1_p(n)$ in this region with $\theta < \frac{29}{56}$. They used Heath-Brown's identity to perform the decompositions. We shall extend the range of θ to $\frac{11}{21} - \varepsilon$ using Harman's sieve.

The Type-II range for this region is

$$\left[\varepsilon,\ \frac{1}{6}(5-8\theta_1-8\theta_2)-\varepsilon\right]\cup\left[\theta_2+\varepsilon,\ \frac{1}{6}(5-8\theta_1-4\theta_2)-\varepsilon\right]\cup\left[\theta_1+\varepsilon,\ \frac{1}{2}(1-\theta_2)-\varepsilon\right]\cup\left[\frac{3}{7}+\varepsilon,\ 1-\theta-\varepsilon\right].$$

We have

$$\kappa=\frac{5-8\theta}{6}-\varepsilon>\frac{17}{126}-\varepsilon>\frac{1}{8}+\varepsilon,$$

$$\frac{5-8\theta_1-4\theta_2}{6}-\varepsilon>\frac{5-8\left(\frac{1}{3}-\varepsilon\right)-4\left(\frac{11}{21}-\varepsilon-\frac{1}{3}+\varepsilon\right)}{6}-\varepsilon=\frac{11}{42}+\frac{1}{3}\varepsilon>\frac{11}{42}>\frac{1}{4}+2\varepsilon$$

and

$$\frac{1-\theta_2}{2}\geqslant\frac{1-\frac{1}{5}}{2}=0.4,$$

and we can simplify our Type-II range in this case to

$$\left[\varepsilon,\ \frac{17}{126}-\varepsilon\right]\cup\left[\frac{1}{5}+\varepsilon,\ \frac{11}{42}\right]\cup\left[\frac{1}{3},\ 0.4-\varepsilon\right]\cup\left[\frac{3}{7}+\varepsilon,\ 1-\theta-\varepsilon\right].$$

Final decompositions, 3-1

Using Buchstab's identity, we have

$$\psi\left(n,(2x)^{\frac{1}{2}}\right)=\psi\left(n,x^{\kappa}\right)-\sum_{\substack{n=p_1\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon}}\psi\left(\beta,x^{\kappa}\right)+\sum_{\substack{n=p_1p_2\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in\boldsymbol{G}_2}}\psi\left(\beta,p_2\right)$$

$$+\sum_{\substack{n=p_1p_2\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in A\cup B}}\psi\left(\beta,x^{\kappa}\right)-\sum_{\substack{n=p_1p_2p_3\beta\\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in A\cup B\cup C\\ \kappa\leqslant\alpha_3<\min\left(\alpha_2,\frac{1}{2}(1-\alpha_1-\alpha_2)\right)}}\psi\left(\beta,p_3\right)$$

$$+\sum_{\substack{n=m_1m_2m_3\\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in C}}\psi\left(m_1m_2m_3,x^{\kappa}\right)-\sum_{\substack{n=m_1m_2m_3\\ \kappa\leqslant\alpha_1<\frac{3}{7}+\varepsilon\\ \kappa\leqslant\alpha_2<\min\left(\alpha_1,\frac{1}{2}(1-\alpha_1)\right)\\ \boldsymbol{\alpha}_2\in C\\ \Omega(m_1m_2)\geqslant 3}}\psi\left(m_1m_2m_3,x^{\kappa}\right)$$

$$-\sum_{\substack{n=p_1\beta\\ \frac{3}{7}+\varepsilon\leqslant\alpha_1\leqslant 1-\theta-\varepsilon}}\psi\left(\beta,p_1\right)-\sum_{\substack{n=p_1\beta\\ 1-\theta-\varepsilon<\alpha_1<\frac{1}{2}}}\psi\left(\beta,p_1\right).$$

We discard the (negative) last term.

We want to show that $(**)$ holds for $f(n) =$ sums that count numbers with 4 or more prime factors. Since $\kappa > \frac{1}{8}$, we have $\Omega(n) \leqslant 7$. Assume that $\Omega(n) \geqslant 4$, and we only need to consider 8 different cases as in the proof of previous two Theorems. We shall prove that $(**)$ holds for $f(n)$ in 7 of the following 8 cases (sometimes with an error $O(\varepsilon)$), and the "loss integrals" for $C_1'(\theta_1, \theta_2)$ will only come from that remaining case and the discarded last term.

Case 1: $\Omega(n) = 7$. Suppose that $n = p_1 \cdots p_7$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_7$, while the remaining part $\alpha_i = \alpha_j$ gives a "loss" of size $O(\varepsilon)$. Now we have $\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 < \frac{4}{7}$. If $\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 \in \left[\theta + \varepsilon,\ \frac{4}{7} - \varepsilon\right]$, then $(**)$ holds for $f(n)$. Assume that $\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 < \theta + \varepsilon$. But since we have $\theta < \frac{11}{21} < \frac{10}{19} - \varepsilon$, we get

$$\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 < \theta + \varepsilon < \frac{10 - 16\theta}{3} - 4\varepsilon = 4\left(\frac{5 - 8\theta}{6} - \varepsilon\right) < \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$$

making a contradiction. Now we can assume that $\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 \in \left(\frac{4}{7} - \varepsilon,\ \frac{4}{7}\right)$, and thus $\alpha_1 + \alpha_2 + \alpha_3 \in \left(\frac{3}{7},\ \frac{3}{7} + \varepsilon\right)$. Now we have $\alpha_3 \leqslant \frac{1}{7} + \frac{1}{3}\varepsilon$ and $\alpha_4 \geqslant \frac{1}{7} - \frac{1}{4}\varepsilon$. Since $\alpha_3 > \alpha_4$, we have $\frac{1}{7} - \frac{1}{4}\varepsilon < \alpha_3 \leqslant \frac{1}{7} + \frac{1}{3}\varepsilon$, leading to a "loss" of size $O(\varepsilon)$. Hence $(**)$ holds for $f(n)$ with an error $O(\varepsilon)$ in **Case 1**.

Case 2: $\Omega(n) = 6$. Suppose that $n = p_1 \cdots p_6$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_6$, while the remaining part $\alpha_i = \alpha_j$ gives a "loss" of size $O(\varepsilon)$. Now we have $\alpha_1 + \alpha_2 > \frac{1}{3}$. If $\alpha_1 + \alpha_2 \in \left[\frac{1}{3},\ 0.4 - \varepsilon\right]$, then $(**)$ holds for $f(n)$. Otherwise we have $\alpha_1 + \alpha_2 > 0.4 - \varepsilon$. If $\alpha_1 + \alpha_2 \in \left[\frac{3}{7} + \varepsilon,\ 1 - \theta - \varepsilon\right]$, then $(**)$ holds for $f(n)$. If $\alpha_1 + \alpha_2 > 1 - \theta - \varepsilon$, we have

$$\alpha_1 + \cdots + \alpha_6 > 1 - \theta - \varepsilon + 4\left(\frac{5 - 8\theta}{6} - \varepsilon\right) > 1,$$

making a contradiction since $\theta < \frac{11}{21} < \frac{10}{19} - \varepsilon$ (see **Case 1**). Thus, we can assume that $\alpha_1 + \alpha_2 \in \left(0.4 - \varepsilon,\ \frac{3}{7} + \varepsilon\right)$.
Suppose that $\alpha_6 \geqslant \frac{1}{7} + \varepsilon$. Since $\alpha_1 > \alpha_2 > \cdots > \alpha_6 \geqslant \frac{1}{7} + \varepsilon$, we have $\alpha_1 + \alpha_2 + \alpha_3 > \frac{3}{7} + 3\varepsilon$ and $\alpha_4 + \alpha_5 + \alpha_6 > \frac{3}{7} + 3\varepsilon$. Hence $\alpha_1 + \alpha_2 + \alpha_3 \in \left[\frac{3}{7} + \varepsilon,\ \frac{4}{7} - \varepsilon\right]$. If $\alpha_1 + \alpha_2 + \alpha_3 \in \left[\frac{3}{7} + \varepsilon,\ 1 - \theta - \varepsilon\right] \cup \left[\theta + \varepsilon,\ \frac{4}{7} - \varepsilon\right]$, then $(**)$ holds for $f(n)$. Otherwise we have $\alpha_1 + \alpha_2 + \alpha_3 \in (1 - \theta - \varepsilon,\ \theta + \varepsilon)$. But since $\alpha_1 + \alpha_2 > 0.4 - \varepsilon$, we have

$$\alpha_1 + \alpha_2 + \alpha_3 > 0.4 - \varepsilon + \frac{1}{7} > 0.54 > \theta + \varepsilon,$$

making a contradiction.
Suppose that $\alpha_6 < \frac{1}{7} - 2\varepsilon$. Now we have $\alpha_1 + \alpha_2 \in \left(0.4 - \varepsilon,\ \frac{3}{7} + \varepsilon\right)$ and $\frac{1}{8} < \alpha_6 < \frac{1}{7} - 2\varepsilon$. Thus,

$$\alpha_1 + \alpha_2 + \alpha_6 \in \left(0.525 - \varepsilon,\ \frac{4}{7} - \varepsilon\right).$$

Since $\theta + \varepsilon < 0.525 - \varepsilon$, we have $(**)$ holds for $f(n)$.
The remaining part is $\frac{1}{7} - 2\varepsilon \leqslant \alpha_6 < \frac{1}{7} + \varepsilon$, leading to a "loss" of size $O(\varepsilon)$. Hence $(**)$ holds for $f(n)$ with an error $O(\varepsilon)$ in **Case 2**.

Case 3: $\Omega(n)=5$**, a product of two variables lies in** $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Suppose that $n=p_1\cdots p_5$ and $\alpha_1+\alpha_2\in(1-\theta-\varepsilon,\ \theta+\varepsilon)$ (of course, $\alpha_3+\alpha_4+\alpha_5\in(1-\theta-\varepsilon,\ \theta+\varepsilon)$ too). Without loss of generality, we further assume that $\alpha_1\geqslant\alpha_2$ and $\alpha_3\geqslant\alpha_4\geqslant\alpha_5$. Now we have $\alpha_1>\frac{1-\theta-\varepsilon}{2}$ and $\alpha_2<\frac{\theta+\varepsilon}{2}$. Since $\frac{1}{5}+\varepsilon<\frac{1-\theta-\varepsilon}{2}<\frac{\theta+\varepsilon}{2}\leqslant\frac{11}{42}$, we can assume that $\alpha_1>\frac{11}{42}-\varepsilon$ and $\alpha_2<\frac{1}{5}+\varepsilon$. Now we have $\alpha_1>(1-\theta)-\left(\frac{1}{5}+\varepsilon\right)=\frac{4}{5}-\theta-\varepsilon>\frac{4}{5}-\frac{11}{21}-2\varepsilon>0.276$. If $\alpha_1\geqslant\frac{1}{3}$, we can assume that $\alpha_1>0.4-\varepsilon$ and $\alpha_1+\alpha_2>0.4-\varepsilon+\frac{1}{8}=0.525-\varepsilon>\theta+\varepsilon$, making a contradiction with the assumption $\alpha_1+\alpha_2\in[1-\theta,\ \theta]$. Hence we also assume that $\alpha_1<\frac{1}{3}$.
Since $\alpha_3+\alpha_4+\alpha_5\in(1-\theta-\varepsilon,\ \theta+\varepsilon)$, we also have $\alpha_3+\alpha_4\geqslant\frac{2}{3}(1-\theta-\varepsilon)>\frac{20}{63}-\varepsilon$ and $\alpha_3\geqslant\frac{10}{63}-\varepsilon$. If $\alpha_3+\alpha_4\geqslant\frac{1}{3}$, then we can assume that $\alpha_3+\alpha_4>0.4-\varepsilon$. Thus we have $\alpha_3+\alpha_4+\alpha_5>0.4-\varepsilon+\frac{1}{8}=0.525-\varepsilon>\theta+\varepsilon$, contradicting with the assumption $\alpha_3+\alpha_4+\alpha_5\in(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Hence we can also assume that $\alpha_3+\alpha_4\in\left(\frac{20}{63}-\varepsilon,\ \frac{1}{3}\right)$.
Since $\alpha_1>0.276$ and $\alpha_3\geqslant\frac{10}{63}-\varepsilon$, we have $\alpha_1+\alpha_3>0.276+\frac{10}{63}-\varepsilon>\frac{3}{7}+\varepsilon$, and we can assume that $\alpha_1+\alpha_3>1-\theta-\varepsilon$. If $\alpha_1+\alpha_3\geqslant\theta+\varepsilon$, then we can assume that $\alpha_1+\alpha_3>\frac{4}{7}-\varepsilon$. Since $\alpha_1<\frac{1}{3}$, we have $\alpha_3>\frac{4}{7}-\varepsilon-\frac{1}{3}=\frac{5}{21}-\varepsilon>\frac{1}{5}+\varepsilon$, and we can assume that $\alpha_3>\frac{11}{42}-\varepsilon$. But now we have $\alpha_3+\alpha_4+\alpha_5>\frac{11}{42}-\varepsilon+2\left(\frac{17}{126}-\varepsilon\right)>0.53>\theta+\varepsilon$, making a contradiction with the assumption $\alpha_3+\alpha_4+\alpha_5\in(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Thus, we assume that $\alpha_1+\alpha_3\in(1-\theta-\varepsilon,\ \theta+\varepsilon)$ and $\alpha_2+\alpha_4+\alpha_5\in(1-\theta-\varepsilon,\ \theta+\varepsilon)$.
When $\alpha_3\geqslant\alpha_4\geqslant\alpha_5\geqslant\alpha_2$, $\alpha_3\geqslant\alpha_4\geqslant\alpha_2\geqslant\alpha_5$ or $\alpha_3\geqslant\alpha_2\geqslant\alpha_4\geqslant\alpha_5$ holds, we know that

$$\alpha_3+\alpha_4\geqslant\frac{1}{2}(1-\alpha_1)>\frac{1}{3},$$

and we can assume that $\alpha_3+\alpha_4>0.4-\varepsilon$. But now we have $\alpha_3+\alpha_4+\alpha_5>0.4-\varepsilon+\frac{1}{8}=0.525-\varepsilon>\theta+\varepsilon$, contradicting with the assumption $\alpha_3+\alpha_4+\alpha_5\in(1-\theta-\varepsilon,\ \theta+\varepsilon)$.
When $\alpha_2\geqslant\alpha_3\geqslant\alpha_4\geqslant\alpha_5$ holds, we know that

$$\alpha_2+\alpha_4\geqslant\frac{1}{2}(1-\alpha_1)>\frac{1}{3},$$

and we can assume that $\alpha_2+\alpha_4>0.4-\varepsilon$. But now we have $\alpha_2+\alpha_4+\alpha_5>0.4-\varepsilon+\frac{1}{8}=0.525-\varepsilon>\theta+\varepsilon$, contradicting with the assumption $\alpha_2+\alpha_4+\alpha_5\in(1-\theta-\varepsilon,\ \theta+\varepsilon)$.
Hence $(**)$ holds for $f(n)$ in **Case 3**.

Case 4: $\Omega(n) = 5$**, one variable lies in** $(1-\theta-\varepsilon,\ \theta+\varepsilon)$**.** Suppose that $n = p_1 \cdots p_5$ and $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_5$. Now we have $\alpha_1 > 1-\theta-\varepsilon$ and $\alpha_2 \geqslant \cdots \geqslant \alpha_5 \geqslant \frac{5-8\theta}{6} - \varepsilon$. But since $\theta < \frac{11}{21} < \frac{10}{19} - \varepsilon$, we have

$$1-\theta-\varepsilon+4\left(\frac{5-8\theta}{6}-\varepsilon\right) > 1,$$

making a contradiction. Hence **Case 4** is empty.

Case 5: $\Omega(n)=5$**, no product of variables lies in** $(1-\theta-\varepsilon,\ \theta+\varepsilon)$**.** Suppose that $n=p_1\cdots p_5$ and $\alpha_1>\alpha_2>\cdots>\alpha_5$, while the remaining part $\alpha_i=\alpha_j$ gives a "loss" of size $O(\varepsilon)$. In this case we can "view" $\left[\frac{3}{7}+\varepsilon,\ \frac{4}{7}-\varepsilon\right]$ as a "fake" Type-II range. Since we have $\theta<\frac{11}{21}<\frac{9}{17}$, the results in [Maynard (2025a), Chapter 9] can be applied here if any of the following conditions holds:
(1). $\alpha_5\geqslant\frac{1}{7}+\varepsilon$;
(2). $\alpha_3+\alpha_4+\alpha_5>\frac{4}{7}-\varepsilon$.
By Condition (1), we can assume that $\frac{1}{8}<\alpha_5<\frac{1}{7}+\varepsilon$. By Condition (2) and the "fake" Type-II range $\left[\frac{3}{7}+\varepsilon,\ \frac{4}{7}-\varepsilon\right]$, we can also assume that $\alpha_3+\alpha_4+\alpha_5<\frac{3}{7}+\varepsilon$. This means that $\alpha_1+\alpha_2>\frac{4}{7}-\varepsilon$ and $\alpha_1>\frac{2}{7}-\frac{1}{2}\varepsilon$.
If we have $\alpha_1+\alpha_5\geqslant\frac{3}{7}+\varepsilon$, then we can assume that $\alpha_1+\alpha_5>\frac{4}{7}-\varepsilon$. Since $\alpha_5<\frac{1}{7}+\varepsilon$, we have $\alpha_1>\frac{3}{7}-2\varepsilon$. If $\alpha_1\geqslant\frac{3}{7}+\varepsilon$, then we can assume that $\alpha_1>\frac{4}{7}-\varepsilon$. Now we have $1=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5>\frac{4}{7}-\varepsilon+4\cdot\frac{1}{8}>1$, making a contradiction. The remaining part is $\frac{3}{7}-2\varepsilon<\alpha_1<\frac{3}{7}+\varepsilon$, leading to a "loss" of size $O(\varepsilon)$.
Hence we can assume that $\alpha_1+\alpha_5<\frac{3}{7}+\varepsilon$, and thus $\alpha_2+\alpha_3+\alpha_4>\frac{4}{7}-\varepsilon$. Now we have $\alpha_2+\alpha_3>\frac{2}{3}(\alpha_2+\alpha_3+\alpha_4)>\frac{8}{21}-\varepsilon$. Since $\frac{1}{3}<\frac{8}{21}-\varepsilon<0.4-\varepsilon$, we can assume that $\alpha_2+\alpha_3>0.4-\varepsilon$. If $\alpha_2+\alpha_3\geqslant\frac{3}{7}+\varepsilon$, then we can assume that $\alpha_2+\alpha_3>\frac{4}{7}-\varepsilon$ and $\alpha_1+\alpha_4+\alpha_5<\frac{3}{7}+\varepsilon$. But we also have $\alpha_1+\alpha_4+\alpha_5>\frac{2}{7}-\frac{1}{2}\varepsilon+2\cdot\frac{1}{8}>\frac{3}{7}+\varepsilon$, making a contradiction. Now we assume that $\alpha_2+\alpha_3\in\left(0.4-\varepsilon,\ \frac{3}{7}+\varepsilon\right)$. Since $\frac{1}{8}<\alpha_5<\frac{1}{7}+\varepsilon$, suppose that $\frac{1}{8}<\alpha_5<\frac{1}{7}-2\varepsilon$, we have

$$\alpha_2+\alpha_3+\alpha_5\in\left(0.525-\varepsilon,\ \frac{4}{7}-\varepsilon\right).$$

Since $\theta+\varepsilon<0.525-\varepsilon$, we have $(**)$ holds for $f(n)$. The remaining part is $\frac{1}{7}-2\varepsilon<\alpha_5<\frac{1}{7}+\varepsilon$, leading to a "loss" of size $O(\varepsilon)$.
Hence $(**)$ holds for $f(n)$ with an error $O(\varepsilon)$ in **Case 5**.

Case 6: $\Omega(n)=4$, a product of two variables lies in $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Suppose that $n=p_1p_2p_3p_4$ and $\alpha_1+\alpha_2 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ (of course, $\alpha_3+\alpha_4 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ too). Without loss of generality, we further assume that $\alpha_1>\alpha_2$ and $\alpha_3>\alpha_4$, while the remaining part $\alpha_i=\alpha_j$ gives a "loss" of size $O(\varepsilon)$. Now we have $\alpha_2, \alpha_4 < \frac{\theta}{2} \leqslant \frac{11}{42}-\varepsilon$, and we can assume that $\alpha_2, \alpha_4 < \frac{1}{5}+\varepsilon$. If $\alpha_4 < \frac{1}{5}-2\varepsilon$, then we have $\alpha_2+\alpha_4 < 0.4-\varepsilon$, and we can assume that $\alpha_2+\alpha_4 < \frac{1}{3}$. If $\alpha_1 \geqslant \frac{1}{3}$, we can assume that $\alpha_1 > 0.4-\varepsilon$. Thus we have $\alpha_1+\alpha_2 > 0.4-\varepsilon+\frac{1}{8} = 0.525-\varepsilon > \theta+\varepsilon$, making a contradiction with the assumption $\alpha_1+\alpha_2 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$. Similar arguments also hold if $\alpha_3 \geqslant \frac{1}{3}$. Hence, we can assume that $\alpha_1, \alpha_3 < \frac{1}{3}$, and thus

$$1=\alpha_1+\alpha_2+\alpha_3+\alpha_4=\alpha_1+\alpha_3+(\alpha_2+\alpha_4) < \frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1,$$

making a contradiction. The remaining part is $\frac{1}{5}-2\varepsilon < \alpha_4 < \frac{1}{5}+\varepsilon$, leading to a "loss" of size $O(\varepsilon)$. Hence $(**)$ holds for $f(n)$ with an error $O(\varepsilon)$ in **Case 6**.

Case 7: $\Omega(n)=4$**, one variable lies in** $(1-\theta-\varepsilon,\ \theta+\varepsilon)$**.** Suppose that $n=p_0p_1p_2p_3$ and $\alpha_0 \geqslant \alpha_1 \geqslant \alpha_2 \geqslant \alpha_3$. Now we have $\alpha_0 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ and $\alpha_1+\alpha_2+\alpha_3 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$. We also have $\alpha_1+\alpha_3 \leqslant \frac{1}{2}$ and $\alpha_2 \leqslant \frac{1}{3}$. Note that the sum of $\psi\left(m_1m_2m_3, x^{\kappa}\right)$ only counts numbers with all prime factors smaller than $\max(m_1,m_2,m_3) \leqslant x^{\frac{3}{7}+\varepsilon}$, one can easily find that this type of n will not be counted in the sum of $\psi\left(m_1m_2m_3, x^{\kappa}\right)$ by a simple observation. Hence, this type of n will only be counted in one part of the sum of $\psi\left(\beta, p_3\right)$:

$$\sum_{\substack{n=p_1p_2p_3\beta \\ 1-\theta-\varepsilon<\alpha_1+\alpha_2+\alpha_3<\theta+\varepsilon}} \psi\left(\beta, p_3\right),$$

where β here is a large prime. Using Buchstab's identity, we have

$$\sum_{\substack{n=p_1p_2p_3\beta \\ 1-\theta-\varepsilon<\alpha_1+\alpha_2+\alpha_3<\theta+\varepsilon}} \psi\left(\beta, p_3\right) = \sum_{\substack{n=p_1p_2p_3\beta \\ 1-\theta-\varepsilon<\alpha_1+\alpha_2+\alpha_3<\theta+\varepsilon}} \psi\left(\beta, x^{\kappa}\right) - \sum_{\substack{n=p_1p_2p_3p_4\beta \\ 1-\theta-\varepsilon<\alpha_1+\alpha_2+\alpha_3<\theta+\varepsilon \\ \kappa\leqslant\alpha_4<\min\left(\alpha_3,\frac{1}{2}(1-\alpha_1-\alpha_2-\alpha_3)\right)}} \psi\left(\beta, p_4\right).$$

We know that $(**)$ holds for the first sum since $(\alpha_1+\alpha_2, \alpha_3) \in A$, and we only need to deal with the second sum that counts numbers with 5 or more prime factors, with a product of 3 variables lies in $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Now we reduce this case to **Case 3** and parts of **Cases 1 and 2** that a product of 3 variables lies in $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Since the second sum is a positive sum that cannot be dropped directly, we need $(**)$ holds for $f(n)$ in **Case 3** and parts of **Cases 1 and 2** without an error $O(\varepsilon)$. By the discussions above, we know that $(**)$ holds for $f(n)$ in **Case 3** without an error $O(\varepsilon)$, and we only need to consider the following 2 subcases.

Case 7.1: $\Omega(n)=7$, a product of three variables lies in $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Suppose that $n=p_1\cdots p_7$ and $\alpha_1+\alpha_2+\alpha_3 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ (of course, $\alpha_4+\alpha_5+\alpha_6+\alpha_7 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ too). Without loss of generality, we further assume that $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3$ and $\alpha_4 \geqslant \alpha_5 \geqslant \alpha_6 \geqslant \alpha_7$. Since $\alpha_4+\alpha_5+\alpha_6 \geqslant \frac{3}{4}(1-\theta-\varepsilon) \geqslant \frac{3}{4}\left(1-\frac{11}{21}-\varepsilon\right) > 0.35 > \frac{1}{3}$, we can assume that $\alpha_4+\alpha_5+\alpha_6 > 0.4-\varepsilon$. Now we have $1=\alpha_1+\cdots+\alpha_7=(\alpha_1+\alpha_2+\alpha_3)+(\alpha_4+\alpha_5+\alpha_6)+\alpha_7 > (1-\theta-\varepsilon)+(0.4-\varepsilon)+\left(\frac{5-8\theta}{6}-\varepsilon\right) > 1$, since $\theta < \frac{11}{21} < \frac{37}{70} - 100\varepsilon$. This makes a contradiction, and we know that $(**)$ holds for $f(n)$ in **Case 7.1**.

Case 7.2: $\Omega(n)=6$, a product of three variables lies in $(1-\theta-\varepsilon,\ \theta+\varepsilon)$. Suppose that $n=p_1\cdots p_6$ and $\alpha_1+\alpha_2+\alpha_3 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ (of course, $\alpha_4+\alpha_5+\alpha_6 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$ too). Without loss of generality, we further assume that $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3$ and $\alpha_4 \geqslant \alpha_5 \geqslant \alpha_6$. Now we have $\alpha_1+\alpha_4 \geqslant \frac{1}{3}$ and $\alpha_1+\alpha_4 \geqslant \alpha_2+\alpha_5 \geqslant \alpha_3+\alpha_6$, and we can assume that $\alpha_1+\alpha_4 > 0.4-\varepsilon$. If $\alpha_2+\alpha_5 \geqslant \frac{1}{3}$, we can assume that $\alpha_2+\alpha_5 > 0.4-\varepsilon$, and thus $1=\alpha_1+\cdots+\alpha_6 > (0.4-\varepsilon)+(0.4-\varepsilon)+2\cdot\frac{1}{8} = 1.05-2\varepsilon > 1$, making a contradiction. Now we can assume that $\alpha_2+\alpha_5 < \frac{1}{3}$, and thus $\alpha_2+\alpha_6, \alpha_3+\alpha_5 < \alpha_2+\alpha_5 < \frac{1}{3}$. If $\alpha_4+\alpha_5 < \frac{1}{3}$, we know that $\alpha_1+\alpha_3 = 1-(\alpha_2+\alpha_6)-(\alpha_4+\alpha_5) > \frac{1}{3}$, and we can assume that $\alpha_1+\alpha_3 > 0.4-\varepsilon$. But now we have $\alpha_1+\alpha_2+\alpha_3 > 0.4-\varepsilon+\frac{1}{8} = 0.525-\varepsilon > \theta+\varepsilon$, making a contradiction with the assumption $\alpha_1+\alpha_2+\alpha_3 \in (1-\theta-\varepsilon,\ \theta+\varepsilon)$. Thus, we can assume that $\alpha_4+\alpha_5 \geqslant \frac{1}{3}$. Similarly, we can assume that $\alpha_1+\alpha_2 \geqslant \frac{1}{3}$. Now we can assume that $\alpha_1+\alpha_2, \alpha_4+\alpha_5 > 0.4-\varepsilon$. But now we have $1=\alpha_1+\cdots+\alpha_6 > (0.4-\varepsilon)+(0.4-\varepsilon)+2\cdot\frac{1}{8} = 1.05-2\varepsilon > 1$, making a contradiction. Hence $(**)$ holds for $f(n)$ in **Case 7.2**.

Case 8: $\Omega(n)=4$, no product of variables lies in $[1-\theta,\ \theta]$. Suppose that $n=p_1p_2p_3p_4$ and $\alpha_1>\alpha_2>\alpha_3>\alpha_4$, while the remaining part $\alpha_i=\alpha_j$ gives a "loss" of size $O(\varepsilon)$. In this case we can "view" $\left[\frac{3}{7}+\varepsilon,\ \frac{4}{7}-\varepsilon\right]$ as a "fake" Type-II range. Since we have $\theta<\frac{11}{21}<\frac{9}{17}$, the results in Maynard (2025a) can be applied here if any of the following conditions holds:
(1). $\alpha_4 \geqslant \frac{1}{7}+\varepsilon$;
(2). $\alpha_1 \leqslant \frac{3}{7}+\varepsilon$, $\alpha_1+\alpha_4>\frac{4}{7}-\varepsilon$.
By Condition (1), we can assume that $\frac{1}{8}<\alpha_4<\frac{1}{7}+\varepsilon$. Suppose that $\alpha_1>\frac{4}{7}-\varepsilon$. Note that the sum of $\psi\left(m_1m_2m_3,x^{\kappa}\right)$ only counts numbers with all prime factors smaller than $\max(m_1,m_2,m_3)\leqslant x^{\frac{3}{7}+\varepsilon}$, one can easily find that this type of n will not be counted in the sum of $\psi\left(m_1m_2m_3,x^{\kappa}\right)$ by a simple observation. Hence, this type of n will only be counted in one part of the sum of $\psi\left(\beta,p_3\right)$:

$$\sum_{\substack{n=p_2p_3p_4\beta \\ \alpha_2+\alpha_3+\alpha_4<\frac{3}{7}+\varepsilon}} \psi\left(\beta,p_3\right),$$

where β here is a large prime.

Using Buchstab's identity twice, we have

$$
\begin{aligned}
\sum_{\substack{n=p_2p_3p_4\beta \\ \alpha_2+\alpha_3+\alpha_4<\frac{3}{7}+\varepsilon}} \psi(\beta, p_4) &= \sum_{\substack{n=p_2p_3p_4\beta \\ \alpha_2+\alpha_3+\alpha_4<\frac{3}{7}+\varepsilon}} \psi(\beta, x^{\kappa}) - \sum_{\substack{n=p_2p_3p_4p_5\beta \\ \alpha_2+\alpha_3+\alpha_4<\frac{3}{7}+\varepsilon \\ \kappa\leqslant\alpha_5<\min\left(\alpha_4, \frac{1}{2}(1-\alpha_2-\alpha_3-\alpha_4)\right)}} \psi(\beta, p_5) \\
&= \sum_{\substack{n=p_2p_3p_4\beta \\ \alpha_2+\alpha_3+\alpha_4<\frac{3}{7}+\varepsilon}} \psi(\beta, x^{\kappa}) - \sum_{\substack{n=p_2p_3p_4p_5\beta \\ \alpha_2+\alpha_3+\alpha_4<\frac{3}{7}+\varepsilon \\ \kappa\leqslant\alpha_5<\min\left(\alpha_4, \frac{1}{2}(1-\alpha_2-\alpha_3-\alpha_4)\right)}} \psi(\beta, x^{\kappa}) \\
&\quad + \sum_{\substack{n=p_2p_3p_4p_5p_6\beta \\ \alpha_2+\alpha_3+\alpha_4<\frac{3}{7}+\varepsilon \\ \kappa\leqslant\alpha_5<\min\left(\alpha_3, \frac{1}{2}(1-\alpha_2-\alpha_3-\alpha_4)\right) \\ \kappa\leqslant\alpha_6<\min\left(\alpha_5, \frac{1}{2}(1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)\right)}} \psi(\beta, p_6).
\end{aligned}
$$

Note that we have $\alpha_2+\alpha_3+\alpha_4 < \frac{3}{7}+\varepsilon$ and $\alpha_5 < \alpha_4 < \frac{1}{7}+\varepsilon$. Since we have $\left(\frac{3}{7}+\varepsilon, \frac{1}{7}+\varepsilon, \varepsilon\right) \in \boldsymbol{S}_3$ when $\theta < \frac{11}{20}$, by the Type-II range $\left[\varepsilon, \frac{17}{126}-\varepsilon\right]$, $(**)$ holds for the first and the second sums. Now we only need to deal with the last sum that counts numbers with 6 or more prime factors. Since this sum is negative, we know that $(**)$ holds for $f(n)$ with an error $O(\varepsilon)$ in **Case 8** if $\alpha_4 < \frac{1}{7}+\varepsilon$ and $\alpha_1 > \frac{4}{7}-\varepsilon$ by the discussions in **Cases 1 and 2**.

Now by Condition (2) and the "fake" Type-II range $\left[\frac{3}{7}+\varepsilon,\ \frac{4}{7}-\varepsilon\right]$, we only need to consider the case $\alpha_1+\alpha_4<\frac{3}{7}+\varepsilon$. Now we have $\alpha_2+\alpha_3>\frac{4}{7}-\varepsilon$, and thus $\alpha_1+\alpha_2>\alpha_1+\alpha_3>\alpha_2+\alpha_3>\frac{4}{7}-\varepsilon$. However, we cannot show that $(**)$ holds for all $f(n)$ in **Case 8**. An obvious counterexample is

$$\boldsymbol{\alpha}_4=\left(\frac{2}{7}+3\varepsilon,\frac{2}{7}+2\varepsilon,\frac{2}{7}+\varepsilon,\frac{1}{7}-6\varepsilon\right).$$

By a simple observation, one can easily find that such $f(n)$ that $(**)$ does not hold will only be negative parts of the sums of $\psi\left(\beta,p_3\right)$ and $\psi\left(m_1m_2m_3,x^{\kappa}\right)$ since $\Omega(n)=4$. We can discard them in the upper bound case. We also have the following conditions on $n=p_1p_2p_3p_4$ counted in such $f(n)$:

$$\frac{5-8\theta}{6}-\varepsilon\leqslant\alpha_4<\frac{1}{7}+\varepsilon,\quad \frac{3}{7}+\varepsilon>\alpha_1+\alpha_4>\alpha_2+\alpha_4>\alpha_3+\alpha_4,$$

$$\frac{4}{7}-\varepsilon<\alpha_2+\alpha_3<\alpha_1+\alpha_3<\alpha_1+\alpha_2.$$

By the conditions above, we can deduce an upper bound for α_1, α_2 and α_3:

$$\alpha_3 < \alpha_2 < \alpha_1 \leqslant \frac{3}{7} - \alpha_4 \leqslant \frac{3}{7} - \frac{5-8\theta}{6} + \varepsilon.$$

Since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$, by the conditions above, we can deduce a lower bound for α_1, α_2 and α_3:

$$\alpha_1 > \alpha_2 > \alpha_3 = 1 - (\alpha_1 + \alpha_2 + \alpha_4) > 1 - 2\left(\frac{3}{7} - \frac{5-8\theta}{6} + \varepsilon\right) - \frac{1}{7} = \frac{5-8\theta}{3} - 2\varepsilon.$$

Now, we know that such part of the sum of $\psi(\beta, p_3)$ that $(**)$ does not hold is no more than

$$\sum_{\substack{n = p_1 p_2 p_3 \beta \\ \frac{5-8\theta}{3} - 2\varepsilon < \alpha_1 \leqslant \frac{3}{7} - \frac{5-8\theta}{6} + \varepsilon \\ \frac{5-8\theta}{3} - 2\varepsilon < \alpha_2 \leqslant \frac{3}{7} - \frac{5-8\theta}{6} + \varepsilon \\ \frac{5-8\theta}{6} - \varepsilon \leqslant \alpha_3 < \frac{1}{7} + \varepsilon \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3}} \psi\left(\beta, \left(\frac{2x}{p_1 p_2 p_3}\right)^{\frac{1}{2}}\right) = \sum_{\substack{n = p_1 p_2 p_3 p_4 \\ \frac{5-8\theta}{3} - 2\varepsilon < \alpha_1 \leqslant \frac{3}{7} - \frac{5-8\theta}{6} + \varepsilon \\ \frac{5-8\theta}{3} - 2\varepsilon < \alpha_2 \leqslant \frac{3}{7} - \frac{5-8\theta}{6} + \varepsilon \\ \frac{5-8\theta}{6} - \varepsilon \leqslant \alpha_3 < \frac{1}{7} + \varepsilon \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3}} 1.$$

For such part of the sum of $\psi\left(m_1 m_2 m_3, x^{\kappa}\right)$ that $(**)$ does not hold, we have an extra condition $\alpha_1 \geqslant \tau = \frac{3(1-\theta)}{5} - \varepsilon$ since $\theta < \frac{11}{21}$. Note that we have $\frac{3(1-\theta)}{5} > \frac{5-8\theta}{3}$ when $\theta > \frac{16}{31}$, and we can assume that $\theta \geqslant \frac{29}{56} > \frac{16}{31}$. Let $i, j \in \{1, 2, 3\}$ and $i \neq j$. By above conditions and the condition $\boldsymbol{\alpha}_2 \in C$, we know that $m_1 \neq p_i p_j$ and $m_2 \neq p_i p_j$. Since we have $\Omega(m_1 m_2) \geqslant 3$, we know that $m_3 \neq p_i p_j$ and $m_3 \neq p_i p_4$. Now the only possible cases are $m_1 = p_i p_4$ or $m_2 = p_i p_4$, and such part of the sum of $\psi\left(m_1 m_2 m_3, x^{\kappa}\right)$ that $(**)$ does not hold is no more than

$$\sum_{\substack{n=p_1 p_2 p_3 p_4 \\ \tau \leqslant \alpha_1 \leqslant \frac{3}{7} - \frac{5-8\theta}{6} + \varepsilon \\ \frac{5-8\theta}{3} - 2\varepsilon < \alpha_2 \leqslant \frac{3}{7} - \frac{5-8\theta}{6} + \varepsilon \\ \frac{5-8\theta}{6} - \varepsilon \leqslant \alpha_3 < \frac{1}{7} + \varepsilon \\ (\alpha_1 + \alpha_3, \alpha_2) \in C \text{ or } (\alpha_1, \alpha_2 + \alpha_3) \in C \\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3}} 1.$$

Finally, we get the following theorem by combining all the 8 cases above.

Theorem (L. (2026))

Define

$$\boldsymbol{D}_5 = \boldsymbol{D}_5(\theta) = \left\{ \boldsymbol{\alpha}_3 : \frac{5-8\theta}{3} \leqslant \alpha_1 \leqslant \frac{3}{7} - \frac{5-8\theta}{6},\ \frac{5-8\theta}{3} \leqslant \alpha_2 \leqslant \frac{3}{7} - \frac{5-8\theta}{6},\ \frac{5-8\theta}{6} \leqslant \alpha_3 \leqslant \frac{1}{7},\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3 \right\}$$

$$\boldsymbol{D}_6 = \boldsymbol{D}_6(\theta) = \left\{ \boldsymbol{\alpha}_3 : \tau \leqslant \alpha_1 \leqslant \frac{3}{7} - \frac{5-8\theta}{6},\ \frac{5-8\theta}{3} \leqslant \alpha_2 \leqslant \frac{3}{7} - \frac{5-8\theta}{6},\ \frac{5-8\theta}{6} \leqslant \alpha_3 \leqslant \frac{1}{7},\ \boldsymbol{\alpha}_3 \notin \boldsymbol{G}_3, \right.$$
$$\left. (\alpha_1+\alpha_3, \alpha_2) \in C \text{ or } (\alpha_1, \alpha_2+\alpha_3) \in C \right\}.$$

Suppose that we have

$$\theta_1 \leqslant \frac{1}{3} - \varepsilon, \quad \theta_2 \leqslant \frac{1}{5} \quad \text{and} \quad \frac{29}{56} \leqslant \theta_1 + \theta_2 \leqslant \frac{11}{21} - \varepsilon.$$

Then we have

$$C_1'(\theta_1, \theta_2) \leqslant 1 + \int_{1-\theta}^{\frac{1}{2}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt + I_5 + I_6 + O(\varepsilon) = 1 + \log\left(\frac{\theta}{1-\theta}\right) + I_5 + I_6 + O(\varepsilon),$$

where

$$I_5 = \int_{(t_1,t_2,t_3)\in \boldsymbol{D}_5} \frac{1}{t_1 t_2 t_3 (1-t_1-t_2-t_3)} dt_3 dt_2 dt_1$$

and

$$I_6 = \int_{(t_1,t_2,t_3)\in \boldsymbol{D}_6} \frac{1}{t_1 t_2 t_3 (1-t_1-t_2-t_3)} dt_3 dt_2 dt_1.$$

The above arguments also show that

$$C_1'(\theta_1,\theta_2) \leqslant 1 + \int_{1-\theta}^{\frac{1}{2}} \frac{\omega\left(\frac{1-t}{t}\right)}{t^2} dt + O(\varepsilon) = 1 + \log\left(\frac{\theta}{1-\theta}\right) + O(\varepsilon)$$

if $\theta_1 \leqslant \frac{1}{3} - \varepsilon$, $\theta_2 \leqslant \frac{1}{5}$ and $\theta_1 + \theta_2 \leqslant \frac{29}{56} - \varepsilon$, since we have $\kappa > \frac{1}{7}$ in this case and $(**)$ holds for $f(n)$ with an error $O(\varepsilon)$ in **Case 8**.
By this Theorem, we get

$$C_1'(0.32, 0.20) \leqslant 1 + \log\left(\frac{13}{12}\right) + 10^{-5} + O(\varepsilon) < 1.0801$$

and

$$C_1'(0.33, 0.19) \leqslant 1 + \log\left(\frac{13}{12}\right) + 10^{-5} + O(\varepsilon) < 1.0801.$$

$\theta_2 \backslash \theta_1$	0.26	0.27	0.28	0.29	0.30	0.31	0.32	0.33	0.34	0.35	0.36	0.37	0.38	0.39	0.40
0.28	1.5641	1.9602	2.3301	—	—	—	—	—	—	—	—	—	—	—	—
0.27	1.4833	1.6632	1.9602	2.1636	—	—	—	—	—	—	—	—	—	—	—
0.26	1.0830	1.4833	1.5641	1.7620	2.1185	—	—	—	—	—	—	—	—	—	—
0.25	1.0401	1.0805	1.4115	1.4795	1.8120	2.0732	—	—	—	—	—	—	—	—	—
0.24	1	1	1.0801	1.3433	1.5084	1.7924	2.0309	—	—	—	—	—	—	—	—
0.23	1	1	1	1.0801	1.3938	1.4975	1.7977	2.0268	—	—	—	—	—	—	—
0.22	1	1	1	1	1.0810	1.4315	1.5398	1.8781	2.0672	—	—	—	—	—	—
0.21	1	1	1	1	1	1.0817	1.4644	1.5605	1.9836	2.1666	—	—	—	—	—
0.20	1	1	1	1	1	1	1.0801	1.4925	1.5939	2.0468	2.2182	—	—	—	—
0.19	1	1	1	1	1	1	1	1.0801	1.5168	1.6144	2.0702	2.2490	—	—	—
0.18	1	1	1	1	1	1	1	1	1.0805	1.5407	1.6360	2.0714	2.2632	—	—
0.17	1	1	1	1	1	1	1	1	1.0401	1.0825	1.5635	1.6513	2.1058	2.2974	—
0.16	1	1	1	1	1	1	1	1	1	1.0405	1.0837	1.1274	1.6749	2.1300	2.3159
0.15	1	1	1	1	1	1	1	1	1	1	1.0401	1.0827	1.1549	1.6998	2.1540
0.14	1	1	1	1	1	1	1	1	1	1	1	1.0401	1.0810	1.1368	1.7389
0.13	1	1	1	1	1	1	1	1	1	1	1	1	1.0407	1.0811	1.1274
0.12	1	1	1	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0418	1.0808
0.11	1	1	1	1	1	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.0401

$\theta_2 \backslash \theta_1$	0.41	0.42	0.43	0.44	0.45	0.46	0.47	0.48	0.49	0.50	0.51	0.52	0.53	0.54	0.55
0.15	2.3347	—	—	—	—	—	—	—	—	—	—	—	—	—	—
0.14	2.1645	2.3628	—	—	—	—	—	—	—	—	—	—	—	—	—
0.13	1.7782	2.2026	2.3900	—	—	—	—	—	—	—	—	—	—	—	—
0.12	1.1249	1.8132	2.2388	2.3930	—	—	—	—	—	—	—	—	—	—	—
0.11	1.0807	1.1266	1.8455	2.2375	2.3891	—	—	—	—	—	—	—	—	—	—
0.10	1.0408	1.0812	1.1320	1.8464	2.2577	2.3885	—	—	—	—	—	—	—	—	—
0.09	1	1.0401	1.0828	1.1386	1.8463	2.2082	2.3888	—	—	—	—	—	—	—	—
0.08	1	1	1	1.0847	1.1467	1.8476	2.2383	2.3907	—	—	—	—	—	—	—
0.07	1	1	1	1	1	1.1491	1.8498	2.2535	2.4108	—	—	—	—	—	—
0.06	1	1	1	1	1	1	1.7993	1.8561	2.2675	2.4313	—	—	—	—	—
0.05	1	1	1	1	1	1	1	1.7183	1.8642	2.2987	2.4608	—	—	—	—
0.04	1	1	1	1	1	1	1	$1+\varepsilon$	1.7618	1.8806	2.3027	2.4918	—	—	—
0.03	1	1	1	1	1	1	1	1	1.0452	1.8057	1.8963	2.3271	2.5324	—	—
0.02	1	1	1	1	1	1	1	1	$1+\varepsilon$	1.1022	1.8111	1.9316	2.3483	2.5345	—
0.01	1	1	1	1	1	1	1	1	1	1.0458	1.7453	1.8181	1.9658	2.3633	2.5345

Upper Bounds for $C_1'(\theta_1, \theta_2)$ $(0.5 < \theta \leqslant 0.56)$

We give lower bounds for $C_0'(\theta_1, \theta_2)$.
Known results:
(1). $C_0'(\theta_1, \theta_2) = C_0'(\theta_2, \theta_1)$;
(2). $C_0'(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $\theta_1 + \theta_2 \leqslant 0.5 - \varepsilon$;
(3). $C_0'(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $2\theta_1 + \theta_2 \leqslant 1 - \varepsilon$ and $7\theta_1 + 12\theta_2 \leqslant 4 - \varepsilon$;
(4). $C_0'(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $\theta_1 < \frac{1}{3}$, $\theta_2 < \frac{1}{5}$ and $\theta_1 + \theta_2 < \frac{29}{56}$;
(5). $C_0'(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $\theta_1 + 3\theta_2 < 1$, $\theta_1 + \theta_2 < \frac{29}{56}$ and $\theta_2 < \max\left(\frac{1-2\theta_1}{2}, \frac{2-2\theta_1}{5}\right)$;
(6). $C_0'(\theta_1, \theta_2) = 1$ for all θ_1, θ_2 satisfy $\theta_1 + 3\theta_2 < 1$, $\theta_1 + \theta_2 < \frac{29}{56}$, $4\theta_1 + \theta_2 < \frac{403}{266}$ and $\frac{7}{4}\theta_1 + \theta_2 < \frac{403}{532}$;
(7). $C_0'(\theta_1, \theta_2) \geqslant C_0(\theta_1, \theta_2) \geqslant C_0(\theta_1 + \theta_2)$ for $0.5 \leqslant \theta_1 + \theta_2 \leqslant 1$.

By the discussions in Final decompositions 1-2, we need to show that $(**)$ holds for

$$f(n)=\sum_{\substack{n=p_1\beta\\ 1-\theta-\varepsilon<\alpha_1<\frac{1}{2}}}\psi(\beta,p_1).$$

We need

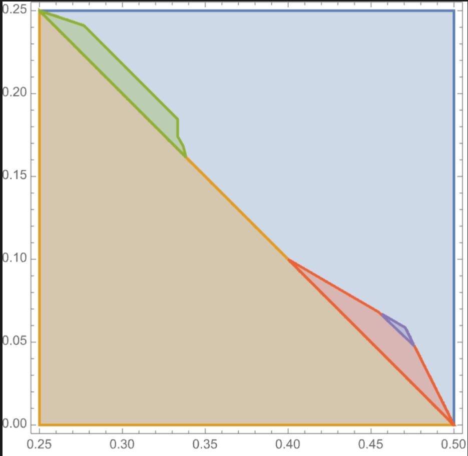
$$2\theta_1+\theta_2\leqslant 1-\varepsilon \quad \text{and} \quad 7\theta_1+12\theta_2\leqslant 4-\varepsilon$$

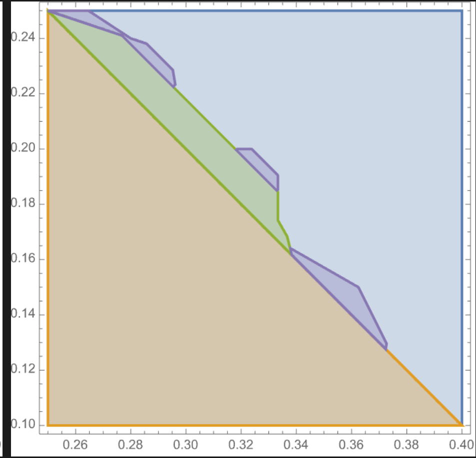
to show that $(**)$ holds for this sum, and we have $C_0'(\theta_1,\theta_2)=1$ in this case. By the discussions in Final decompositions 2-2, we cannot get any nontrivial lower bound for $C_0'(\theta_1,\theta_2)$ if either

$$2\theta_1+\theta_2\leqslant 1-\varepsilon \quad \text{or} \quad 7\theta_1+12\theta_2\leqslant 4-\varepsilon$$

is not fulfilled.

Plots


0.25
0.20
0.15
0.10
0.05
0.00
0.25 0.30 0.35 0.40 0.45 0.50


0.24
0.22
0.20
0.18
0.16
0.14
0.12
0.10
0.26 0.28 0.30 0.32 0.34 0.36 0.38 0.40

Now we focus on the trilinear case, or the 3-factored case with divisor-bounded coefficients. We want to show that

$$\sum_{\substack{q_1\leqslant Q_1\\ q_2\leqslant Q_2\\ q_3\leqslant Q_3\\ (q_1q_2q_3,a)=1}} \lambda_{1,q_1}\lambda_{2,q_2}\lambda_{3,q_3}\left(\sum_{\substack{n\leqslant x\\ n\equiv a(\bmod q_1q_2q_3)}} \rho_j(n)-\frac{1}{\varphi(q_1q_2q_3)}\sum_{\substack{n\leqslant x\\ (n,q_1q_2q_3)=1}} \rho_j(n)\right)\ll \frac{x}{(\log x)^A}$$

for some functions $\rho_j(n)$ $(j\in\{0,1\})$ satisfy

$$\rho_0(n)\leqslant 1_p(n)\leqslant \rho_1(n).$$

We want the minorants are always positive and the majorants are not too large. That is, we have

$$\sum_{n\leqslant x}\rho_0(n)\geqslant (1+o(1))\frac{C_0'(\theta_1,\theta_2,\theta_3)x}{\log x}\quad\text{and}\quad \sum_{n\leqslant x}\rho_1(n)\leqslant (1+o(1))\frac{C_1'(\theta_1,\theta_2,\theta_3)x}{\log x}$$

for two functions $C_0'(\theta_1,\theta_2,\theta_3)$ and $C_1'(\theta_1,\theta_2,\theta_3)$ satisfy $0<C_0'(\theta_1,\theta_2,\theta_3)\leqslant 1$ and $C_1'(\theta_1,\theta_2,\theta_3)\geqslant 1$.

Now we need the following type of asymptotic formulas.

$$\sum_{\substack{q_1 \sim Q_1 \\ q_2 \sim Q_2 \\ q_3 \sim Q_3 \\ (q_1 q_2 q_3, a)=1}} \lambda_{1,q_1} \lambda_{2,q_2} \lambda_{3,q_3} \left(\sum_{\substack{n \sim x \\ n \equiv a (\bmod q_1 q_2 q_3)}} f(n) - \frac{1}{\varphi(q_1 q_2 q_3)} \sum_{\substack{n \sim x \\ (n, q_1 q_2 q_3)=1}} f(n) \right) \ll \frac{x}{(\log x)^A}. \quad (***)$$

Note that asymptotic formulas of the forms (C), (C_2) and $(**)$ also imply asymptotic formulas of the form $(***)$.

Arithmetic information

Type-II information: Asymptotic formula of the form $(***)$ for Type-II sums. Let $M_1M_2 \asymp x$. Suppose that a_{2,m_2} satisfies **Conditions A and B**.

$$\sum_{\substack{q_1\sim Q_1\\ q_2\sim Q_2\\ q_3\sim Q_3\\ (q_1q_2q_3,a)=1}} \lambda_{1,q_1}\lambda_{2,q_2}\lambda_{3,q_3}\left(\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ m_1m_2\equiv a(\bmod q_1q_2q_3)}} a_{1,m_1}a_{2,m_2} - \frac{1}{\varphi(q_1q_2q_3)}\sum_{\substack{m_1\sim M_1\\ m_2\sim M_2\\ (m_1m_2,q_1q_2q_3)=1}} a_{1,m_1}a_{2,m_2}\right) \ll \frac{x}{(\log x)^A}.$$

(Lichtman (2022))

$$Q_1^7Q_2^{12}Q_3^{10} < x^{4-20\varepsilon},\ Q_2 < Q_1Q_3^3,\ Q_1x^{\varepsilon} < M_2 < Q_1^{-1}Q_3^{-2}x^{1-6\varepsilon}.$$

Define

$$\boldsymbol{k} = \boldsymbol{k}(\theta_1, \theta_2, \theta_3) = \{(s,t) : 1 - \theta_1 - \theta_2 - \theta_3 - \varepsilon < s < \theta_1 + \theta_2 + \theta_3 + \varepsilon\},$$
$$\boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{K}}(\theta_1, \theta_2, \theta_3) = \{\boldsymbol{\alpha}_j : \boldsymbol{\alpha}_j \text{ partitions into } \boldsymbol{k}\}.$$

Let $P_1 P_2 \cdots P_j \asymp x$ and $P_1 \geqslant P_2 \geqslant \cdots \geqslant P_j \geqslant x^{\frac{1}{7}+10\varepsilon}$. Suppose that

$$\theta_2 < \theta_1 + 3\theta_3, \quad \theta_1 + 2\theta_3 < \frac{1}{2} - 20\varepsilon, \quad 2\theta_1 + \theta_2 + \theta_3 < 1 - 20\varepsilon \quad \text{and} \quad 7\theta_1 + 12\theta_2 + 10\theta_3 < 4 - 20\varepsilon.$$

Then

$$f(n) = \sum_{\substack{n = p_1 \cdots p_j \\ p_i \sim P_i,\ 1 \leqslant i \leqslant j \\ \boldsymbol{\alpha}_j \in \boldsymbol{\mathcal{K}}}} 1$$

has an asymptotic formula of the form $(***)$.

Known results:

(1.1). $C_1'(\theta_1,\theta_2,\theta_3)=C_1'(\theta_1,\theta_3,\theta_2)=C_1'(\theta_2,\theta_1,\theta_3)=C_1'(\theta_2,\theta_3,\theta_1)=C_1'(\theta_3,\theta_1,\theta_2)=C_1'(\theta_3,\theta_2,\theta_1)$;
(1.2). $C_0'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_3,\theta_2)=C_0'(\theta_2,\theta_1,\theta_3)=C_0'(\theta_2,\theta_3,\theta_1)=C_0'(\theta_3,\theta_1,\theta_2)=C_0'(\theta_3,\theta_2,\theta_1)$;
(2). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1+\theta_2+\theta_3\leqslant 0.5-\varepsilon$;
(3). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1+\theta_2+\theta_3=0.5+r$, $0<r<0.001$, $40r<\theta_2<\frac{1}{20}-7r$ and $\frac{1}{10}-\theta_2+12r<\theta_3<\frac{1}{10}-\frac{3}{5}\theta_2-4r$;
(4.1). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1<\frac{1}{3}$, $\theta_2+\theta_3<\frac{1}{5}$ and $\theta_1+\theta_2+\theta_3<\frac{29}{56}$;
(4.2). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1+\theta_3<\frac{1}{3}$, $\theta_2<\frac{1}{5}$ and $\theta_1+\theta_2+\theta_3<\frac{29}{56}$;
(5.1). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1+3\theta_2+3\theta_3<1$, $\theta_1+\theta_2+\theta_3<\frac{29}{56}$ and $\theta_2+\theta_3<\max\left(\frac{1-2\theta_1}{2},\frac{2-2\theta_1}{5}\right)$;
(5.2). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1+3\theta_2+\theta_3<1$, $\theta_1+\theta_2+\theta_3<\frac{29}{56}$ and $\theta_2<\max\left(\frac{1-2\theta_1-2\theta_3}{2},\frac{2-2\theta_1-2\theta_3}{5}\right)$;
(6.1). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1+3\theta_2+3\theta_3<1$, $\theta_1+\theta_2+\theta_3<\frac{29}{56}$, $4\theta_1+\theta_2+\theta_3<\frac{403}{266}$ and $\frac{7}{4}\theta_1+\theta_2+\theta_3<\frac{403}{532}$;
(6.2). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1+3\theta_2+\theta_3<1$, $\theta_1+\theta_2+\theta_3<\frac{29}{56}$, $4\theta_1+\theta_2+4\theta_3<\frac{403}{266}$ and $\frac{7}{4}\theta_1+\theta_2+\frac{7}{4}\theta_3<\frac{403}{532}$;
(7.1). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ for all $\theta_1,\theta_2,\theta_3$ satisfy $2\theta_1+\theta_2+\theta_3\leqslant 1-\varepsilon$ and $7\theta_1+12\theta_2+12\theta_3\leqslant 4-\varepsilon$;
(7.2). $C_1'(\theta_1,\theta_2,\theta_3)=C_0'(\theta_1,\theta_2,\theta_3)=1$ for all $\theta_1,\theta_2,\theta_3$ satisfy $2\theta_1+\theta_2+2\theta_3\leqslant 1-\varepsilon$ and $7\theta_1+12\theta_2+7\theta_3\leqslant 4-\varepsilon$;
(8.1). $C_1'(\theta_1,\theta_2,\theta_3)\leqslant\min\left(C_1'(\theta_1,\theta_2+\theta_3),C_1'(\theta_2,\theta_1+\theta_3),C_1'(\theta_3,\theta_1+\theta_2),C_1'(\theta_1+\theta_2,\theta_3),C_1'(\theta_1+\theta_3,\theta_2),C_1'(\theta_2+\theta_3,\theta_1)\right)$ for $0.5\leqslant\theta_1+\theta_2\leqslant 1$;
(8.2). $C_0'(\theta_1,\theta_2,\theta_3)\geqslant\max\left(C_0'(\theta_1,\theta_2+\theta_3),C_0'(\theta_2,\theta_1+\theta_3),C_0'(\theta_3,\theta_1+\theta_2),C_0'(\theta_1+\theta_2,\theta_3),C_0'(\theta_1+\theta_3,\theta_2),C_0'(\theta_2+\theta_3,\theta_1)\right)$ for $0.5\leqslant\theta_1+\theta_2\leqslant 1$;
(9) $C_1'(\theta_1,\theta_2,\theta_3)\leqslant 1+\varepsilon$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1+\theta_2+\theta_3=0.5$;
(10.1). $C_1'(\theta_1,\theta_2,\theta_3)\leqslant 1+\varepsilon$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1\leqslant 0.5$ and $\theta_2+\theta_3=\min\left(1-2\theta_1,\frac{4-7\theta_1}{12}\right)$;
(10.2). $C_1'(\theta_1,\theta_2,\theta_3)\leqslant 1+\varepsilon$ for all $\theta_1,\theta_2,\theta_3$ satisfy $\theta_1+\theta_3\leqslant 0.5$ and $\theta_2=\min\left(1-2\theta_1-2\theta_3,\frac{4-7\theta_1-7\theta_3}{12}\right)$.

Final decompositions, 4

Define

$$\boldsymbol{Y} = \boldsymbol{E}_{0201} \cup \boldsymbol{E}_{0301} \cup \boldsymbol{E}_{0401} \cup \boldsymbol{E}_{0501} \cup \boldsymbol{E}_{0601} \cup \boldsymbol{E}_{0801} \cup \boldsymbol{E}_{0901},$$

$$\begin{aligned}\boldsymbol{J}' = \Bigg\{(\theta_1,\theta_2) : (\theta_1,\theta_2) \in \boldsymbol{U};\ & \theta_1+\theta_2 \leqslant \frac{1}{2}-\varepsilon \\ & \text{or } 2\theta_1+\theta_2 \leqslant 1-\varepsilon,\ 7\theta_1+12\theta_2 \leqslant 4-\varepsilon \\ & \text{or } \theta_1 < \frac{1}{3},\ \theta_2 < \frac{1}{5},\ \theta_1+\theta_2 < \frac{29}{56} \\ & \text{or } \theta_1+3\theta_2 < 1,\ \theta_1+\theta_2 < \frac{29}{56},\ \theta_2 < \max\left(\frac{1-2\theta_1}{2}, \frac{2-2\theta_1}{5}\right) \\ & \text{or } \theta_1+3\theta_2 < 1,\ \theta_1+\theta_2 < \frac{29}{56},\ 4\theta_1+\theta_2 < \frac{403}{266},\ \frac{7}{4}\theta_1+\theta_2 < \frac{403}{532}\Bigg\},\end{aligned}$$

$$\boldsymbol{K} = \left\{(\theta_1,\theta_2,\theta_3) : \theta_2 < \theta_1+3\theta_3,\ \theta_1+2\theta_3 < \frac{1}{2}-\varepsilon,\ 2\theta_1+\theta_2+\theta_3 < 1-\varepsilon,\ 7\theta_1+12\theta_2+10\theta_3 < 4-\varepsilon\right\},$$

$$\begin{aligned}\boldsymbol{\mathcal{W}} = \{(\theta_1,\theta_2,\theta_3) : & (\theta_1,\theta_2,\theta_3) \text{ partitions into } \boldsymbol{Y},\ (\theta_1,\theta_2,\theta_3) \text{ partitions into } \boldsymbol{T}, \\ & (\theta_1,\theta_2,\theta_3) \text{ does not partition into } \boldsymbol{J}',\ (\theta_1,\theta_2,\theta_3) \in \boldsymbol{K}\Big\}.\end{aligned}$$

The explicit definition of the region $\boldsymbol{\mathcal{W}}$ is very complicated (about 50 pages).

Theorem (L. (2026))

Fix $a \in \mathbb{Z}\backslash\{0\}$. Let $Q_i = x^{\theta_i}$. Suppose that $(\theta_1, \theta_2, \theta_3) \in \boldsymbol{\mathcal{W}}$. Then we have

$$\sum_{\substack{q_1 \leqslant Q_1 \\ q_2 \leqslant Q_2 \\ q_3 \leqslant Q_3 \\ (q_1 q_2 q_3, a)=1}} \lambda_{1,q_1} \lambda_{2,q_2} \lambda_{3,q_3} \left(\pi(x; q_1 q_2 q_3, a) - \frac{\pi(x)}{\varphi(q_1 q_2 q_3)} \right) \ll \frac{x}{(\log x)^A}.$$

Since

$$\frac{17}{32} > \frac{9}{17}, \quad \left(\frac{15}{32} + \varepsilon, \frac{1}{16} - 3\varepsilon \right) \in \boldsymbol{E}_{0901} \cap \boldsymbol{T} \quad \text{and} \quad \left(\frac{15}{32} + \varepsilon, \frac{3}{64} - 2\varepsilon, \frac{1}{64} - \varepsilon \right) \in \boldsymbol{K},$$

we have

$$\left(\frac{15}{32} + \varepsilon, \frac{3}{64} - 2\varepsilon, \frac{1}{64} - \varepsilon \right) \in \boldsymbol{\mathcal{W}}.$$

Hence, this Theorem generalizes the main result of Lichtman (2022).

Proof. One of the available Type-II ranges for $\boldsymbol{Y}$ is

$$\left[\varepsilon,\ \frac{1}{6}(5-8\theta_1-4\theta_2)-\varepsilon\right].$$

When $(\theta_1,\theta_2,\theta_3)$ partitions into $\boldsymbol{Y}$, then at least one of $(\theta_1+\theta_2,\theta_3)$, $(\theta_1+\theta_3,\theta_2)$, $(\theta_2+\theta_3,\theta_1)$, $(\theta_1,\theta_2+\theta_3)$, $(\theta_2,\theta_1+\theta_3)$ and $(\theta_3,\theta_1+\theta_2)$ is in $\boldsymbol{Y}$. Thus, we can assume that $(\theta_1,\theta_2,\theta_3)$ partitions into $(s,t)\in\boldsymbol{Y}$. By the definition of the region $\boldsymbol{Y}$, we know that $2s+t\leqslant 1-\varepsilon$. Also, for $(s,t)\in\boldsymbol{Y}$ we have a Type-II range

$$\left[\varepsilon,\ \frac{1}{6}(5-8s-4t)-\varepsilon\right].$$

Since the available Type-II range starts from ε and $(\theta_1,\theta_2,\theta_3)$ partitions into $\boldsymbol{T}$, we can use the new three-dimensional Harman's sieve. Moreover, we have

$$\frac{5-8s-4t}{6}-\varepsilon\geqslant\frac{5-4(1-\varepsilon)}{6}-\varepsilon>\frac{1}{6}-\varepsilon>\frac{1}{7}+100\varepsilon,$$

and the Type-II range can be simplified to $\left[\varepsilon,\ \frac{1}{6}-\varepsilon\right]$.

Write $\kappa = \frac{1}{6} - \varepsilon$. Since $(\theta_1, \theta_2, \theta_3) \in \boldsymbol{K}$ implies $\theta = \theta_1 + \theta_2 + \theta_3 \leqslant \frac{17}{32} - \varepsilon$, we can decompose our $\psi\left(n, (2x)^{\frac{1}{2}}\right)$ in a way similar to the decompositions in the proof of Theorem $\boldsymbol{\mathcal{Z}_4}$. By the discussions in the proof of Theorem $\boldsymbol{\mathcal{Z}_4}$, we only need to show that $(***)$ holds for $f(n) =$ sums that count numbers with 4 or more prime factors. Now we can use the arguments in [Maynard (2025a), Chapter 9] to complete the proof of this Theorem. We only need the following two modifications:

In *Case 1* $(J = 4)$ we have

$$Q_1 Q_2 K L^4 < x^{\frac{17}{32}+\varepsilon} P_1 P_4 (P_2 P_3)^{\frac{3}{2}} \ll x^{\frac{17}{32}+\varepsilon+1+\frac{3}{14}+\varepsilon} = x^{\frac{391}{224}+2\varepsilon} < x^{\frac{57}{32}-10\varepsilon}.$$

In *Case 2* $(J = 5)$ we have

$$Q_1 Q_2 K L^4 < x^{\frac{17}{32}+\varepsilon} \frac{(P_1 P_2 P_3 P_4 P_5)^{\frac{4}{3}}}{(P_4 P_5)^{\frac{1}{3}}} \ll x^{\frac{17}{32}+\varepsilon+\frac{4}{3}-\frac{2}{7}\cdot\frac{1}{3}} = x^{\frac{1189}{672}+\varepsilon} < x^{\frac{57}{32}-10\varepsilon}.$$

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Thank you!