# PRIMES IN ALMOST ALL SHORT INTERVALS

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Abstract. The author sharpens a result of Jia (1996), showing that the interval $[n, n + n^{\frac{1}{21.5}+\varepsilon}]$ contains prime numbers for almost all $n$. Watt's mean value bound, a delicate sieve decomposition and more accurate estimates for integrals are used to good effect.

Contents



## 1. Introduction

One of the famous topics in number theory is to find prime numbers in short intervals. In 1937, Cramér [7] conjectured that every interval $[n, n+f(n)(\log n)^2]$ contains prime numbers for some $f(n) \to 1$ as $n \to \infty$. The Riemann Hypothesis implies that for all large $n$, the interval $[n, n + n^{\theta}]$ contains $\sim n^{\theta}(\log n)^{-1}$ prime numbers for every $\frac{1}{2} + \varepsilon \leqslant \theta \leqslant 1$. The first unconditional result of this asymptotic formula was proved by Hoheisel [19] in 1930 with $\theta = 1 - \frac{1}{33000}$. After the works of Hoheisel [19], Heilbronn [18], Chudakov [6], Ingham [21] and Montgomery [37], Huxley [20] proved in 1972 that the above asymptotic formula holds when $\theta > \frac{7}{12}$ by his zero density estimate. Very recently, Guth and Maynard [9] improved this to $\theta > \frac{17}{30}$ by a new zero density estimate.

In 1979, Iwaniec and Jutila [22] first introduced a sieve method into this problem. They established a lower bound with correct order (instead of an asymptotic formula) with $\theta = \frac{13}{23}$. After that breakthrough, many improvements were made and the value of $\theta$ was reduced successively to

$$\frac{5}{9} = 0.5556,\ \frac{11}{20} = 0.5500,\ \frac{17}{31} = 0.5484,\ \frac{23}{42} = 0.5476,$$
$$\frac{1051}{1920} = 0.5474,\ \frac{6}{11} = 0.5455 \text{ and } \frac{7}{13} = 0.5385$$

by Iwaniec and Jutila [22], Heath–Brown and Iwaniec [16], Pintz [39] [40], Iwaniec and Pintz [23], Mozzochi [38] and Lou and Yao [34] [35] [36] respectively. In 1996, Baker and Harman [2] presented an alternative

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approach to this problem. They used the alternative sieve developed by Harman [11] [12] to reduce $\theta$ to 0.535. Finally, Baker, Harman and Pintz [4] further developed this sieve process and combined it with Watt's theorem and showed $\theta = 0.525$.

However, if we only consider the prime numbers in "almost all" intervals instead of "all" intervals, the intervals will be much shorter than $n^{0.525}$. In 1943, under the Riemann Hypothesis, Selberg [44] showed that Cramér's interval contains primes for almost all $n$ if $f(n) \to \infty$ as $n \to \infty$. In the same paper, he also showed unconditionally that the interval $[n, n + n^{\frac{19}{77}+\varepsilon}]$ contains prime numbers for almost all $n$. In 1971, Montgomery [37] improved the exponent $\frac{19}{77}$ to $\frac{1}{5}$ with an asymptotic formula. The zero density estimate of Huxley [20] gives the exponent $\frac{1}{6}$ with an asymptotic formula, and the best asymptotic result now is also due to Guth and Maynard [9], where they proved the exponent $\frac{2}{15}$.

In 1982, Harman [10] used his alternative sieve method to showed that the interval $[n, n+n^{\frac{1}{10}+\varepsilon}]$ contains prime numbers for almost all $n$. His method can only provide a lower bound instead of an asymptotic formula. The exponent $\frac{1}{10}$ was reduced successively to

$$\frac{1}{12} = 0.0833,\ \frac{14}{159} = 0.0881,\ \frac{1}{13} = 0.0769,\ \frac{17}{227} = 0.0749,\ \frac{1}{13.5} = 0.0740,$$

$$\frac{1}{14} = 0.0714,\ \frac{1}{15} = 0.0667,\ \frac{1}{16} = 0.0625,\ \frac{1}{18} = 0.0556 \text{ and } \frac{1}{20} = 0.0500$$

by Harman [11] (and Heath–Brown [15]), Lou and Yao [33], Jia [25] [26], Lou and Yao [33], Li [31], Jia [24] (and Watt [46]), Li [32], Baker, Harman and Pintz [3], Wong [47] (and Jia [28], Harman [[14], Chapter 9]) and Jia [27] respectively. As the strongest result above, Jia [27] used many powerful tools and arguments in his proof, both analytic and combinatorial, and these are extremely complicated. In this paper, we further develop the sieve machinery used in [27] and obtain the following result.

**Theorem 1.1.** *The interval* $[n, n + n^{\frac{1}{21.5}+\varepsilon}]$ *contains prime numbers for almost all* $n$. *Specifically, suppose that* $B$ *is a sufficiently large positive constant,* $\varepsilon$ *is a sufficiently small positive constant and* $X$ *is sufficiently large. Then for positive integers* $n \in [X, 2X]$, *except for* $O\left(X(\log X)^{-B}\right)$ *values, the interval* $[n, n+n^{\frac{1}{21.5}+\varepsilon}]$ *contains* $\gg n^{1/21.5+\varepsilon}(\log n)^{-1}$ *prime numbers.*

Throughout this paper, we always suppose that $B$ is a sufficiently large positive constant, $\varepsilon$ is a sufficiently small positive constant, $X$ is sufficiently large and $x \in [X, 2X]$. The letter $p$, with or without subscript, is reserved for prime numbers. Let $c_0$, $c_1$ and $c_2$ denote positive constants which may have different values at different places, and we write $m \sim M$ to mean that $c_1 M < m \leqslant c_2 M$. Let $\varepsilon_1 = \varepsilon^2$, $\delta = \varepsilon^{1/3}$ and $\eta = \frac{1}{2} X^{-\frac{20.5}{21.5}+\varepsilon}$. We use $M(s)$, $N(s)$ and some other capital letters to denote the Dirichlet polynomials

$$M(s) = \sum_{m \sim M} a(m) m^{-s}, \quad N(s) = \sum_{n \sim N} b(n) n^{-s}$$

where $a(m)$, $b(n)$ are complex numbers with $a(m) = O(1)$ and $b(n) = O(1)$. We define the boolean function as

$$\texttt{Boole}[\mathbf{X}] = \begin{cases} 1 & \text{if } \mathbf{X} \text{ is true,} \\ 0 & \text{if } \mathbf{X} \text{ is false.} \end{cases}$$

## 2. An outline of the proof

Let $p_j = X^{t_j}$ in the following sections and put

$$\mathcal{A} = \{n : x < n \leqslant x + \eta x\}, \quad \mathcal{B} = \{n : x < n \leqslant 2x\},$$

$$\mathcal{A}_d = \{a : a \in \mathcal{A},\ d \mid a\}, \quad P(z) = \prod_{p<z} p, \quad S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z))=1}} 1.$$

Then we have

$$\pi(x + \eta x) - \pi(x) = S\left(\mathcal{A}, (2X)^{\frac{1}{2}}\right). \tag{1}$$

In order to prove Theorem 1.1, we only need to show that $S\left(\mathcal{A},(2X)^{\frac{1}{2}}\right)>0$. By Buchstab's identity, we have

$$\begin{aligned}
S\left(\mathcal{A},(2X)^{\frac{1}{2}}\right)=&\ S\left(\mathcal{A},X^{\frac{79}{817}}\right)-\sum_{\frac{79}{817}\leqslant t_1<\frac{1}{2}}S\left(\mathcal{A}_{p_1},X^{\frac{79}{817}}\right)\\
&+\sum_{\substack{\frac{79}{817}\leqslant t_1<\frac{1}{2}\\ \frac{79}{817}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)}}S\left(\mathcal{A}_{p_1p_2},p_2\right)\\
=&\ S_1-S_2+S_3. \qquad (2)
\end{aligned}$$

Our aim is to show that the sparser set $\mathcal{A}$ contains the expected proportion of primes compared to the bigger set $\mathcal{B}$, which requires us to decompose $S\left(\mathcal{A},(2X)^{\frac{1}{2}}\right)$ and prove asymptotic formulas (for almost $x\in[X,2X]$ except for $O\left(X(\log X)^{-B}\right)$ values) of the form

$$S\left(\mathcal{A},z\right)=\eta(1+o(1))S\left(\mathcal{B},z\right) \qquad (3)$$

for some parts of it, and drop the other positive parts. In Sections 3 and 4 we provide some arithmetic information of Dirichlet polynomials, and we shall use them to prove the asymptotic formulas for type–I terms $S\left(\mathcal{A}_{p_1\ldots p_n},X^{c_0}\right)$ in Section 5 and the asymptotic formulas for type–II terms $S\left(\mathcal{A}_{p_1\ldots p_n},p_n\right)$ in Section 6. In Section 7 we will make further use of Buchstab's identity to decompose $S\left(\mathcal{A},(2X)^{\frac{1}{2}}\right)$ and prove Theorem 1.1.

## 3. Arithmetic Information I

In the following two sections we provide some arithmetic information (i.e. mean value bounds for some Dirichlet polynomials) which will help us prove the asymptotic formulas for sieve functions. In this section we only use the classical mean value estimate and Halász method. In next section we will provide a crucial result which will help us give asymptotic formulas for even more terms.

**Lemma 3.1.** *Suppose that $MH=X$ where $M(s)$ is a Dirichlet polynomial, $H(s)=\sum_{h\sim H}\Lambda(h)h^{-s}$ and $X^{\delta}\ll H\ll X^{\frac{79}{817}}$. Let $b=1+\frac{1}{\log X}$, $T_0=(\log X)^{B/\varepsilon}$, then for $T_0\leqslant T\leqslant X$ we have*

$$\left(\min\left(\eta,\frac{1}{T}\right)\right)^2\int_T^{2T}|M(b+it)H(b+it)|^2dt\ll\eta^2(\log x)^{-10B}.$$

*Proof.* The proof is similar to that of [[27], Lemma 1]. Let $s=b+it$ and by the zero–free region of the $\zeta$ function, for $|t|\leqslant 2X$ we have

$$H(s)=\frac{(c_2H)^{1-s}-(c_1H)^{1-s}}{1-s}+O\left((\log x)^{-2B/\varepsilon}\right). \qquad (4)$$

So, for $T_0\leqslant|t|\leqslant 2X$ we have $H(s)\ll(\log x)^{-B/\varepsilon}$. According to the discussion in , there are $O\left((\log X)^2\right)$ sets $S(V,W)$, where $S(V,W)$ is the set of $t_k (k=1,\ldots,K)$ with the property $|t_r-t_s|\geqslant 1$ $(r\neq s)$. Moreover,

$$V\leqslant M^{\frac{1}{2}}\left|M\left(b+it_k\right)\right|<2V,\quad W\leqslant H^{\frac{1}{2}}\left|H\left(b+it_k\right)\right|<2W,$$

where $X^{-1}\leqslant M^{-\frac{1}{2}}V, X^{-1}\leqslant H^{-\frac{1}{2}}W$ and $V\ll M^{\frac{1}{2}}, W\ll H^{\frac{1}{2}}(\log x)^{-B/\varepsilon}$. Then we have

$$\int_T^{2T}|M(b+it)H(b+it)|^2dt\ll V^2W^2x^{-1}(\log x)^2|S(V,W)|+O\left(x^{-2\varepsilon_1}\right), \qquad (5)$$

where $S(V,W)$ is one of sets with the above properties.

Assume $X^{\frac{1}{k+1}}\leqslant H<X^{\frac{1}{k}}$, where $k$ is a positive integer, $k\geqslant 10$ and $k\delta\ll 1$. Applying the mean value estimate (see Lemma 7 of [25]) to $M(s)$ and $H^k(s)$, we have

$$|S(V,W)|\ll V^{-2}(M+T)(\log x)^d,$$
$$|S(V,W)|\ll W^{-2k}\left(H^k+T\right)(\log x)^d,$$

where $d = c/\delta^2$. Applying the Halász method (see Lemma 7 of [25]) to $M(s)$ and $H^k(s)$, we have

$$|S(V,W)| \ll \left(V^{-2}M + V^{-6}MT\right)(\log x)^d,$$
$$|S(V,W)| \ll \left(W^{-2k}H^k + W^{-6k}H^kT\right)(\log x)^d.$$

Thus,

$$V^2W^2|S(V,W)| \ll V^2W^2F(\log x)^d,$$

where

$$F = \min\left\{V^{-2}(M+T), V^{-2}M + V^{-6}MT, W^{-2k}(H^k+T), W^{-2k}H^k + W^{-6k}H^kT\right\}.$$

It will be proved that

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 V^2W^2F \ll \eta^2 x(\log x)^{-B/\varepsilon}. \tag{6}$$

We consider four cases.

(a) $F \leqslant 2V^{-2}M, F \leqslant 2W^{-2k}H^k$. Then

$$\begin{aligned} V^2W^2F &\ll V^2W^2 \min\left\{V^{-2}M, W^{-2k}H^k\right\} \\ &\leqslant V^2W^2\left(V^{-2}M\right)^{1-\frac{1}{2k}}\left(W^{-2k}H^k\right)^{\frac{1}{2k}} \\ &= V^{\frac{1}{k}}WM^{1-\frac{1}{2k}}H^{\frac{1}{2}} \\ &\ll x(\log x)^{-B/\varepsilon} \end{aligned}$$

and so

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 V^2W^2F \ll \eta^2 x(\log x)^{-B/\varepsilon}.$$

(b) $F > 2V^{-2}M, F > 2W^{-2k}H^k$. Then

$$\begin{aligned} V^2W^2F &\ll V^2W^2 \min\left\{V^{-2}T, V^{-6}MT, W^{-2k}T, W^{-6k}H^kT\right\} \\ &\leqslant V^2W^2\left(V^{-2}\right)^{1-\frac{3}{2k}}\left(V^{-6}M\right)^{\frac{1}{2k}}\left(W^{-2k}\right)^{\frac{1}{k}}T \\ &= M^{\frac{1}{2k}}T. \end{aligned}$$

Since $k \geqslant 10$, we have $H \geqslant X^{\frac{1}{k+1}} \geqslant X^{1-\frac{k}{11}}, M^{\frac{1}{2k}} \ll X^{\frac{1}{22}}$, and so

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 V^2W^2F \ll \frac{\eta}{T}x^{\frac{1}{22}}T \ll \eta^2 x^{1-\varepsilon_1}.$$

(c) $F \leqslant 2V^{-2}M, F > 2W^{-2k}H^k$. Then

$$\begin{aligned} V^2W^2F &\ll V^2W^2 \min\left\{V^{-2}M, W^{-2k}T, W^{-6k}H^kT\right\} \\ &\leqslant V^2W^2\left(V^{-2}M\right)^{1-\frac{1}{3k}}\left(W^{-6k}H^kT\right)^{\frac{1}{3k}} \\ &\ll MH^{\frac{1}{3}}T^{\frac{1}{3k}}, \end{aligned}$$

since $V \ll M^{\frac{1}{2}}$. As $H \geqslant X^{\frac{1}{k+1}} \geqslant X^{\frac{21}{44k}+\varepsilon}$ and $M \leqslant X^{1-\frac{21}{44k}}$, we have

$$\begin{aligned} \left(\min\left(\eta, \frac{1}{T}\right)\right)^2 V^2W^2F &\ll \eta^{2-\frac{1}{3k}}\left(\frac{1}{T}\right)^{\frac{1}{3k}}(MH)^{\frac{1}{3}}M^{\frac{2}{3}}T^{\frac{1}{3k}} \\ &\ll x^{\frac{1}{3}-\frac{20.5}{21.5}(2-\frac{1}{3k})+\frac{2}{3}(1-\frac{21}{44k})} \\ &\ll \eta^2 x^{1-\varepsilon_1}. \end{aligned}$$

(d) $F > 2V^{-2}M, F \leqslant 2W^{-2k}H^k$. Then

$$\begin{aligned} V^2W^2F &\ll V^2W^2 \min\left\{V^{-2}T, V^{-6}MT, W^{-2k}H^k\right\} \\ &\leqslant V^2W^2\left(V^{-2}T\right)^{1-\frac{3}{2k}}\left(V^{-6}MT\right)^{\frac{1}{2k}}\left(W^{-2k}H^k\right)^{\frac{1}{k}} \\ &= M^{\frac{1}{2k}}HT^{1-\frac{1}{k}}. \end{aligned}$$

If $k \geqslant 11$, then $H \leqslant X^{\frac{1}{k}} \leqslant X^{1-\frac{21(k-1)}{11(2k-1)}}, M \gg X^{\frac{21(k-1)}{11(2k-1)}}$, and so

$$\begin{aligned}\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 V^2W^2F &\ll \eta^{1+\frac{1}{k}}\left(\frac{1}{T}\right)^{1-\frac{1}{k}}(MH)\,M^{-\frac{2k-1}{2k}}T^{1-\frac{1}{k}} \\ &\ll x^{1-\frac{20.5}{21.5}(1+\frac{1}{k})-\frac{21(k-1)}{22k}} \\ &\ll \eta^2 x^{1-\varepsilon_1}.\end{aligned}$$

If $k = 10$, then $X^{\frac{1}{11}} \leqslant H \ll X^{\frac{79}{817}}, M \gg X^{\frac{738}{817}}$, and so

$$\begin{aligned}\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 V^2W^2F &\ll \eta^{\frac{11}{10}}\left(\frac{1}{T}\right)^{\frac{9}{10}}(MH)\,M^{-\frac{19}{20}}T^{\frac{9}{10}} \\ &\ll x^{1-\frac{11}{10}\cdot\frac{20.5}{21.5}-\frac{19}{20}\cdot\frac{738}{817}} \\ &\ll \eta^2 x^{1-\varepsilon_1}.\end{aligned}$$

Combining the above cases, Lemma 3.1 is proved. □

**Lemma 3.2.** *Suppose that $MHK = X$ where $M(s), H(s)$ and $K(s)$ are Dirichlet polynomials and $G(s) = M(s)H(s)K(s)$. Let $b = 1 + \frac{1}{\log X}$, $T_0 = (\log X)^{B/\varepsilon}$. Assume further that for $T_0 \leqslant |t| \leqslant 2X, M(b+it) \ll (\log x)^{-B/\varepsilon}$ and $H(b+it) \ll (\log x)^{-B/\varepsilon}$. Moreover, suppose that $M$ and $H$ satisfy one of the following 9 conditions:*

*(1) $MH \ll X^{\frac{673}{1247}},\ X^{\frac{82}{473}} \ll H,\ M^{29}H^{-1} \ll X^{\frac{427}{43}},\ X^{\frac{246}{817}} \ll M,\ M^{-1}H^{29} \ll X^{\frac{263}{43}},\ X^{\frac{246}{43}} \ll M^{12}H^{11}$;*

*(2) $MH \ll X^{\frac{223}{387}},\ M^{29}H^{19} \ll X^{\frac{632}{43}},\ X^{\frac{328}{387}} \ll M^2H,\ M^2H^{11} \ll X^{\frac{120}{43}},\ X^{\frac{1230}{43}} \ll M^{58}H^{49}$;*

*(3) $MH \ll X^{\frac{268}{473}},\ X^{\frac{41}{215}} \ll H,\ M^6H \ll X^{\frac{94}{43}},\ X^{\frac{82}{301}} \ll M \ll X^{\frac{16}{43}},\ MH^8 \ll X^{\frac{98}{43}},\ X^{\frac{123}{43}} \ll M^6H^5$;*

*(4) $MH \ll X^{\frac{571}{817}},\ X^{\frac{82}{473}} \ll H,\ M^{12}H \ll X^{\frac{270}{43}},\ X^{\frac{574}{1247}} \ll M,\ M^{-1}H^{19} \ll X^{\frac{161}{43}},\ X^{\frac{328}{43}} \ll M^{12}H^{11}$;*

*(5) $M^2H \ll X^{\frac{446}{387}},\ M^{58}H^9 \ll X^{\frac{1264}{43}},\ X^{\frac{164}{387}} \ll M,\ MH^5 \ll X^{\frac{60}{43}},\ X^{\frac{615}{43}} \ll M^{29}H^{10}$;*

*(6) $X^{\frac{27}{43}} \ll MH \ll X^{\frac{219}{301}},\ X^{\frac{41}{215}} \ll H,\ M^6H \ll X^{\frac{135}{43}},\ X^{\frac{205}{473}} \ll M,\ M^{-1}H^7 \ll X^{\frac{55}{43}},\ X^{\frac{164}{43}} \ll M^6H^5$;*

*(7) $MH \ll X^{\frac{268}{473}},\ M^{35}H^{23} \ll X^{\frac{767}{43}},\ X^{\frac{410}{473}} \ll M^2H,\ M^2H^{13} \ll X^{\frac{130}{43}},\ X^{\frac{1476}{43}} \ll M^{70}H^{59}$;*

*(8) $MH \ll X^{\frac{313}{559}},\ M^{41}H^{27} \ll X^{\frac{902}{43}},\ X^{\frac{492}{559}} \ll M^2H,\ M^2H^{15} \ll X^{\frac{138}{43}},\ X^{\frac{1722}{43}} \ll M^{82}H^{69}$;*

*(9) $M^2H \ll X^{\frac{536}{473}},\ M^{70}H^{11} \ll X^{\frac{1534}{43}},\ X^{\frac{205}{473}} \ll M,\ MH^6 \ll X^{\frac{65}{43}},\ X^{\frac{738}{43}} \ll M^{35}H^{12}$.*

*Then for $T_0 \leqslant T \leqslant X$, we have*

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |G(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}. \tag{7}$$

*Proof.* The proof is similar to that of [[27], Lemmas 5,7,9,10,12]. Using the method of Lemma 3.1, we only need to show that for $T = 1/\eta = 2X^{\frac{20.5}{21.5}-\varepsilon}$,

$$I = \int_T^{2T} |G(b+it)|^2 dt \ll (\log x)^{-10B}. \tag{8}$$

Assuming the condition (1). By applying the mean value estimate and Halász method to $M^3(s), H^5(s)$ and $K^2(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F(\log x)^c$$

where

$$\begin{aligned}F = \min\{&V^{-6}\left(M^3+T\right), V^{-6}M^3 + V^{-18}M^3T, W^{-10}\left(H^5+T\right), \\ &W^{-10}H^5 + W^{-30}H^5T, U^{-4}\left(K^2+T\right), U^{-4}K^2 + U^{-12}K^2T\}.\end{aligned}$$

We consider 8 cases:

(a) $F \leqslant 2V^{-6}M^3, F \leqslant 2W^{-10}H^5, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F &\ll U^2V^2W^2 \min\{V^{-4}W^{-2}M^2H, W^{-10}H^5, U^{-4}K^2\} \\ &\leqslant U^2V^2W^2\left(V^{-4}W^{-2}M^2H\right)^{\frac{3}{8}}\left(W^{-10}H^5\right)^{\frac{1}{8}}\left(U^{-4}K^2\right)^{\frac{1}{2}} \\ &= V^{\frac{1}{2}}M^{\frac{3}{4}}HK\end{aligned}$$

$$\ll x(\log x)^{-11B}.$$

(b) $F \leqslant 2V^{-6}M^3, F \leqslant 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}M^3, W^{-10}H^5, U^{-4}T, U^{-12}K^2T\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-6}M^3\right)^{\frac{1}{3}} \left(W^{-10}H^5\right)^{\frac{1}{5}} \left(U^{-4}T\right)^{\frac{9}{20}} \left(U^{-12}K^2T\right)^{\frac{1}{60}} \\
&= T^{\frac{7}{15}} MHK^{\frac{1}{30}} \\
&= T^{\frac{7}{15}} (MHK)^{\frac{1}{30}} \left(M^{\frac{29}{30}} H^{\frac{29}{30}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $MH \ll X^{\frac{673}{1247}}$ is required.

(c) $F \leqslant 2V^{-6}M^3, F > 2W^{-10}H^5, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}M^3, W^{-10}T, W^{-30}H^5T, U^{-4}K^2\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-6}M^3\right)^{\frac{1}{3}} \left(W^{-10}T\right)^{\frac{3}{20}} \left(W^{-30}H^5T\right)^{\frac{1}{60}} \left(U^{-4}K^2\right)^{\frac{1}{2}} \\
&= T^{\frac{1}{6}} MH^{\frac{1}{12}} K \\
&= T^{\frac{1}{6}} (MHK) H^{-\frac{11}{12}} \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $X^{\frac{82}{473}} \ll H$ is required.

(d) $F \leqslant 2V^{-6}M^3, F > 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}M^3, W^{-10}T, W^{-30}H^5T, U^{-4}T, U^{-12}K^2T\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-6}M^3\right)^{\frac{1}{3}} \left(W^{-10}T\right)^{\frac{1}{5}} \left(U^{-4}T\right)^{\frac{9}{20}} \left(U^{-12}K^2T\right)^{\frac{1}{60}} \\
&= T^{\frac{2}{3}} MK^{\frac{1}{30}} \\
&= T^{\frac{2}{3}} (MHK)^{\frac{1}{30}} \left(M^{\frac{29}{30}} H^{-\frac{1}{30}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $M^{29}H^{-1} \ll X^{\frac{427}{43}}$ is required.

(e) $F > 2V^{-6}M^3, F \leqslant 2W^{-10}H^5, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}T, V^{-18}M^3T, W^{-10}H^5, U^{-4}K^2\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-6}T\right)^{\frac{17}{60}} \left(V^{-18}M^3T\right)^{\frac{1}{60}} \left(W^{-10}H^5\right)^{\frac{1}{5}} \left(U^{-4}K^2\right)^{\frac{1}{2}} \\
&= T^{\frac{3}{10}} M^{\frac{1}{20}} HK \\
&= T^{\frac{3}{10}} (MHK) M^{-\frac{19}{20}} \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $X^{\frac{246}{817}} \ll M$ is required.

(f) $F > 2V^{-6}M^3, F \leqslant 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}T, V^{-18}M^3T, W^{-10}H^5, U^{-4}T, U^{-12}K^2T\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-6}T\right)^{\frac{1}{3}} \left(W^{-10}H^5\right)^{\frac{1}{5}} \left(U^{-4}T\right)^{\frac{9}{20}} \left(U^{-12}K^2T\right)^{\frac{1}{60}} \\
&= T^{\frac{4}{5}} HK^{\frac{1}{30}} \\
&= T^{\frac{4}{5}} (MHK)^{\frac{1}{30}} \left(M^{-\frac{1}{30}} H^{\frac{29}{30}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $M^{-1}H^{29} \ll X^{\frac{263}{43}}$ is required.

(g) $F > 2V^{-6}M^3, F > 2W^{-10}H^5, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}T, V^{-18}M^3T, W^{-10}T, W^{-30}H^5T, U^{-4}K^2\right\} \\ &\leqslant U^2V^2W^2\left(V^{-6}T\right)^{\frac{1}{3}}\left(W^{-10}T\right)^{\frac{3}{20}}\left(W^{-30}H^5T\right)^{\frac{1}{60}}\left(U^{-4}K^2\right)^{\frac{1}{2}} \\ &= T^{\frac{1}{2}}H^{\frac{1}{12}}K \\ &= T^{\frac{1}{2}}(MHK)\left(M^{-1}H^{-\frac{11}{12}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $X^{\frac{246}{43}} \ll M^{12}H^{11}$ is required.

(h) $F > 2V^{-6}M^3, F > 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}, V^{-18}M^3, W^{-10}, W^{-30}H^5, U^{-4}, U^{-12}K^2\right\}T \\ &\leqslant U^2V^2W^2\left(V^{-6}\right)^{\frac{170}{60}}\left(V^{-18}M^3\right)^{\frac{1}{60}}\left(W^{-10}\right)^{\frac{1}{5}}\left(U^{-4}\right)^{\frac{1}{2}}T \\ &= TM^{\frac{1}{20}} \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $M \ll X^{\frac{5025}{13717}}$ and $TM^{\frac{1}{20}} \ll X^{\frac{20.5}{21.5}+\frac{1}{20}\cdot\frac{5025}{13717}} \ll X^{\frac{53321}{54868}}$ are used. The former one can be obtained by using $MH \ll X^{\frac{673}{1247}}$ and $X^{\frac{82}{473}} \ll H$.

Assuming the condition (2). By applying the mean value estimate and Halász method to $M^2(s)H(s), H^5(s)$ and $K^2(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F(\log x)^c$$

where

$$\begin{aligned} F = \min\Big\{&V^{-4}W^{-2}\left(M^2H+T\right), V^{-4}W^{-2}M^2H + V^{-12}W^{-6}M^2HT, \\ &W^{-10}\left(H^5+T\right), W^{-10}H^5+W^{-30}H^5T, U^{-4}\left(K^2+T\right), U^{-4}K^2+U^{-12}K^2T\Big\}. \end{aligned}$$

We consider 8 cases:

(a) $F \leqslant 2V^{-4}W^{-2}M^2H, F \leqslant 2W^{-10}H^5, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}M^2H, W^{-10}H^5, U^{-4}K^2\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}W^{-2}M^2H\right)^{\frac{3}{8}}\left(W^{-10}H^5\right)^{\frac{1}{8}}\left(U^{-4}K^2\right)^{\frac{1}{2}} \\ &= V^{\frac{1}{2}}M^{\frac{3}{4}}HK \\ &\ll x(\log x)^{-11B}. \end{aligned}$$

(b) $F \leqslant 2V^{-4}W^{-2}M^2H, F \leqslant 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}M^2H, W^{-10}H^5, U^{-4}T, U^{-12}K^2T\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}W^{-2}M^2H\right)^{\frac{1}{2}}\left(W^{-10}H^5\right)^{\frac{1}{10}}\left(U^{-4}T\right)^{\frac{7}{20}}\left(U^{-12}K^2T\right)^{\frac{1}{20}} \\ &= T^{\frac{2}{5}}MHK^{\frac{1}{10}} \\ &= T^{\frac{2}{5}}(MHK)^{\frac{1}{10}}\left(M^{\frac{9}{10}}H^{\frac{9}{10}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $MH \ll X^{\frac{223}{387}}$ is required.

(c) $F \leqslant 2V^{-4}W^{-2}M^2H, F > 2W^{-10}H^5, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}M^2H, W^{-10}T, W^{-30}H^5T, U^{-4}K^2\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}W^{-2}M^2H\right)^{\frac{1}{2}}\left(U^{-4}K^2\right)^{\frac{1}{2}} \\ &= WMH^{\frac{1}{2}}K \\ &\ll x(\log x)^{-11B}. \end{aligned}$$

(d) $F \leqslant 2V^{-4}W^{-2}M^2H, F > 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}M^2H, W^{-10}T, W^{-30}H^5T, U^{-4}T, U^{-12}K^2T\right\} \\
&\leqslant U^2V^2W^2\left(V^{-4}W^{-2}M^2H\right)^{\frac{1}{2}}\left(W^{-30}H^5T\right)^{\frac{1}{30}}\left(U^{-4}T\right)^{\frac{9}{20}}\left(U^{-12}K^2T\right)^{\frac{1}{10}} \\
&= T^{\frac{1}{2}}MH^{\frac{2}{3}}K^{\frac{1}{30}} \\
&= T^{\frac{1}{2}}(MHK)^{\frac{1}{30}}\left(M^{\frac{29}{30}}H^{\frac{19}{30}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $M^{29}H^{19} \ll X^{\frac{632}{43}}$ is required.

(e) $F > 2V^{-4}W^{-2}M^2H, F \leqslant 2W^{-10}H^5, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-10}H^5, U^{-4}K^2\right\} \\
&\leqslant U^2V^2W^2\left(V^{-4}W^{-2}T\right)^{\frac{7}{20}}\left(V^{-12}W^{-6}M^2HT\right)^{\frac{1}{20}}\left(W^{-10}H^5\right)^{\frac{1}{10}}\left(U^{-4}K^2\right)^{\frac{1}{2}} \\
&= T^{\frac{2}{5}}M^{\frac{1}{10}}H^{\frac{11}{20}}K \\
&= T^{\frac{2}{5}}(MHK)\left(M^{-\frac{9}{10}}H^{-\frac{9}{20}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $X^{\frac{328}{387}} \ll M^2H$ is required.

(f) $F > 2V^{-4}W^{-2}M^2H, F \leqslant 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-10}H^5, U^{-4}T, U^{-12}K^2T\right\} \\
&\leqslant U^2V^2W^2\left(V^{-4}W^{-2}T\right)^{\frac{7}{20}}\left(V^{-12}W^{-6}M^2HT\right)^{\frac{1}{20}}\left(W^{-10}H^5\right)^{\frac{1}{10}}\left(U^{-4}T\right)^{\frac{1}{2}} \\
&= T^{\frac{9}{10}}\left(M^{\frac{1}{10}}H^{\frac{11}{20}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $M^2H^{11} \ll X^{\frac{122}{43}}$ is required. This can be obtained by using $M^2H^{11} \ll X^{\frac{120}{43}}$.

(g) $F > 2V^{-4}W^{-2}M^2H, F > 2W^{-10}H^5, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-10}T, W^{-30}H^5T, U^{-4}K^2\right\} \\
&\leqslant U^2V^2W^2\left(V^{-4}W^{-2}T\right)^{\frac{9}{20}}\left(V^{-12}W^{-6}M^2HT\right)^{\frac{1}{60}}\left(W^{-30}H^5T\right)^{\frac{1}{30}}\left(U^{-4}K^2\right)^{\frac{1}{2}} \\
&= T^{\frac{1}{2}}M^{\frac{1}{30}}H^{\frac{11}{60}}K \\
&= T^{\frac{1}{2}}(MHK)\left(M^{-\frac{29}{30}}H^{-\frac{49}{60}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $X^{\frac{1230}{43}} \ll M^{58}H^{49}$ is required.

(h) $F > 2V^{-4}W^{-2}M^2H, F > 2W^{-10}H^5, F > 2U^{-4}K^2$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}, V^{-12}W^{-6}M^2H, W^{-10}, W^{-30}H^5, U^{-4}, U^{-12}K^2\right\}T \\
&\leqslant U^2V^2W^2\left(V^{-4}W^{-2}\right)^{\frac{9}{20}}\left(V^{-12}W^{-6}M^2H\right)^{\frac{1}{60}}\left(W^{-30}H^5\right)^{\frac{1}{30}}\left(U^{-4}\right)^{\frac{1}{2}}T \\
&= T\left(M^{\frac{1}{30}}H^{\frac{11}{60}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $M^2H^{11} \ll X^{\frac{120}{43}}$ is required.

Assuming the condition (3). By applying the mean value estimate and Halász method to $M^3(s), H^4(s)$ and $K^2(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F(\log x)^c$$

where

$$F = \min\left\{V^{-6}\left(M^3+T\right), V^{-6}M^3+V^{-18}M^3T, W^{-8}\left(H^4+T\right),\right.$$
$$\left. W^{-8}H^4+W^{-24}H^4T, U^{-4}\left(K^2+T\right), U^{-4}K^2+U^{-12}K^2T\right\}.$$

We consider 8 cases:

(a) $F \leqslant 2V^{-6}M^3, F \leqslant 2W^{-8}H^4, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}M^3, W^{-8}H^4, U^{-4}K^2\right\} \\ &\leqslant U^2V^2W^2\left(V^{-6}M^3\right)^{\frac{1}{4}}\left(W^{-8}H^4\right)^{\frac{1}{4}}\left(U^{-4}K^2\right)^{\frac{1}{2}} \\ &= V^{\frac{1}{2}}M^{\frac{3}{4}}HK \\ &\ll x(\log x)^{-11B}. \end{aligned}$$

(b) $F \leqslant 2V^{-6}M^3, F \leqslant 2W^{-8}H^4, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}M^3, W^{-8}H^4, U^{-4}T, U^{-12}K^2T\right\} \\ &\leqslant U^2V^2W^2\left(V^{-6}M^3\right)^{\frac{1}{3}}\left(W^{-8}H^4\right)^{\frac{1}{4}}\left(U^{-4}T\right)^{\frac{3}{8}}\left(U^{-12}K^2T\right)^{\frac{1}{24}} \\ &= T^{\frac{5}{12}}MHK^{\frac{1}{12}} \\ &= T^{\frac{5}{12}}(MHK)^{\frac{1}{12}}\left(M^{\frac{11}{12}}H^{\frac{11}{12}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $MH \ll X^{\frac{268}{473}}$ is required.

(c) $F \leqslant 2V^{-6}M^3, F > 2W^{-8}H^4, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}M^3, W^{-8}T, W^{-24}H^4T, U^{-4}K^2\right\} \\ &\leqslant U^2V^2W^2\left(V^{-6}M^3\right)^{\frac{1}{3}}\left(W^{-8}T\right)^{\frac{1}{8}}\left(W^{-24}H^4T\right)^{\frac{1}{24}}\left(U^{-4}K^2\right)^{\frac{1}{2}} \\ &= T^{\frac{1}{6}}MH^{\frac{1}{6}}K \\ &= T^{\frac{1}{6}}(MHK)H^{-\frac{5}{6}} \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $X^{\frac{41}{215}} \ll H$ is required.

(d) $F \leqslant 2V^{-6}M^3, F > 2W^{-8}H^4, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}M^3, W^{-8}T, W^{-24}H^4T, U^{-4}T, U^{-12}K^2T\right\} \\ &\leqslant U^2V^2W^2\left(V^{-6}M^3\right)^{\frac{1}{3}}\left(W^{-8}T\right)^{\frac{1}{8}}\left(W^{-24}H^4T\right)^{\frac{1}{24}}\left(U^{-4}T\right)^{\frac{1}{2}} \\ &= T^{\frac{2}{3}}\left(MH^{\frac{1}{6}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $M^6H \ll X^{\frac{94}{43}}$ is required.

(e) $F > 2V^{-6}M^3, F \leqslant 2W^{-8}H^4, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}T, V^{-18}M^3T, W^{-8}H^4, U^{-4}K^2\right\} \\ &\leqslant U^2V^2W^2\left(V^{-6}T\right)^{\frac{5}{24}}\left(V^{-18}M^3T\right)^{\frac{1}{24}}\left(W^{-8}H^4\right)^{\frac{1}{4}}\left(U^{-4}K^2\right)^{\frac{1}{2}} \\ &= T^{\frac{1}{4}}M^{\frac{1}{8}}HK \\ &= T^{\frac{1}{4}}(MHK)M^{-\frac{7}{8}} \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $X^{\frac{82}{301}} \ll M$ is required.

(f) $F > 2V^{-6}M^3, F \leqslant 2W^{-8}H^4, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}T, V^{-18}M^3T, W^{-8}H^4, U^{-4}T, U^{-12}K^2T\right\} \\ &\leqslant U^2V^2W^2 \left(V^{-6}T\right)^{\frac{5}{24}} \left(V^{-18}M^3T\right)^{\frac{1}{24}} \left(W^{-8}H^4\right)^{\frac{1}{4}} \left(U^{-4}T\right)^{\frac{1}{2}} \\ &= T^{\frac{3}{4}} \left(M^{\frac{1}{8}}H\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $MH^8 \ll X^{\frac{98}{43}}$ is required.

(g) $F > 2V^{-6}M^3, F > 2W^{-8}H^4, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}T, V^{-18}M^3T, W^{-8}T, W^{-24}H^4T, U^{-4}K^2\right\} \\ &\leqslant U^2V^2W^2 \left(V^{-6}T\right)^{\frac{1}{3}} \left(W^{-8}T\right)^{\frac{1}{8}} \left(W^{-24}H^4T\right)^{\frac{1}{24}} \left(U^{-4}K^2\right)^{\frac{1}{2}} \\ &= T^{\frac{1}{2}}H^{\frac{1}{6}}K \\ &= T^{\frac{1}{2}}(MHK)\left(M^{-1}H^{-\frac{5}{6}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $X^{\frac{123}{43}} \ll M^6H^5$ is required.

(h) $F > 2V^{-6}M^3, F > 2W^{-8}H^4, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-6}, V^{-18}M^3, W^{-8}, W^{-24}H^4, U^{-4}, U^{-12}K^2\right\}T \\ &\leqslant U^2V^2W^2 \left(V^{-6}\right)^{\frac{5}{24}} \left(V^{-18}M^3\right)^{\frac{1}{24}} \left(W^{-8}\right)^{\frac{1}{4}} \left(U^{-4}\right)^{\frac{1}{2}} T \\ &= TM^{\frac{1}{8}} \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $M \ll X^{\frac{16}{43}}$ is required.

Assuming the condition (4). By applying the mean value estimate and Halász method to $M^2(s), H^5(s)$ and $K^3(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F(\log x)^c$$

where

$$\begin{aligned} F = \min\big\{&V^{-4}\left(M^2+T\right), V^{-4}M^2+V^{-12}M^2T, W^{-10}\left(H^5+T\right), \\ &W^{-10}H^5+W^{-30}H^5T, U^{-6}\left(K^3+T\right), U^{-6}K^3+U^{-18}K^3T\big\}. \end{aligned}$$

We consider 8 cases:

(a) $F \leqslant 2V^{-4}M^2, F \leqslant 2W^{-10}H^5, F \leqslant 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}M^2, W^{-10}H^5, U^{-6}K^3\right\} \\ &\leqslant U^2V^2W^2 \left(V^{-4}M^2\right)^{\frac{1}{2}} \left(W^{-10}H^5\right)^{\frac{1}{6}} \left(U^{-6}K^3\right)^{\frac{1}{3}} \\ &= W^{\frac{1}{3}}MH^{\frac{5}{6}}K \\ &\ll x(\log x)^{-11B}. \end{aligned}$$

(b) $F \leqslant 2V^{-4}M^2, F \leqslant 2W^{-10}H^5, F > 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}M^2, W^{-10}H^5, U^{-6}T, U^{-18}K^3T\right\} \\ &\leqslant U^2V^2W^2 \left(V^{-4}M^2\right)^{\frac{1}{2}} \left(W^{-10}H^5\right)^{\frac{1}{5}} \left(U^{-6}T\right)^{\frac{17}{60}} \left(U^{-18}K^3T\right)^{\frac{1}{60}} \\ &= T^{\frac{3}{10}}MHK^{\frac{1}{20}} \\ &= T^{\frac{3}{10}}(MHK)^{\frac{1}{20}}\left(M^{\frac{19}{20}}H^{\frac{19}{20}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $MH \ll X^{\frac{571}{817}}$ is required.

(c) $F \leqslant 2V^{-4}M^2, F > 2W^{-10}H^5, F \leqslant 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}M^2, W^{-10}T, W^{-30}H^5T, U^{-6}K^3\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-4}M^2\right)^{\frac{1}{2}} \left(W^{-10}T\right)^{\frac{3}{20}} \left(W^{-30}H^5T\right)^{\frac{1}{60}} \left(U^{-6}K^3\right)^{\frac{1}{3}} \\
&= T^{\frac{1}{6}} M H^{\frac{1}{12}} K \\
&= T^{\frac{1}{6}} (MHK) H^{-\frac{11}{12}} \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $X^{\frac{82}{473}} \ll H$ is required.

(d) $F \leqslant 2V^{-4}M^2, F > 2W^{-10}H^5, F > 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}M^2, W^{-10}T, W^{-30}H^5T, U^{-6}T, U^{-18}K^3T\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-4}M^2\right)^{\frac{1}{2}} \left(W^{-10}T\right)^{\frac{3}{20}} \left(W^{-30}H^5T\right)^{\frac{1}{60}} \left(U^{-6}T\right)^{\frac{1}{3}} \\
&= T^{\frac{1}{2}} \left(M H^{\frac{1}{12}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $M^{12}H \ll X^{\frac{270}{43}}$ is required.

(e) $F > 2V^{-4}M^2, F \leqslant 2W^{-10}H^5, F \leqslant 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}T, V^{-12}M^2T, W^{-10}H^5, U^{-6}K^3\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-4}T\right)^{\frac{9}{20}} \left(V^{-12}M^2T\right)^{\frac{1}{60}} \left(W^{-10}H^5\right)^{\frac{1}{5}} \left(U^{-6}K^3\right)^{\frac{1}{3}} \\
&= T^{\frac{7}{15}} M^{\frac{1}{30}} HK \\
&= T^{\frac{7}{15}} (MHK) M^{-\frac{29}{30}} \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $X^{\frac{574}{1247}} \ll M$ is required.

(f) $F > 2V^{-4}M^2, F \leqslant 2W^{-10}H^5, F > 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}T, V^{-12}M^2T, W^{-10}H^5, U^{-6}T, U^{-18}K^3T\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-4}T\right)^{\frac{1}{2}} \left(W^{-10}H^5\right)^{\frac{1}{5}} \left(U^{-6}T\right)^{\frac{17}{60}} \left(U^{-18}K^3T\right)^{\frac{1}{60}} \\
&= T^{\frac{4}{5}} H K^{\frac{1}{20}} \\
&= T^{\frac{4}{5}} (MHK)^{\frac{1}{20}} \left(M^{-\frac{1}{20}} H^{\frac{19}{20}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $M^{-1}H^{19} \ll X^{\frac{161}{43}}$ is required.

(g) $F > 2V^{-4}M^2, F > 2W^{-10}H^5, F \leqslant 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}T, V^{-12}M^2T, W^{-10}T, W^{-30}H^5T, U^{-6}K^3\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-4}T\right)^{\frac{1}{2}} \left(W^{-10}T\right)^{\frac{3}{20}} \left(W^{-30}H^5T\right)^{\frac{1}{60}} \left(U^{-6}K^3\right)^{\frac{1}{3}} \\
&= T^{\frac{2}{3}} H^{\frac{1}{12}} K \\
&= T^{\frac{2}{3}} (MHK) \left(M^{-1} H^{-\frac{11}{12}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $X^{\frac{328}{43}} \ll M^{12}H^{11}$ is required.

(h) $F > 2V^{-4}M^2, F > 2W^{-10}H^5, F > 2U^{-6}K^3$. Then

$$U^2V^2W^2F \ll U^2V^2W^2 \min\left\{V^{-4}, V^{-12}M^2, W^{-10}, W^{-30}H^5, U^{-6}, U^{-18}K^3\right\} T$$

$$\begin{aligned}
&\leqslant U^2V^2W^2\left(V^{-4}\right)^{\frac{1}{2}}\left(W^{-10}\right)^{\frac{1}{5}}\left(U^{-6}\right)^{\frac{17}{60}}\left(U^{-18}K^3\right)^{\frac{1}{60}}T\\
&=TK^{\frac{1}{20}}\\
&=T(MHK)^{\frac{1}{20}}\left(M^{-\frac{1}{20}}H^{-\frac{1}{20}}\right)\\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $X^{\frac{3}{43}} \ll MH$ is required. This can be obtained by using $X^{\frac{574}{1247}} \ll M$ and $X^{\frac{82}{473}} \ll H$.

Assuming the condition (5). By applying the mean value estimate and Halász method to $M^2(s), H^5(s)$ and $K^2(s)H(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F(\log x)^c$$

where

$$\begin{aligned}
F=\min\{&V^{-4}\left(M^2+T\right),V^{-4}M^2+V^{-12}M^2T,W^{-10}\left(H^5+T\right),\\
&W^{-10}H^5+W^{-30}H^5T,U^{-4}W^{-2}\left(K^2H+T\right),U^{-4}W^{-2}K^2H+U^{-12}W^{-6}K^2HT\}.
\end{aligned}$$

We consider 8 cases:

(a) $F\leqslant 2V^{-4}M^2, F\leqslant 2W^{-10}H^5, F\leqslant 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\left\{V^{-4}M^2,W^{-10}H^5,U^{-4}W^{-2}K^2H\right\}\\
&\leqslant U^2V^2W^2\left(V^{-4}M^2\right)^{\frac{1}{2}}\left(U^{-4}W^{-2}K^2H\right)^{\frac{1}{2}}\\
&=WMH^{\frac{1}{2}}K\\
&\ll x(\log x)^{-11B}.
\end{aligned}$$

(b) $F\leqslant 2V^{-4}M^2, F\leqslant 2W^{-10}H^5, F> 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\left\{V^{-4}M^2,W^{-10}H^5,U^{-4}W^{-2}T,U^{-12}W^{-6}K^2HT\right\}\\
&\leqslant U^2V^2W^2\left(V^{-4}M^2\right)^{\frac{1}{2}}\left(W^{-10}H^5\right)^{\frac{1}{10}}\left(U^{-4}W^{-2}T\right)^{\frac{7}{20}}\left(U^{-12}W^{-6}K^2HT\right)^{\frac{1}{20}}\\
&=T^{\frac{2}{5}}MH^{\frac{11}{20}}K^{\frac{1}{10}}\\
&=T^{\frac{2}{5}}(MHK)^{\frac{1}{10}}\left(M^{\frac{9}{10}}H^{\frac{9}{20}}\right)\\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $M^2H \ll X^{\frac{446}{387}}$ is required.

(c) $F\leqslant 2V^{-4}M^2, F> 2W^{-10}H^5, F\leqslant 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\left\{V^{-4}M^2,W^{-10}T,W^{-30}H^5T,U^{-4}W^{-2}K^2H\right\}\\
&\leqslant U^2V^2W^2\left(V^{-4}M^2\right)^{\frac{1}{2}}\left(U^{-4}W^{-2}K^2H\right)^{\frac{1}{2}}\\
&=WMH^{\frac{1}{2}}K\\
&\ll x(\log x)^{-11B}.
\end{aligned}$$

(d) $F\leqslant 2V^{-4}M^2, F> 2W^{-10}H^5, F> 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\left\{V^{-4}M^2,W^{-10}T,W^{-30}H^5T,U^{-4}W^{-2}T,U^{-12}W^{-6}K^2HT\right\}\\
&\leqslant U^2V^2W^2\left(V^{-4}M^2\right)^{\frac{1}{2}}\left(W^{-30}H^5T\right)^{\frac{1}{30}}\left(U^{-4}W^{-2}T\right)^{\frac{9}{20}}\left(U^{-12}W^{-6}K^2HT\right)^{\frac{1}{60}}\\
&=T^{\frac{1}{2}}MH^{\frac{11}{60}}K^{\frac{1}{30}}\\
&=T^{\frac{1}{2}}(MHK)^{\frac{1}{30}}\left(M^{\frac{29}{30}}H^{\frac{9}{60}}\right)\\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $M^{58}H^9 \ll X^{\frac{1264}{43}}$ is required.

(e) $F > 2V^{-4}M^2, F \leqslant 2W^{-10}H^5, F \leqslant 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}T, V^{-12}M^2T, W^{-10}H^5, U^{-4}W^{-2}K^2H\right\} \\ &\leqslant U^2V^2W^2 \left(V^{-4}T\right)^{\frac{7}{20}} \left(V^{-12}M^2T\right)^{\frac{1}{20}} \left(W^{-10}H^5\right)^{\frac{1}{10}} \left(U^{-4}W^{-2}K^2H\right)^{\frac{1}{2}} \\ &= T^{\frac{2}{5}} M^{\frac{1}{10}} HK \\ &= T^{\frac{2}{5}} (MHK) M^{-\frac{9}{10}} \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $X^{\frac{164}{387}} \ll M$ is required.

(f) $F > 2V^{-4}M^2, F \leqslant 2W^{-10}H^5, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}T, V^{-12}M^2T, W^{-10}H^5, U^{-4}W^{-2}T, U^{-12}W^{-6}K^2HT\right\} \\ &\leqslant U^2V^2W^2 \left(V^{-4}T\right)^{\frac{7}{20}} \left(V^{-12}M^2T\right)^{\frac{1}{20}} \left(W^{-10}H^5\right)^{\frac{1}{10}} \left(U^{-4}W^{-2}T\right)^{\frac{1}{2}} \\ &= T^{\frac{9}{10}} \left(M^{\frac{1}{10}} H^{\frac{1}{2}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $MH^5 \ll X^{\frac{61}{43}}$ is required. This can be obtained by using $MH^5 \ll X^{\frac{60}{43}}$.

(g) $F > 2V^{-4}M^2, F > 2W^{-10}H^5, F \leqslant 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}T, V^{-12}M^2T, W^{-10}T, W^{-30}H^5T, U^{-4}W^{-2}K^2H\right\} \\ &\leqslant U^2V^2W^2 \left(V^{-4}T\right)^{\frac{9}{20}} \left(V^{-12}M^2T\right)^{\frac{1}{60}} \left(W^{-30}H^5T\right)^{\frac{1}{30}} \left(U^{-4}W^{-2}K^2H\right)^{\frac{1}{2}} \\ &= T^{\frac{1}{2}} M^{\frac{1}{30}} H^{\frac{2}{3}} K \\ &= T^{\frac{1}{2}} (MHK) \left(M^{-\frac{29}{30}} H^{-\frac{1}{3}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $X^{\frac{615}{43}} \ll M^{29}H^{10}$ is required.

(h) $F > 2V^{-4}M^2, F > 2W^{-10}H^5, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}, V^{-12}M^2, W^{-10}, W^{-30}H^5, U^{-4}W^{-2}, U^{-12}W^{-6}K^2H\right\} T \\ &\leqslant U^2V^2W^2 \left(V^{-4}\right)^{\frac{9}{20}} \left(V^{-12}M^2\right)^{\frac{1}{60}} \left(W^{-30}H^5\right)^{\frac{1}{30}} \left(U^{-4}W^{-2}\right)^{\frac{1}{2}} T \\ &= T \left(M^{\frac{1}{30}} H^{\frac{1}{6}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $MH^5 \ll X^{\frac{60}{43}}$ is required.

Assuming the condition (6). By applying the mean value estimate and Halász method to $M^2(s), H^4(s)$ and $K^3(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F(\log x)^c$$

where

$$\begin{aligned} F = \min\Big\{ & V^{-4}\left(M^2+T\right), V^{-4}M^2 + V^{-12}M^2T, W^{-8}\left(H^4+T\right), \\ & W^{-8}H^4 + W^{-24}H^4T, U^{-6}\left(K^3+T\right), U^{-6}K^3 + U^{-18}K^3T\Big\}. \end{aligned}$$

We consider 8 cases:

(a) $F \leqslant 2V^{-4}M^2, F \leqslant 2W^{-8}H^4, F \leqslant 2U^{-6}K^3$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}M^2, W^{-8}H^4, U^{-6}K^3\right\} \\ &\leqslant U^2V^2W^2 \left(V^{-4}M^2\right)^{\frac{1}{2}} \left(W^{-8}H^4\right)^{\frac{1}{6}} \left(U^{-6}K^3\right)^{\frac{1}{3}} \\ &= W^{\frac{2}{3}} M H^{\frac{2}{3}} K \\ &\ll x(\log x)^{-11B}. \end{aligned}$$

(b) $F \leqslant 2V^{-4}M^2, F \leqslant 2W^{-8}H^4, F > 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}M^2, W^{-8}H^4, U^{-6}T, U^{-18}K^3T\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-4}M^2\right)^{\frac{1}{2}} \left(W^{-8}H^4\right)^{\frac{1}{4}} \left(U^{-6}T\right)^{\frac{5}{24}} \left(U^{-18}K^3T\right)^{\frac{1}{24}} \\
&= T^{\frac{1}{4}} MHK^{\frac{1}{8}} \\
&= T^{\frac{1}{4}} (MHK)^{\frac{1}{8}} \left(M^{\frac{7}{8}}H^{\frac{7}{8}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $MH \ll X^{\frac{219}{301}}$ is required.

(c) $F \leqslant 2V^{-4}M^2, F > 2W^{-8}H^4, F \leqslant 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}M^2, W^{-8}T, W^{-24}H^4T, U^{-6}K^3\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-4}M^2\right)^{\frac{1}{2}} \left(W^{-8}T\right)^{\frac{1}{8}} \left(W^{-24}H^4T\right)^{\frac{1}{24}} \left(U^{-6}K^3\right)^{\frac{1}{3}} \\
&= T^{\frac{1}{6}} MH^{\frac{1}{6}}K \\
&= T^{\frac{1}{6}} (MHK)H^{-\frac{5}{6}} \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $X^{\frac{41}{215}} \ll H$ is required.

(d) $F \leqslant 2V^{-4}M^2, F > 2W^{-8}H^4, F > 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}M^2, W^{-8}T, W^{-24}H^4T, U^{-6}T, U^{-18}K^3T\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-4}M^2\right)^{\frac{1}{2}} \left(W^{-8}T\right)^{\frac{1}{8}} \left(W^{-24}H^4T\right)^{\frac{1}{24}} \left(U^{-6}T\right)^{\frac{1}{3}} \\
&= T^{\frac{1}{2}} \left(MH^{\frac{1}{6}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $M^6H \ll X^{\frac{135}{43}}$ is required.

(e) $F > 2V^{-4}M^2, F \leqslant 2W^{-8}H^4, F \leqslant 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}T, V^{-12}M^2T, W^{-8}H^4, U^{-6}K^3\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-4}T\right)^{\frac{3}{8}} \left(V^{-12}M^2T\right)^{\frac{1}{24}} \left(W^{-8}H^4\right)^{\frac{1}{4}} \left(U^{-6}K^3\right)^{\frac{1}{3}} \\
&= T^{\frac{5}{12}} M^{\frac{1}{12}}HK \\
&= T^{\frac{5}{12}} (MHK)M^{-\frac{11}{12}} \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $X^{\frac{205}{473}} \ll M$ is required.

(f) $F > 2V^{-4}M^2, F \leqslant 2W^{-8}H^4, F > 2U^{-6}K^3$. Then

$$\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}T, V^{-12}M^2T, W^{-8}H^4, U^{-6}T, U^{-18}K^3T\right\} \\
&\leqslant U^2V^2W^2 \left(V^{-4}T\right)^{\frac{1}{2}} \left(W^{-8}H^4\right)^{\frac{1}{4}} \left(U^{-6}T\right)^{\frac{5}{24}} \left(U^{-18}K^3T\right)^{\frac{1}{24}} \\
&= T^{\frac{3}{4}} HK^{\frac{1}{8}} \\
&= T^{\frac{3}{4}} (MHK)^{\frac{1}{8}} \left(M^{-\frac{1}{8}}H^{\frac{7}{8}}\right) \\
&\ll x^{1-\varepsilon_1},
\end{aligned}$$

where $M^{-1}H^7 \ll X^{\frac{55}{43}}$ is required.

(g) $F > 2V^{-4}M^2, F > 2W^{-8}H^4, F \leqslant 2U^{-6}K^3$. Then

$$U^2V^2W^2F \ll U^2V^2W^2 \min\left\{V^{-4}T, V^{-12}M^2T, W^{-8}T, W^{-24}H^4T, U^{-6}K^3\right\}$$

$$
\begin{aligned}
&\leqslant U^2V^2W^2\left(V^{-4}T\right)^{\frac{1}{2}}\left(W^{-8}T\right)^{\frac{1}{8}}\left(W^{-24}H^4T\right)^{\frac{1}{24}}\left(U^{-6}K^3\right)^{\frac{1}{3}}\\
&=T^{\frac{2}{3}}H^{\frac{1}{6}}K\\
&=T^{\frac{2}{3}}(MHK)\left(M^{-1}H^{-\frac{5}{6}}\right)\\
&\ll x^{1-\varepsilon_1},
\end{aligned}
$$

where $X^{\frac{164}{43}} \ll M^6H^5$ is required.

(h) $F > 2V^{-4}M^2, F > 2W^{-8}H^4, F > 2U^{-6}K^3$. Then

$$
\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\left\{V^{-4},V^{-12}M^2,W^{-8},W^{-24}H^4,U^{-6},U^{-18}K^3\right\}T\\
&\leqslant U^2V^2W^2\left(V^{-4}\right)^{\frac{1}{2}}\left(W^{-8}\right)^{\frac{1}{4}}\left(U^{-6}\right)^{\frac{5}{24}}\left(U^{-18}K^3\right)^{\frac{1}{24}}T\\
&=TK^{\frac{1}{8}}\\
&=T(MHK)^{\frac{1}{8}}\left(M^{-\frac{1}{8}}H^{-\frac{1}{8}}\right)\\
&\ll x^{1-\varepsilon_1},
\end{aligned}
$$

where $X^{\frac{27}{43}} \ll MH$ is required.

Assuming the condition (7). By applying the mean value estimate and Halász method to $M^2(s)H(s), H^6(s)$ and $K^2(s)$, we get

$$
I \ll U^2V^2W^2x^{-1}F(\log x)^c
$$

where

$$
\begin{aligned}
F=\min\big\{&V^{-4}W^{-2}\left(M^2H+T\right),V^{-4}W^{-2}M^2H+V^{-12}W^{-6}M^2HT,\\
&W^{-12}\left(H^6+T\right),W^{-12}H^6+W^{-36}H^6T,U^{-4}\left(K^2+T\right),U^{-4}K^2+U^{-12}K^2T\big\}.
\end{aligned}
$$

We consider 8 cases:

(a) $F \leqslant 2V^{-4}W^{-2}M^2H, F \leqslant 2W^{-12}H^6, F \leqslant 2U^{-4}K^2$. Then

$$
\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\left\{V^{-4}W^{-2}M^2H,W^{-12}H^6,U^{-4}K^2\right\}\\
&\leqslant U^2V^2W^2\left(V^{-4}W^{-2}M^2H\right)^{\frac{1}{2}}\left(U^{-4}K^2\right)^{\frac{1}{2}}\\
&=WMH^{\frac{1}{2}}K\\
&\ll x(\log x)^{-11B}.
\end{aligned}
$$

(b) $F \leqslant 2V^{-4}W^{-2}M^2H, F \leqslant 2W^{-12}H^6, F > 2U^{-4}K^2$. Then

$$
\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\left\{V^{-4}W^{-2}M^2H,W^{-12}H^6,U^{-4}T,U^{-12}K^2T\right\}\\
&\leqslant U^2V^2W^2\left(V^{-4}W^{-2}M^2H\right)^{\frac{1}{2}}\left(W^{-12}H^6\right)^{\frac{1}{12}}\left(U^{-4}T\right)^{\frac{3}{8}}\left(U^{-12}K^2T\right)^{\frac{1}{24}}\\
&=T^{\frac{5}{12}}MHK^{\frac{1}{12}}\\
&=T^{\frac{5}{12}}(MHK)^{\frac{1}{12}}\left(M^{\frac{11}{12}}H^{\frac{11}{12}}\right)\\
&\ll x^{1-\varepsilon_1},
\end{aligned}
$$

where $MH \ll X^{\frac{268}{473}}$ is required.

(c) $F \leqslant 2V^{-4}W^{-2}M^2H, F > 2W^{-12}H^6, F \leqslant 2U^{-4}K^2$. Then

$$
\begin{aligned}
U^2V^2W^2F &\ll U^2V^2W^2\min\left\{V^{-4}W^{-2}M^2H,W^{-12}T,W^{-36}H^6T,U^{-4}K^2\right\}\\
&\leqslant U^2V^2W^2\left(V^{-4}W^{-2}M^2H\right)^{\frac{1}{2}}\left(U^{-4}K^2\right)^{\frac{1}{2}}\\
&=WMH^{\frac{1}{2}}K\\
&\ll x(\log x)^{-11B}.
\end{aligned}
$$

(d) $F \leqslant 2V^{-4}W^{-2}M^2H, F > 2W^{-12}H^6, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}M^2H, W^{-12}T, W^{-36}H^6T, U^{-4}T, U^{-12}K^2T\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}W^{-2}M^2H\right)^{\frac{1}{2}}\left(W^{-36}H^6T\right)^{\frac{1}{36}}\left(U^{-4}T\right)^{\frac{11}{24}}\left(U^{-12}K^2T\right)^{\frac{1}{72}} \\ &= T^{\frac{1}{2}}MH^{\frac{2}{3}}K^{\frac{1}{36}} \\ &= T^{\frac{1}{2}}(MHK)^{\frac{1}{36}}\left(M^{\frac{35}{36}}H^{\frac{23}{36}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $M^{35}H^{23} \ll X^{\frac{767}{43}}$ is required.

(e) $F > 2V^{-4}W^{-2}M^2H, F \leqslant 2W^{-12}H^6, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-12}H^6, U^{-4}K^2\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}W^{-2}T\right)^{\frac{3}{8}}\left(V^{-12}W^{-6}M^2HT\right)^{\frac{1}{24}}\left(W^{-12}H^6\right)^{\frac{1}{12}}\left(U^{-4}K^2\right)^{\frac{1}{2}} \\ &= T^{\frac{5}{12}}M^{\frac{1}{12}}H^{\frac{13}{24}}K \\ &= T^{\frac{5}{12}}(MHK)\left(M^{-\frac{11}{12}}H^{-\frac{11}{24}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $X^{\frac{410}{473}} \ll M^2H$ is required.

(f) $F > 2V^{-4}W^{-2}M^2H, F \leqslant 2W^{-12}H^6, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-12}H^6, U^{-4}T, U^{-12}K^2T\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}W^{-2}T\right)^{\frac{3}{8}}\left(V^{-12}W^{-6}M^2HT\right)^{\frac{1}{24}}\left(W^{-12}H^6\right)^{\frac{1}{12}}\left(U^{-4}T\right)^{\frac{1}{2}} \\ &= T^{\frac{11}{12}}\left(M^{\frac{1}{12}}H^{\frac{13}{24}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $M^2H^{13} \ll X^{\frac{130}{43}}$ is required.

(g) $F > 2V^{-4}W^{-2}M^2H, F > 2W^{-12}H^6, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-12}T, W^{-36}H^6T, U^{-4}K^2\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}W^{-2}T\right)^{\frac{11}{24}}\left(V^{-12}W^{-6}M^2HT\right)^{\frac{1}{72}}\left(W^{-36}H^6T\right)^{\frac{1}{36}}\left(U^{-4}K^2\right)^{\frac{1}{2}} \\ &= T^{\frac{1}{2}}M^{\frac{1}{36}}H^{\frac{13}{72}}K \\ &= T^{\frac{1}{2}}(MHK)\left(M^{-\frac{35}{36}}H^{-\frac{59}{72}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $X^{\frac{1476}{43}} \ll M^{70}H^{59}$ is required.

(h) $F > 2V^{-4}W^{-2}M^2H, F > 2W^{-12}H^6, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}, V^{-12}W^{-6}M^2H, W^{-12}, W^{-36}H^6, U^{-4}, U^{-12}K^2\right\}T \\ &\leqslant U^2V^2W^2\left(V^{-4}W^{-2}\right)^{\frac{1}{2}}\left(W^{-12}\right)^{\frac{1}{12}}\left(U^{-4}\right)^{\frac{3}{8}}\left(U^{-12}K^2\right)^{\frac{1}{24}}T \\ &= TK^{\frac{1}{12}} \\ &= T(MHK)^{\frac{1}{12}}\left(M^{-\frac{1}{12}}H^{-\frac{1}{12}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $X^{\frac{19}{43}} \ll MH$ is required. This can be obtained by using $X^{\frac{738}{1505}} \ll MH^{\frac{59}{70}} \ll MH$.

Assuming the condition (8). By applying the mean value estimate and Halász method to $M^2(s)H(s), H^7(s)$ and $K^2(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F(\log x)^c$$

where

$$F=\min\left\{V^{-4}W^{-2}\left(M^2H+T\right),V^{-4}W^{-2}M^2H+V^{-12}W^{-6}M^2HT,\right.$$
$$\left.W^{-14}\left(H^7+T\right),W^{-14}H^7+W^{-42}H^7T,U^{-4}\left(K^2+T\right)U^{-4}K^2+U^{-12}K^2T\right\}.$$

We consider 8 cases:

(a) $F\leqslant 2V^{-4}W^{-2}M^2H, F\leqslant 2W^{-14}H^7, F\leqslant 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F&\ll U^2V^2W^2\min\left\{V^{-4}W^{-2}M^2H,W^{-14}H^7,U^{-4}K^2\right\}\\&\leqslant U^2V^2W^2\left(V^{-4}W^{-2}M^2H\right)^{\frac12}\left(U^{-4}K^2\right)^{\frac12}\\&=WMH^{\frac12}K\\&\ll x(\log x)^{-11B}.\end{aligned}$$

(b) $F\leqslant 2V^{-4}W^{-2}M^2H, F\leqslant 2W^{-14}H^7, F>2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F&\ll U^2V^2W^2\min\left\{V^{-4}W^{-2}M^2H,W^{-14}H^7,U^{-4}T,U^{-12}K^2T\right\}\\&\leqslant U^2V^2W^2\left(V^{-4}W^{-2}M^2H\right)^{\frac12}\left(W^{-14}H^7\right)^{\frac1{14}}\left(U^{-4}T\right)^{\frac{11}{28}}\left(U^{-12}K^2T\right)^{\frac1{28}}\\&=T^{\frac37}MHK^{\frac1{14}}\\&=T^{\frac37}(MHK)^{\frac1{14}}\left(M^{\frac{13}{14}}H^{\frac{13}{14}}\right)\\&\ll x^{1-\varepsilon_1},\end{aligned}$$

where $MH\ll X^{\frac{313}{559}}$ is required.

(c) $F\leqslant 2V^{-4}W^{-2}M^2H, F>2W^{-14}H^7, F\leqslant 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F&\ll U^2V^2W^2\min\left\{V^{-4}W^{-2}M^2H,W^{-14}T,W^{-42}H^7T,U^{-4}K^2\right\}\\&\leqslant U^2V^2W^2\left(V^{-4}W^{-2}M^2H\right)^{\frac12}\left(U^{-4}K^2\right)^{\frac12}\\&=WMH^{\frac12}K\\&\ll x(\log x)^{-11B}.\end{aligned}$$

(d) $F\leqslant 2V^{-4}W^{-2}M^2H, F>2W^{-14}H^7, F>2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F&\ll U^2V^2W^2\min\left\{V^{-4}W^{-2}M^2H,W^{-14}T,W^{-42}H^7T,U^{-4}T,U^{-12}K^2T\right\}\\&\leqslant U^2V^2W^2\left(V^{-4}W^{-2}M^2H\right)^{\frac12}\left(W^{-42}H^7T\right)^{\frac1{42}}\left(U^{-4}T\right)^{\frac{13}{28}}\left(U^{-12}K^2T\right)^{\frac1{84}}\\&=T^{\frac12}MH^{\frac23}K^{\frac1{42}}\\&=T^{\frac12}(MHK)^{\frac1{42}}\left(M^{\frac{41}{42}}H^{\frac9{14}}\right)\\&\ll x^{1-\varepsilon_1},\end{aligned}$$

where $M^{41}H^{27}\ll X^{\frac{902}{43}}$ is required.

(e) $F>2V^{-4}W^{-2}M^2H, F\leqslant 2W^{-14}H^7, F\leqslant 2U^{-4}K^2$. Then

$$\begin{aligned}U^2V^2W^2F&\ll U^2V^2W^2\min\left\{V^{-4}W^{-2}T,V^{-12}W^{-6}M^2HT,W^{-14}H^7,U^{-4}K^2\right\}\\&\leqslant U^2V^2W^2\left(V^{-4}W^{-2}T\right)^{\frac{11}{28}}\left(V^{-12}W^{-6}M^2HT\right)^{\frac1{28}}\left(W^{-14}H^7\right)^{\frac1{14}}\left(U^{-4}K^2\right)^{\frac12}\\&=T^{\frac37}M^{\frac1{14}}H^{\frac{15}{28}}K\\&=T^{\frac37}(MHK)\left(M^{-\frac{13}{14}}H^{-\frac{13}{28}}\right)\\&\ll x^{1-\varepsilon_1},\end{aligned}$$

where $X^{\frac{492}{559}}\ll M^2H$ is required.

(f) $F > 2V^{-4}W^{-2}M^2H, F \leqslant 2W^{-14}H^7, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-14}H^7, U^{-4}T, U^{-12}K^2T\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}W^{-2}T\right)^{\frac{11}{28}}\left(V^{-12}W^{-6}M^2HT\right)^{\frac{1}{28}}\left(W^{-14}H^7\right)^{\frac{1}{14}}\left(U^{-4}T\right)^{\frac{1}{2}} \\ &= T^{\frac{13}{14}}\left(M^{\frac{1}{14}}H^{\frac{15}{28}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $M^2H^{15} \ll X^{\frac{138}{43}}$ is required.

(g) $F > 2V^{-4}W^{-2}M^2H, F > 2W^{-14}H^7, F \leqslant 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}T, V^{-12}W^{-6}M^2HT, W^{-14}T, W^{-42}H^7T, U^{-4}K^2\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}W^{-2}T\right)^{\frac{13}{88}}\left(V^{-12}W^{-6}M^2HT\right)^{\frac{1}{84}}\left(W^{-42}H^7T\right)^{\frac{1}{42}}\left(U^{-4}K^2\right)^{\frac{1}{2}} \\ &= T^{\frac{1}{2}}M^{\frac{1}{42}}H^{\frac{5}{28}}K \\ &= T^{\frac{1}{2}}(MHK)\left(M^{-\frac{41}{42}}H^{-\frac{23}{28}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $X^{\frac{1722}{43}} \ll M^{82}H^{69}$ is required.

(h) $F > 2V^{-4}W^{-2}M^2H, F > 2W^{-14}H^7, F > 2U^{-4}K^2$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}W^{-2}, V^{-12}W^{-6}M^2H, W^{-14}, W^{-42}H^7, U^{-4}, U^{-12}K^2\right\}T \\ &\leqslant U^2V^2W^2\left(V^{-4}W^{-2}\right)^{\frac{1}{2}}\left(W^{-14}\right)^{\frac{1}{14}}\left(U^{-4}\right)^{\frac{11}{28}}\left(U^{-12}K^2\right)^{\frac{1}{28}}T \\ &= TK^{\frac{1}{14}} \\ &= T(MHK)^{\frac{1}{14}}\left(M^{-\frac{1}{14}}H^{-\frac{1}{14}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $X^{\frac{15}{43}} \ll MH$ is required. This can be obtained by using $X^{\frac{246}{559}} \ll MH^{\frac{1}{2}} \ll MH$.

Assuming the condition (9). By applying the mean value estimate and Halász method to $M^2(s), H^6(s)$ and $K^2(s)H(s)$, we get

$$I \ll U^2V^2W^2x^{-1}F(\log x)^c$$

where

$$\begin{aligned} F = \min\{&V^{-4}\left(M^2+T\right), V^{-4}M^2+V^{-12}M^2T, W^{-12}\left(H^6+T\right), W^{-12}H^6+W^{-36}H^6T, \\ &U^{-4}W^{-2}\left(K^2H+T\right), U^{-4}W^{-2}K^2H+U^{-12}W^{-6}K^2HT\}. \end{aligned}$$

We consider 8 cases:

(a) $F \leqslant 2V^{-4}M^2, F \leqslant 2W^{-12}H^6, F \leqslant 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}M^2, W^{-12}H^6, U^{-4}W^{-2}K^2H\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}M^2\right)^{\frac{1}{2}}\left(U^{-4}W^{-2}K^2H\right)^{\frac{1}{2}} \\ &= WMH^{\frac{1}{2}}K \\ &\ll x(\log x)^{-11B}. \end{aligned}$$

(b) $F \leqslant 2V^{-4}M^2, F \leqslant 2W^{-12}H^6, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}M^2, W^{-12}H^6, U^{-4}W^{-2}T, U^{-12}W^{-6}K^2HT\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}M^2\right)^{\frac{1}{2}}\left(W^{-12}H^6\right)^{\frac{1}{12}}\left(U^{-4}W^{-2}T\right)^{\frac{3}{8}}\left(U^{-12}W^{-6}K^2HT\right)^{\frac{1}{24}} \\ &= T^{\frac{5}{12}}MH^{\frac{13}{24}}K^{\frac{1}{12}} \\ &= T^{\frac{5}{12}}(MHK)^{\frac{1}{12}}\left(M^{\frac{11}{12}}H^{\frac{11}{24}}\right) \end{aligned}$$

$$\ll x^{1-\varepsilon_1},$$

where $M^2H \ll X^{\frac{536}{473}}$ is required.

(c) $F \leqslant 2V^{-4}M^2, F > 2W^{-12}H^6, F \leqslant 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}M^2, W^{-12}T, W^{-36}H^6T, U^{-4}W^{-2}K^2H\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}M^2\right)^{\frac{1}{2}}\left(U^{-4}W^{-2}K^2H\right)^{\frac{1}{2}} \\ &= WMH^{\frac{1}{2}}K \\ &\ll x(\log x)^{-11B}. \end{aligned}$$

(d) $F \leqslant 2V^{-4}M^2, F > 2W^{-12}H^6, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}M^2, W^{-12}T, W^{-36}H^6T, U^{-4}W^{-2}T, U^{-12}W^{-6}K^2HT\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}M^2\right)^{\frac{1}{2}}\left(W^{-36}H^6T\right)^{\frac{1}{36}}\left(U^{-4}W^{-2}T\right)^{\frac{11}{24}}\left(U^{-12}W^{-6}K^2HT\right)^{\frac{1}{72}} \\ &= T^{\frac{1}{2}}MH^{\frac{13}{72}}K^{\frac{1}{36}} \\ &= T^{\frac{1}{2}}(MHK)^{\frac{1}{36}}\left(M^{\frac{35}{36}}H^{\frac{11}{72}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $M^{70}H^{11} \ll X^{\frac{1534}{43}}$ is required.

(e) $F > 2V^{-4}M^2, F \leqslant 2W^{-12}H^6, F \leqslant 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}T, V^{-12}M^2T, W^{-12}H^6, U^{-4}W^{-2}K^2H\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}T\right)^{\frac{3}{8}}\left(V^{-12}M^2T\right)^{\frac{1}{24}}\left(W^{-12}H^6\right)^{\frac{1}{12}}\left(U^{-4}W^{-2}K^2H\right)^{\frac{1}{2}} \\ &= T^{\frac{5}{12}}M^{\frac{1}{12}}HK \\ &= T^{\frac{5}{12}}(MHK)M^{-\frac{11}{12}} \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $X^{\frac{205}{473}} \ll M$ is required.

(f) $F > 2V^{-4}M^2, F \leqslant 2W^{-12}H^6, F > 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}T, V^{-12}M^2T, W^{-12}H^6, U^{-4}W^{-2}T, U^{-12}W^{-6}K^2HT\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}T\right)^{\frac{3}{8}}\left(V^{-12}M^2T\right)^{\frac{1}{24}}\left(W^{-12}H^6\right)^{\frac{1}{12}}\left(U^{-4}W^{-2}T\right)^{\frac{1}{2}} \\ &= T^{\frac{11}{12}}\left(M^{\frac{1}{12}}H^{\frac{1}{2}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $MH^6 \ll X^{\frac{65}{43}}$ is required.

(g) $F > 2V^{-4}M^2, F > 2W^{-12}H^6, F \leqslant 2U^{-4}W^{-2}K^2H$. Then

$$\begin{aligned} U^2V^2W^2F &\ll U^2V^2W^2 \min\left\{V^{-4}T, V^{-12}M^2T, W^{-12}T, W^{-36}H^6T, U^{-4}W^{-2}K^2H\right\} \\ &\leqslant U^2V^2W^2\left(V^{-4}T\right)^{\frac{11}{24}}\left(V^{-12}M^2T\right)^{\frac{1}{72}}\left(W^{-36}H^6T\right)^{\frac{1}{36}}\left(U^{-4}W^{-2}K^2H\right)^{\frac{1}{2}} \\ &= T^{\frac{1}{2}}M^{\frac{1}{36}}H^{\frac{2}{3}}K \\ &= T^{\frac{1}{2}}(MHK)\left(M^{-\frac{35}{36}}H^{-\frac{1}{3}}\right) \\ &\ll x^{1-\varepsilon_1}, \end{aligned}$$

where $X^{\frac{738}{43}} \ll M^{35}H^{12}$ is required.

(h) $F > 2V^{-4}M^2, F > 2W^{-12}H^6, F > 2U^{-4}W^{-2}K^2H$. Then

$$U^2V^2W^2F \ll U^2V^2W^2 \min\left\{V^{-4}, V^{-12}M^2, W^{-12}, W^{-36}H^6, U^{-4}W^{-2}, U^{-12}W^{-6}K^2H\right\}T$$

$$\begin{aligned}&\leqslant U^2V^2W^2\left(V^{-4}\right)^{\frac{3}{8}}\left(V^{-12}M^2\right)^{\frac{1}{24}}\left(W^{-12}\right)^{\frac{1}{12}}\left(U^{-4}W^{-2}\right)^{\frac{1}{2}}T\\&=TM^{\frac{1}{12}}\\&\ll x^{1-\varepsilon_1},\end{aligned}$$

where $M \ll X^{\frac{24}{43}}$ is required. This can be obtained by using $M \ll MH^{\frac{11}{70}} \ll X^{\frac{767}{1505}}$.

Finally, by combining all the cases above, Lemma 3.2 is proved. □

**Lemma 3.3.** *Suppose that $NQRK = X$ where $N(s), Q(s), R(s)$ and $K(s)$ are Dirichlet polynomials and $G(s) = N(s)Q(s)R(s)K(s)$. Let $b = 1 + \frac{1}{\log X}$, $T_0 = (\log X)^{B/\varepsilon}$. Assume further that for $T_0 \leqslant |t| \leqslant 2X$, $N(b+it)Q(b+it) \ll (\log x)^{-B/\varepsilon}$ and $R(b+it) \ll (\log x)^{-B/\varepsilon}$. Moreover, suppose that $N$, $Q$ and $R$ satisfy one of the following 10 conditions:*

*(1)* $X^{\frac{27267}{66994}} \ll NQ \ll X^{\frac{231}{559}},\ X^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-\frac{35}{23}} X^{\frac{767}{989}}$;

*(2)* $X^{\frac{231}{559}} \ll NQ \ll X^{\frac{3275}{7826}},\ X^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-1} X^{\frac{313}{559}}$;

*(3)* $X^{\frac{3275}{7826}} \ll NQ \ll X^{\frac{15005}{33497}},\ X^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-\frac{41}{27}} X^{\frac{902}{1161}}$;

*(4)* $X^{\frac{13074}{28595}} \ll NQ \ll X^{\frac{20}{43}},\ X^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-\frac{1}{5}} X^{\frac{12}{43}}$;

*(5)* $X^{\frac{20}{43}} \ll NQ \ll X^{\frac{41}{86}},\ X^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{\frac{1}{7}} X^{\frac{55}{301}}$;

*(6)* $X^{\frac{41}{86}} \ll NQ \ll X^{\frac{227}{473}},\ X^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-1} X^{\frac{219}{301}}$;

*(7)* $X^{\frac{227}{473}} \ll NQ \ll X^{\frac{2333}{4859}},\ X^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-\frac{58}{9}} X^{\frac{1264}{387}}$;

*(8)* $X^{\frac{2333}{4859}} \ll NQ \ll X^{\frac{2501}{5203}},\ X^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-\frac{1}{6}} X^{\frac{65}{258}}$;

*(9)* $X^{\frac{2501}{5203}} \ll NQ \ll X^{\frac{499}{1032}},\ X^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-2} X^{\frac{536}{473}}$;

*(10)* $X^{\frac{499}{1032}} \ll NQ \ll X^{\frac{28277}{57190}},\ X^{\frac{79}{817}} \ll R \ll Q \ll (NQ)^{-\frac{70}{11}} X^{\frac{1534}{473}}$.

*Then for $T_0 \leqslant T \leqslant X$, we have*

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |G(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}. \tag{9}$$

*Proof.* The proof is similar to that of [[27], Lemma 13]. Let $M(s) = N(s)Q(s)$ and $H(s) = R(s)$. Then by Lemma 3.2, Lemma 3.3 is proved. □

**Lemma 3.4.** *Suppose that $NQRK = X$ where $N(s), Q(s), R(s)$ and $K(s)$ are Dirichlet polynomials and $G(s) = N(s)Q(s)R(s)K(s)$. Let $b = 1 + \frac{1}{\log X}$, $T_0 = (\log X)^{B/\varepsilon}$. Assume further that for $T_0 \leqslant |t| \leqslant 2X$, $N(b+it)R(b+it) \ll (\log x)^{-B/\varepsilon}$ and $Q(b+it) \ll (\log x)^{-B/\varepsilon}$. Moreover, suppose that $N$, $Q$ and $R$ satisfy one of the following 9 conditions:*

*(1)* $X^{\frac{79}{817}} \ll R \ll Q,\ (NR)Q \ll X^{\frac{673}{1247}},\ X^{\frac{82}{473}} \ll Q,\ (NR)^{29}Q^{-1} \ll X^{\frac{427}{43}},\ X^{\frac{246}{817}} \ll (NR),\ (NR)^{-1}Q^{29} \ll X^{\frac{263}{43}},\ X^{\frac{246}{43}} \ll (NR)^{12}Q^{11}$;

*(2)* $X^{\frac{79}{817}} \ll R \ll Q,\ (NR)Q \ll X^{\frac{223}{387}},\ (NR)^{29}Q^{19} \ll X^{\frac{632}{43}},\ X^{\frac{328}{387}} \ll (NR)^2Q,\ (NR)^2Q^{11} \ll X^{\frac{120}{43}},\ X^{\frac{1230}{43}} \ll (NR)^{58}Q^{49}$;

*(3)* $X^{\frac{79}{817}} \ll R \ll Q,\ (NR)Q \ll X^{\frac{268}{473}},\ X^{\frac{41}{215}} \ll Q,\ (NR)^6Q \ll X^{\frac{94}{43}},\ X^{\frac{82}{301}} \ll (NR) \ll X^{\frac{16}{43}},\ (NR)Q^8 \ll X^{\frac{98}{43}},\ X^{\frac{123}{43}} \ll (NR)^6Q^5$;

*(4)* $X^{\frac{79}{817}} \ll R \ll Q,\ (NR)Q \ll X^{\frac{571}{817}},\ X^{\frac{82}{473}} \ll Q,\ (NR)^{12}Q \ll X^{\frac{270}{43}},\ X^{\frac{574}{1247}} \ll (NR),\ (NR)^{-1}Q^{19} \ll X^{\frac{161}{43}},\ X^{\frac{328}{43}} \ll (NR)^{12}Q^{11}$;

*(5)* $X^{\frac{79}{817}} \ll R \ll Q,\ (NR)^2Q \ll X^{\frac{446}{387}},\ (NR)^{58}Q^9 \ll X^{\frac{1264}{43}},\ X^{\frac{164}{387}} \ll (NR),\ (NR)Q^5 \ll X^{\frac{60}{43}},\ X^{\frac{615}{43}} \ll (NR)^{29}Q^{10}$;

*(6)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{27}{43}} \ll (NR)Q \ll X^{\frac{219}{301}},\ X^{\frac{41}{215}} \ll Q,\ (NR)^6Q \ll X^{\frac{135}{43}},\ X^{\frac{205}{473}} \ll (NR),\ (NR)^{-1}Q^7 \ll X^{\frac{55}{43}},\ X^{\frac{164}{43}} \ll (NR)^6Q^5$;

*(7)* $X^{\frac{79}{817}} \ll R \ll Q,\ (NR)Q \ll X^{\frac{268}{473}},\ (NR)^{35}Q^{23} \ll X^{\frac{767}{43}},\ X^{\frac{410}{473}} \ll (NR)^2Q,\ (NR)^2Q^{13} \ll X^{\frac{130}{43}},\ X^{\frac{1476}{43}} \ll (NR)^{70}Q^{59}$;

*(8)* $X^{\frac{79}{817}} \ll R \ll Q,\ (NR)Q \ll X^{\frac{313}{559}},\ (NR)^{41}Q^{27} \ll X^{\frac{902}{43}},\ X^{\frac{492}{559}} \ll (NR)^2Q,\ (NR)^2Q^{15} \ll X^{\frac{138}{43}},\ X^{\frac{1722}{43}} \ll (NR)^{82}Q^{69}$;

*(9)* $X^{\frac{79}{817}} \ll R \ll Q,\ (NR)^2Q \ll X^{\frac{536}{473}},\ (NR)^{70}Q^{11} \ll X^{\frac{1534}{43}},\ X^{\frac{205}{473}} \ll (NR),\ (NR)Q^6 \ll X^{\frac{65}{43}},\ X^{\frac{738}{43}} \ll (NR)^{35}Q^{12}$.

*Then for $T_0 \leqslant T \leqslant X$, we have*

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |G(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}. \tag{10}$$

*Proof.* Let $M(s) = N(s)R(s)$ and $H(s) = Q(s)$ and by Lemma 3.2, Lemma 3.4 is proved. □

**Lemma 3.5.** *Suppose that $NQRK = X$ where $N(s), Q(s), R(s)$ and $K(s)$ are Dirichlet polynomials and $G(s) = N(s)Q(s)R(s)K(s)$. Let $b = 1 + \frac{1}{\log X}$, $T_0 = (\log X)^{B/\varepsilon}$. Assume further that for $T_0 \leqslant |t| \leqslant 2X, Q(b+it)R(b+it) \ll (\log x)^{-B/\varepsilon}$ and $N(b+it) \ll (\log x)^{-B/\varepsilon}$. Moreover, suppose that $N$, $Q$ and $R$ satisfy one of the following 9 conditions:*

*(1)* $X^{\frac{79}{817}} \ll R \ll Q,\ (QR)N \ll X^{\frac{673}{1247}},\ X^{\frac{82}{473}} \ll N,\ (QR)^{29}N^{-1} \ll X^{\frac{427}{43}},\ X^{\frac{246}{817}} \ll (QR),\ (QR)^{-1}N^{29} \ll X^{\frac{263}{43}},\ X^{\frac{246}{43}} \ll (QR)^{12}N^{11}$;

*(2)* $X^{\frac{79}{817}} \ll R \ll Q,\ (QR)N \ll X^{\frac{223}{387}},\ (QR)^{29}N^{19} \ll X^{\frac{632}{43}},\ X^{\frac{328}{387}} \ll (QR)^2N,\ (QR)^2N^{11} \ll X^{\frac{120}{43}},\ X^{\frac{1230}{43}} \ll (QR)^{58}N^{49}$;

*(3)* $X^{\frac{79}{817}} \ll R \ll Q,\ (QR)N \ll X^{\frac{268}{473}},\ X^{\frac{41}{215}} \ll N,\ (QR)^6N \ll X^{\frac{94}{43}},\ X^{\frac{82}{301}} \ll (QR) \ll X^{\frac{16}{43}},\ (QR)N^8 \ll X^{\frac{98}{43}},\ X^{\frac{123}{43}} \ll (QR)^6N^5$;

*(4)* $X^{\frac{79}{817}} \ll R \ll Q,\ (QR)N \ll X^{\frac{571}{817}},\ X^{\frac{82}{473}} \ll N,\ (QR)^{12}N \ll X^{\frac{270}{43}},\ X^{\frac{574}{1247}} \ll (QR),\ (QR)^{-1}N^{19} \ll X^{\frac{161}{43}},\ X^{\frac{328}{43}} \ll (QR)^{12}N^{11}$;

*(5)* $X^{\frac{79}{817}} \ll R \ll Q,\ (QR)^2N \ll X^{\frac{446}{387}},\ (QR)^{58}N^9 \ll X^{\frac{1264}{43}},\ X^{\frac{164}{387}} \ll (QR),\ (QR)N^5 \ll X^{\frac{60}{43}},\ X^{\frac{615}{43}} \ll (QR)^{29}N^{10}$;

*(6)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{27}{43}} \ll (QR)N \ll X^{\frac{219}{301}},\ X^{\frac{41}{215}} \ll N,\ (QR)^6N \ll X^{\frac{135}{43}},\ X^{\frac{205}{473}} \ll (QR),\ (QR)^{-1}N^7 \ll X^{\frac{55}{43}},\ X^{\frac{164}{43}} \ll (QR)^6N^5$;

*(7)* $X^{\frac{79}{817}} \ll R \ll Q,\ (QR)N \ll X^{\frac{268}{473}},\ (QR)^{35}N^{23} \ll X^{\frac{767}{43}},\ X^{\frac{410}{473}} \ll (QR)^2N,\ (QR)^2N^{13} \ll X^{\frac{130}{43}},\ X^{\frac{1476}{43}} \ll (QR)^{70}N^{59}$;

*(8)* $X^{\frac{79}{817}} \ll R \ll Q,\ (QR)N \ll X^{\frac{313}{559}},\ (QR)^{41}N^{27} \ll X^{\frac{902}{43}},\ X^{\frac{492}{559}} \ll (QR)^2N,\ (QR)^2N^{15} \ll X^{\frac{138}{43}},\ X^{\frac{1722}{43}} \ll (QR)^{82}N^{69}$;

*(9)* $X^{\frac{79}{817}} \ll R \ll Q,\ (QR)^2N \ll X^{\frac{536}{473}},\ (QR)^{70}N^{11} \ll X^{\frac{1534}{43}},\ X^{\frac{205}{473}} \ll (QR),\ (QR)N^6 \ll X^{\frac{65}{43}},\ X^{\frac{738}{43}} \ll (QR)^{35}N^{12}$.

*Then for $T_0 \leqslant T \leqslant X$, we have*

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |G(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}. \tag{11}$$

*Proof.* Let $M(s) = Q(s)R(s)$ and $H(s) = N(s)$ and by Lemma 3.2, Lemma 3.5 is proved. □

**Lemma 3.6.** *Suppose that $NQRK = X$ where $N(s), Q(s), R(s)$ and $K(s)$ are Dirichlet polynomials and $G(s) = N(s)Q(s)R(s)K(s)$. Let $b = 1 + \frac{1}{\log X}$, $T_0 = (\log X)^{B/\varepsilon}$. Assume further that for $T_0 \leqslant |t| \leqslant 2X, Q(b+it)R(b+it) \ll (\log x)^{-B/\varepsilon}$ and $N(b+it) \ll (\log x)^{-B/\varepsilon}$. Moreover, suppose that $N$, $Q$ and $R$ satisfy one of the following 9 conditions:*

*(1)* $X^{\frac{79}{817}} \ll R \ll Q,\ N(QR) \ll X^{\frac{673}{1247}},\ X^{\frac{82}{473}} \ll (QR),\ N^{29}(QR)^{-1} \ll X^{\frac{427}{43}},\ X^{\frac{246}{817}} \ll N,\ N^{-1}(QR)^{29} \ll X^{\frac{263}{43}},\ X^{\frac{246}{43}} \ll N^{12}(QR)^{11}$;

*(2)* $X^{\frac{79}{817}} \ll R \ll Q,\ N(QR) \ll X^{\frac{223}{387}},\ N^{29}(QR)^{19} \ll X^{\frac{632}{43}},\ X^{\frac{328}{387}} \ll N^2(QR),\ N^2(QR)^{11} \ll X^{\frac{120}{43}},\ X^{\frac{1230}{43}} \ll N^{58}(QR)^{49}$;

*(3)* $X^{\frac{79}{817}} \ll R \ll Q,\ N(QR) \ll X^{\frac{268}{473}},\ X^{\frac{41}{215}} \ll (QR),\ N^6(QR) \ll X^{\frac{94}{43}},\ X^{\frac{82}{301}} \ll N \ll X^{\frac{16}{43}},\ N(QR)^8 \ll X^{\frac{98}{43}},\ X^{\frac{123}{43}} \ll N^6(QR)^5$;

*(4)* $X^{\frac{79}{817}} \ll R \ll Q,\ N(QR) \ll X^{\frac{571}{817}},\ X^{\frac{82}{473}} \ll (QR),\ N^{12}(QR) \ll X^{\frac{270}{43}},\ X^{\frac{574}{1247}} \ll N,\ N^{-1}(QR)^{19} \ll X^{\frac{161}{43}},\ X^{\frac{328}{43}} \ll N^{12}(QR)^{11}$;

*(5)* $X^{\frac{79}{817}} \ll R \ll Q,\ N^2(QR) \ll X^{\frac{446}{387}},\ N^{58}(QR)^9 \ll X^{\frac{1264}{43}},\ X^{\frac{164}{387}} \ll N,\ N(QR)^5 \ll X^{\frac{60}{43}},\ X^{\frac{615}{43}} \ll N^{29}(QR)^{10}$;

*(6)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{27}{43}} \ll N(QR) \ll X^{\frac{219}{301}},\ X^{\frac{41}{215}} \ll (QR),\ N^6(QR) \ll X^{\frac{135}{43}},\ X^{\frac{205}{473}} \ll N,\ N^{-1}(QR)^7 \ll X^{\frac{55}{43}},\ X^{\frac{164}{43}} \ll N^6(QR)^5$;

*(7)* $X^{\frac{79}{817}} \ll R \ll Q,\ N(QR) \ll X^{\frac{268}{473}},\ N^{35}(QR)^{23} \ll X^{\frac{767}{43}},\ X^{\frac{410}{473}} \ll N^2(QR),\ N^2(QR)^{13} \ll X^{\frac{130}{43}},\ X^{\frac{1476}{43}} \ll N^{70}(QR)^{59}$;

*(8)* $X^{\frac{79}{817}} \ll R \ll Q,\ N(QR) \ll X^{\frac{313}{559}},\ N^{41}(QR)^{27} \ll X^{\frac{902}{43}},\ X^{\frac{492}{559}} \ll N^2(QR),\ N^2(QR)^{15} \ll X^{\frac{138}{43}},\ X^{\frac{1722}{43}} \ll N^{82}(QR)^{69}$;

*(9)* $X^{\frac{79}{817}} \ll R \ll Q,\ N^2(QR) \ll X^{\frac{536}{473}},\ N^{70}(QR)^{11} \ll X^{\frac{1534}{43}},\ X^{\frac{205}{473}} \ll N,\ N(QR)^6 \ll X^{\frac{65}{43}},\ X^{\frac{738}{43}} \ll N^{35}(QR)^{12}$.

*Then for* $T_0 \leqslant T \leqslant X$*, we have*

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |G(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}. \tag{12}$$

*Proof.* Let $M(s) = N(s)$ and $H(s) = Q(s)R(s)$ and by Lemma 3.2, Lemma 3.6 is proved. □

## 4. Arithmetic Information II

In this section we are looking for more type–I information. In [27], Jia used a mean value bound of Deshouillers and Iwaniec [8], which makes an approximation to the sixth–power moment of the Riemann zeta–function. Now we shall use another mean value bound of Watt [45], which is stronger than that of Deshouillers and Iwaniec when the length of interval is $n^{\frac{1}{21.5}+\varepsilon}$.

**Lemma 4.1.** *Suppose that* $N(s)$ *is a Dirichlet polynomial and* $L(s) = \sum_{l\sim L} l^{-s}$*. Let* $T \geqslant 1$*, then*

$$\int_T^{2T} \left|L\left(\frac{1}{2}+it\right)\right|^4 \left|N\left(\frac{1}{2}+it\right)\right|^2 dt \ll \left(T + N^2 T^{\frac{1}{2}} + NL^2 T^{-2}\right) T^{\varepsilon_1}.$$

*Proof.* For $c_1 L \leqslant T^{\frac{1}{2}}$ and $N \leqslant T$, by the main theorem proved in [45] we have

$$\int_T^{2T} \left|L\left(\frac{1}{2}+it\right)\right|^4 \left|N\left(\frac{1}{2}+it\right)\right|^2 dt \ll \left(T + N^2 T^{\frac{1}{2}}\right) T^{\varepsilon_1}.$$

For $c_1 L \leqslant T^{\frac{1}{2}}$ and $N > T$, by the classical mean value estimate (see [27]) we have

$$\begin{aligned}\int_T^{2T} \left|L\left(\frac{1}{2}+it\right)\right|^4 \left|N\left(\frac{1}{2}+it\right)\right|^2 dt &\ll \left(L^2 N + T\right)(LN)^{\varepsilon_1} \\ &\ll \left(T + N^2 T^{\frac{1}{2}}\right) T^{\varepsilon_1}.\end{aligned}$$

When $T^{\frac{1}{2}} < c_1 L \leqslant 2T$, by a reflection principle based on an approximate functional equation, we can get the same bound as above. By this process we may replace $L$ by $L_0 \sim T/L$, so that $L_0 \ll T^{\frac{1}{2}}$. For this, one can see [[8], Section 2], [[4], Lemma 2] or [[1], Lemma 5.2] for a detailed proof with $N \leqslant T$. For $N > T$, by the classical mean value estimate we have

$$\begin{aligned}\int_T^{2T} \left|L\left(\frac{1}{2}+it\right)\right|^4 \left|N\left(\frac{1}{2}+it\right)\right|^2 dt &\ll \left(L_0^2 N + T\right)(L_0 N)^{\varepsilon_1} + \text{Error} \\ &\ll \left(T + N^2 T^{\frac{1}{2}}\right) T^{\varepsilon_1}.\end{aligned}$$

When $2T < c_1 L$, by the method in [27] we have

$$\int_T^{2T} \left|L\left(\frac{1}{2}+it\right)\right|^4 \left|N\left(\frac{1}{2}+it\right)\right|^2 dt \ll NL^2 T^{-2}.$$

(Actually we have a better bound $L^2 T^{-4}(N+T)$ in this case, but this won't bring any improvement to the next lemma.)

Finally, by combining all the cases above, Lemma 4.1 is proved. □

**Lemma 4.2.** *Suppose that* $MHL = X$ *where* $M(s), H(s)$ *are Dirichlet polynomials and* $L(s) = \sum_{l\sim L} l^{-s}$*. Let* $b = 1 + \frac{1}{\log X}$*,* $T_1 = \sqrt{L}$*. Moreover, suppose that* $M$ *and* $H$ *satisfy one of the following 2 conditions:*

*(1)* $M^2 H \ll X^{2-\frac{20.5}{21.5}},\ M^4 H^6 \ll X^{4-\frac{20.5}{21.5}},\ H^4 \ll X^{4-\frac{61.5}{21.5}}$;

*(2)* $M \ll X^{\frac{45}{86}},\ H \ll X^{\frac{41}{344}}$.

*Then for* $T_1 \leqslant T \leqslant X$*, we have*

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |M(b+it)H(b+it)L(b+it)|^2 dt \ll \eta^2 x^{-2\varepsilon_1}. \tag{13}$$

*Proof.* We can prove this by using Lemma 4.1 and the methods in [[27], Lemmas 3,4]. For condition (1) we apply Lemma 4.1 to $L$ and $N$, while for condition (2) we apply Lemma 4.1 to $L$ and $N^2$. One can see [27] for an explanation. $\square$

**Lemma 4.3.** *Suppose that $NQRL = X$ where $N(s), Q(s)$ and $R(s)$ are Dirichlet polynomials, $L(s) = \sum_{l\sim L} l^{-s}$ and $G(s) = N(s)Q(s)R(s)L(s)$. Let $b = 1 + \frac{1}{\log X}$, $T_1 = \sqrt{L}$. Assume further that for $T_1 \leqslant |t| \leqslant 2X$, $N(b+it)Q(b+it) \ll (\log x)^{-B/\varepsilon}$ and $R(b+it) \ll (\log x)^{-B/\varepsilon}$. Moreover, suppose that $N$, $Q$ and $R$ satisfy one of the following 9 conditions:*
*(1)* $X^{\frac{79}{817}} \ll R \ll Q,\ (NQ)^2 R \ll X^{2-\frac{20.5}{21.5}},\ (NQ)^4 R^6 \ll X^{4-\frac{20.5}{21.5}},\ R^4 \ll X^{4-\frac{61.5}{21.5}}$;
*(2)* $X^{\frac{79}{817}} \ll R \ll Q,\ (NQ) \ll X^{\frac{45}{86}},\ R \ll X^{\frac{41}{344}}$;
*(3)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{55}{129}} \ll (NQ) \ll X^{\frac{20}{43}},\ R \ll (NQ)^{-\frac{1}{5}} X^{\frac{12}{43}}$;
*(4)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{20}{43}} \ll (NQ) \ll X^{\frac{41}{86}},\ R \ll (NQ)^{\frac{1}{7}} X^{\frac{55}{301}}$;
*(5)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{41}{86}} \ll (NQ) \ll X^{\frac{227}{473}},\ R \ll (NQ)^{-1} X^{\frac{219}{301}}$;
*(6)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{227}{473}} \ll (NQ) \ll X^{\frac{2333}{4859}},\ R \ll (NQ)^{-\frac{58}{9}} X^{\frac{1264}{387}}$;
*(7)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{2333}{4859}} \ll (NQ) \ll X^{\frac{2501}{5203}},\ R \ll (NQ)^{-\frac{1}{6}} X^{\frac{65}{258}}$;
*(8)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{2501}{5203}} \ll (NQ) \ll X^{\frac{499}{1032}},\ R \ll (NQ)^{-2} X^{\frac{536}{473}}$;
*(9)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{499}{1032}} \ll (NQ) \ll X^{\frac{28277}{57190}},\ R \ll (NQ)^{-\frac{70}{11}} X^{\frac{1534}{473}}$.
*Then for $T_1 \leqslant T \leqslant X$, we have*

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |G(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}. \tag{14}$$

*Proof.* The proof is similar to that of [[27], Lemmas 14,15]. Let $M(s) = N(s)Q(s)$ and $H(s) = R(s)$. Then by Lemma 4.2 and Lemma 3.2, Lemma 4.3 is proved. $\square$

**Lemma 4.4.** *Suppose that $NQRL = X$ where $N(s), Q(s)$ and $R(s)$ are Dirichlet polynomials, $L(s) = \sum_{l\sim L} l^{-s}$ and $G(s) = N(s)Q(s)R(s)L(s)$. Let $b = 1 + \frac{1}{\log X}$, $T_1 = \sqrt{L}$. Assume further that for $T_1 \leqslant |t| \leqslant 2X$, $Q(b+it)R(b+it) \ll (\log x)^{-B/\varepsilon}$ and $N(b+it) \ll (\log x)^{-B/\varepsilon}$. Moreover, suppose that $N$, $Q$ and $R$ satisfy one of the following 9 conditions:*
*(1)* $X^{\frac{79}{817}} \ll R \ll Q,\ (QR)^2 N \ll X^{2-\frac{20.5}{21.5}},\ (QR)^4 N^6 \ll X^{4-\frac{20.5}{21.5}},\ N^4 \ll X^{4-\frac{61.5}{21.5}}$;
*(2)* $X^{\frac{79}{817}} \ll R \ll Q,\ (QR) \ll X^{\frac{45}{86}},\ N \ll X^{\frac{41}{344}}$;
*(3)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{55}{129}} \ll (QR) \ll X^{\frac{20}{43}},\ N \ll (QR)^{-\frac{1}{5}} X^{\frac{12}{43}}$;
*(4)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{20}{43}} \ll (QR) \ll X^{\frac{41}{86}},\ N \ll (QR)^{\frac{1}{7}} X^{\frac{55}{301}}$;
*(5)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{41}{86}} \ll (QR) \ll X^{\frac{227}{473}},\ N \ll (QR)^{-1} X^{\frac{219}{301}}$;
*(6)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{227}{473}} \ll (QR) \ll X^{\frac{2333}{4859}},\ N \ll (QR)^{-\frac{58}{9}} X^{\frac{1264}{387}}$;
*(7)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{2333}{4859}} \ll (QR) \ll X^{\frac{2501}{5203}},\ N \ll (QR)^{-\frac{1}{6}} X^{\frac{65}{258}}$;
*(8)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{2501}{5203}} \ll (QR) \ll X^{\frac{499}{1032}},\ N \ll (QR)^{-2} X^{\frac{536}{473}}$;
*(9)* $X^{\frac{79}{817}} \ll R \ll Q,\ X^{\frac{499}{1032}} \ll (QR) \ll X^{\frac{28277}{57190}},\ N \ll (QR)^{-\frac{70}{11}} X^{\frac{1534}{473}}$.
*Then for $T_1 \leqslant T \leqslant X$, we have*

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |G(b+it)|^2 dt \ll \eta^2 (\log x)^{-10B}. \tag{15}$$

*Proof.* Let $M(s) = Q(s)R(s)$ and $H(s) = N(s)$. Then by Lemma 4.2 and Lemma 3.2, Lemma 4.4 is proved. $\square$

## 5. Sieve error terms

For type–I terms we will use Iwaniec's linear sieve method and this requires us to control the errors occured. The following two lemmas will help us deal with the error terms when we apply Iwaniec's linear sieve in Section 7.

**Lemma 5.1.** *Suppose that $M \ll X^{\frac{45}{86}}, H \ll X^{\frac{41}{344}}$ and $a(m)=O(1)$, $b(h)=O(1)$. Then for real numbers $x \in [X,2X]$, except for $O\left(X(\log X)^{-B}\right)$ values, we have*

$$\sum_{\substack{m\sim M\\ h\sim H}} a(m)b(h)\left(\sum_{x<mhl\leqslant x+\eta x} 1-\frac{\eta x}{mh}\right) \ll \eta x(\log x)^{-B}. \tag{16}$$

*Proof.* The proof is similar to that of [[27], Lemma 16] where the condition (2) of Lemma 4.2 is used. □

**Lemma 5.2.** *Suppose that $a(n)=O(1)$, $b(q)=O(1)$, $c(r)=O(1)$ and $N$, $Q$, $R$ satisfy one of the 18 conditions in Lemma 4.3 and Lemma 4.4. Let $b=1+\frac{1}{\log X}$, $T_1=\sqrt{L}$ and $NQRL=X$. Assume further that for $T_1 \leqslant |t| \leqslant 2X, N(b+it)Q(b+it)R(b+it) \ll (\log x)^{-B/\varepsilon}$. Then for real numbers $x\in[X,2X]$, except for $O\left(X(\log X)^{-B}\right)$ values, we have*

$$\sum_{\substack{n\sim N\\ q\sim Q\\ r\sim R}} a(n)b(q)c(r)\left(\sum_{x<nqrl\leqslant x+\eta x} 1-\frac{\eta x}{nqr}\right) \ll \eta x(\log x)^{-B}. \tag{17}$$

*Proof.* The proof is similar to that of [[27], Lemma 17]. □

## 6. Asymptotic formulas for type–II terms

These lemmas will help us transform the sum $S\left(\mathcal{A}_{p_1\cdots p_n}, p_n\right)$ into $S\left(\mathcal{B}_{p_1\cdots p_n}, p_n\right)$ and estimate them.

**Lemma 6.1.** *Suppose that $X^{\frac{738}{817}} \ll M \ll X^{1-\delta}$ and that $0 \leqslant a(m) = O(1)$. If $m$ has a prime factor $< X^{\delta}$, then $a(m)=0$. Then for real numbers $x\in[X,2X]$, except for $O\left(X(\log X)^{-B}\right)$ values, we have*

$$\sum_{\substack{x<mp\leqslant x+\eta x\\ m\sim M}} a(m) = \eta\left(1+O\left(\frac{1}{\log x}\right)\right)\sum_{\substack{x<mp\leqslant 2x\\ m\sim M}} a(m) + O\left(\eta x(\log x)^{-B}\right). \tag{18}$$

*Proof.* The proof is similar to that of [[27], Lemma 18] where Lemma 3.1 is used. □

**Lemma 6.2.** *Suppose that $MHK=X$ where $H(s)$ and $K(s)$ are Dirichlet polynomials, $M(s)=\sum_{m\sim M}\Lambda(m)m^{-s}$. Suppose that $X^{\frac{79}{817}} \ll HK \ll X^{\frac{738}{817}}$ and that $0\leqslant a(h)=O(1)$, $0\leqslant b(k)=O(1)$. If $h$ has a prime factor $< X^{\delta}$, then $a(h)=0$. If $k$ has a prime factor $<X^{\delta}$, then $b(k)=0$. Let $G(s)=M(s)H(s)K(s)$, $b=1+\frac{1}{\log X}$ and $T_0=(\log X)^{B/\varepsilon}$. If we have*

$$\left(\min\left(\eta, \frac{1}{T}\right)\right)^2 \int_T^{2T} |G(b+it)|^2 dt \ll \eta^2(\log x)^{-10B} \tag{19}$$

*for $T_0 \leqslant T \leqslant X$, then for real numbers $x\in[X,2X]$, except for $O\left(X(\log X)^{-B}\right)$ values, we have*

$$\sum_{\substack{x<phk\leqslant x+\eta x\\ h\sim H\\ k\sim K}} a(h)b(k) = \eta\left(1+O\left(\frac{1}{\log x}\right)\right)\sum_{\substack{x<phk\leqslant 2x\\ h\sim H\\ k\sim K}} a(h)b(k) + O\left(\eta x(\log x)^{-B}\right). \tag{20}$$

*Proof.* The proof is similar to that of [[26], Lemma 4]. □

**Lemma 6.3.** *([[27], Lemma 21]). For sufficiently large $x$ and $z$, we have*

$$S(\mathcal{B},z) = \sum_{\substack{x<n\leqslant 2x\\ (n,P(z))=1}} 1 = (1+o(1))\frac{x}{\log z}\omega\left(\frac{\log x}{\log z}\right),$$

*where $\omega(u)$ denote the Buchstab function determined by the following differential–difference equation*

$$\begin{cases} \omega(u)=\frac{1}{u}, & 1\leqslant u\leqslant 2,\\ (u\omega(u))'=\omega(u-1), & u\geqslant 2.\end{cases}$$

## 7. The final decomposition

In this section, sets $\mathcal{A}$ and $\mathcal{B}$ are defined respectively. Let $\gamma$ denote the Euler's constant, $F(s)$ and $f(s)$ are determined by the following differential–difference equation

$$\begin{cases} F(s)=\frac{2e^{\gamma}}{s}, \quad f(s)=0, & 0<s\leqslant 2,\\ (sF(s))'=f(s-1), \quad (sf(s))'=F(s-1), & s\geqslant 2,\end{cases}$$

and $\omega(u)$ denote the Buchstab function defined above. Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\omega(u)\geqslant\omega_0(u)=\begin{cases}\frac{1}{u}, & 1\leqslant u<2,\\ \frac{1+\log(u-1)}{u}, & 2\leqslant u<3,\\ \frac{1+\log(u-1)}{u}+\frac{1}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt, & 3\leqslant u<4,\\ 0.5612, & u\geqslant 4,\end{cases}$$

$$\omega(u)\leqslant\omega_1(u)=\begin{cases}\frac{1}{u}, & 1\leqslant u<2,\\ \frac{1+\log(u-1)}{u}, & 2\leqslant u<3,\\ \frac{1+\log(u-1)}{u}+\frac{1}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt, & 3\leqslant u<4,\\ 0.5617, & u\geqslant 4.\end{cases}$$

We shall use $\omega_0(u)$ and $\omega_1(u)$ to give numerical lower bound for some sieve functions discussed below.

Beginning with (2), we shall use Iwaniec's linear sieve method to estimate $S_1$ and $S_2$. By Buchstab's identity we have

$$S_1=S\left(\mathcal{A},X^{\frac{79}{817}}\right)=S\left(\mathcal{A},X^{\delta}\right)-\sum_{\delta\leqslant t_1<\frac{79}{817}}S\left(\mathcal{A}_{p_1},p_1\right). \tag{21}$$

Let $z=X^{\delta}$ and $D=X^{\frac{45}{86}}$. Then by Iwaniec's linear sieve, Lemma 5.1 and arguments in [27] we have

$$\begin{aligned} S\left(\mathcal{A},X^{\delta}\right) &\geqslant \frac{\eta x}{\log z}f\left(\frac{\log D}{\log z}\right)-O\left(\frac{\varepsilon\eta x}{\log x}\right)-\sum_{m\leqslant D}a(m)\left(|\mathcal{A}_m|-\frac{\eta x}{m}\right)\\ &=\frac{e^{-\gamma}\eta x}{\delta\log x}+O\left(\frac{\varepsilon\eta x}{\log x}\right); \end{aligned}\tag{22}$$

$$\begin{aligned} S\left(\mathcal{A},X^{\delta}\right) &\leqslant \frac{\eta x}{\log z}F\left(\frac{\log D}{\log z}\right)+O\left(\frac{\varepsilon\eta x}{\log x}\right)+\sum_{m\leqslant D}a(m)\left(|\mathcal{A}_m|-\frac{\eta x}{m}\right)\\ &=\frac{e^{-\gamma}\eta x}{\delta\log x}+O\left(\frac{\varepsilon\eta x}{\log x}\right). \end{aligned}\tag{23}$$

So we have the asymptotic formula

$$S\left(\mathcal{A},X^{\delta}\right)=\frac{e^{-\gamma}\eta x}{\delta\log x}+O\left(\frac{\varepsilon\eta x}{\log x}\right). \tag{24}$$

By Lemma 6.1 and Lemma 6.3 we have

$$\begin{aligned} \sum_{\delta\leqslant t_1<\frac{79}{817}}S\left(\mathcal{A}_{p_1},p_1\right) &=\eta\sum_{\delta\leqslant t_1<\frac{79}{817}}S\left(\mathcal{B}_{p_1},p_1\right)+O\left(\frac{\varepsilon\eta x}{\log x}\right)\\ &=\frac{\eta x}{\log x}\int_{\delta}^{\frac{79}{817}}\frac{1}{t^2}\omega\left(\frac{1-t}{t}\right)dt+O\left(\frac{\delta\eta x}{\log x}\right)\\ &=\frac{\eta x}{\log x}\int_{\frac{817}{79}}^{\frac{1}{\delta}}\omega(t-1)dt+O\left(\frac{\delta\eta x}{\log x}\right)\\ &=\frac{\eta x}{\log x}\left(\frac{1}{\delta}\omega\left(\frac{1}{\delta}\right)-\frac{817}{79}\omega\left(\frac{817}{79}\right)\right)+O\left(\frac{\delta\eta x}{\log x}\right) \end{aligned}$$

$$= \frac{e^{-\gamma}\eta x}{\delta \log x} - \frac{817}{79}\omega\left(\frac{817}{79}\right)\frac{\eta x}{\log x} + O\left(\frac{\delta \eta x}{\log x}\right). \tag{25}$$

By (24) and (25) we have

$$\begin{aligned} S_1 &= S\left(\mathcal{A}, X^{\delta}\right) - \sum_{\delta \leqslant t_1 < \frac{79}{817}} S\left(\mathcal{A}_{p_1}, p_1\right) \\ &= \frac{817}{79}\omega\left(\frac{817}{79}\right)\frac{\eta x}{\log x} + O\left(\frac{\delta \eta x}{\log x}\right) \\ &\geqslant 5.806486\frac{\eta x}{\log x}. \end{aligned} \tag{26}$$

Similarly, we have

$$\begin{aligned} S_2 =& \sum_{\frac{79}{817} \leqslant t_1 < \frac{1}{2}} S\left(\mathcal{A}_{p_1}, X^{\frac{79}{817}}\right) \\ =& \sum_{\frac{79}{817} \leqslant t_1 < \frac{1}{2}} S\left(\mathcal{A}_{p_1}, X^{\delta}\right) - \sum_{\delta \leqslant t_2 < \frac{79}{817} \leqslant t_1 < \frac{1}{2}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \qquad (27) \\ =& \frac{e^{-\gamma}\eta x}{\delta \log x}\int_{\frac{79}{817}}^{\frac{1}{2}} \frac{1}{t} dt \\ & - \left(\frac{e^{-\gamma}\eta x}{\delta \log x}\int_{\frac{79}{817}}^{\frac{1}{2}} \frac{1}{t} dt - \frac{\eta x}{\log x}\frac{817}{79}\int_{\frac{79}{817}}^{\frac{1}{2}} \frac{1}{t}\omega\left(\frac{817}{79}(1-t)\right) dt\right) + O\left(\frac{\delta \eta x}{\log x}\right) \\ =& \left(\frac{817}{79}\int_{\frac{79}{817}}^{\frac{1}{2}} \frac{1}{t}\omega\left(\frac{817}{79}(1-t)\right) dt\right)\frac{\eta x}{\log x} + O\left(\frac{\delta \eta x}{\log x}\right) \\ \leqslant& 9.540312\frac{\eta x}{\log x}. \end{aligned} \tag{28}$$

Before estimating $S_3$, we first define the disjoint regions $U_{01}$–$U_{06}$ as

$$\begin{aligned} U_{01}(t_1, t_2) :=& \left\{(t_1, t_2) \in U_{0101} \cup U_{0102} \cup \cdots \cup U_{0109},\right. \\ & \left.\frac{79}{817} \leqslant t_1 < \frac{1}{2},\ \frac{79}{817} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}, \\ U_{02}(t_1, t_2) :=& \left\{(t_1, t_2) \in U_{0201} \cup U_{0202} \cup \cdots \cup U_{0210},\ (t_1, t_2) \notin U_{01},\right. \\ & \left.\frac{79}{817} \leqslant t_1 < \frac{1}{2},\ \frac{79}{817} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}, \\ U_{03}(t_1, t_2) :=& \left\{(t_1, t_2) \in U_{0301} \cup U_{0302} \cup \cdots \cup U_{0327},\ (t_1, t_2) \notin U_{01} \cup U_{02},\right. \\ & \left.\frac{79}{817} \leqslant t_1 < \frac{1}{2},\ \frac{79}{817} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}, \\ U_{04}(t_1, t_2) :=& \left\{(t_1, t_2) \in U_{0401} \cup U_{0402} \cup \cdots \cup U_{0409},\ (t_1, t_2) \notin U_{01} \cup U_{02} \cup U_{03},\right. \\ & \left.\frac{79}{817} \leqslant t_1 < \frac{1}{2},\ \frac{79}{817} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}, \\ U_{05}(t_1, t_2) :=& \left\{(t_1, t_2) \in U_{0501} \cup U_{0502} \cup \cdots \cup U_{0509},\ (t_1, t_2) \notin U_{01} \cup U_{02} \cup U_{03} \cup U_{04},\right. \\ & \left.\frac{79}{817} \leqslant t_1 < \frac{1}{2},\ \frac{79}{817} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}, \\ U_{06}(t_1, t_2) :=& \left\{(t_1, t_2) \in U_{0601} \cup U_{0602},\ (t_1, t_2) \notin U_{01} \cup U_{02} \cup U_{03} \cup U_{04} \cup U_{05},\right. \\ & \left.\frac{79}{817} \leqslant t_1 < \frac{1}{2},\ \frac{79}{817} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}, \end{aligned}$$

where

$$U_{0101}(t_1,t_2) := \left\{t_1+t_2 \leqslant \frac{673}{1247},\ \frac{82}{473} \leqslant t_2,\ 29t_1 - t_2 \leqslant \frac{427}{43},\ \frac{246}{817} \leqslant t_1,\ 29t_2 - t_1 \leqslant \frac{263}{43},\ \frac{246}{43} \leqslant 12t_1 + 11t_2\right\},$$

$$U_{0102}(t_1,t_2) := \left\{t_1+t_2 \leqslant \frac{223}{387},\ 29t_1 + 19t_2 \leqslant \frac{632}{43},\ \frac{328}{387} \leqslant 2t_1 + t_2,\ 2t_1 + 11t_2 \leqslant \frac{120}{43},\ \frac{1230}{43} \leqslant 58t_1 + 49t_2\right\},$$

$$U_{0103}(t_1,t_2) := \left\{t_1+t_2 \leqslant \frac{268}{473},\ \frac{41}{215} \leqslant t_2,\ 6t_1 + t_2 \leqslant \frac{94}{43},\ \frac{82}{301} \leqslant t_1 \leqslant \frac{16}{43},\ t_1 + 8t_2 \leqslant \frac{98}{43},\ \frac{123}{43} \leqslant 6t_1 + 5t_2\right\},$$

$$U_{0104}(t_1,t_2) := \left\{t_1+t_2 \leqslant \frac{571}{817},\ \frac{82}{473} \leqslant t_2,\ 12t_1 + t_2 \leqslant \frac{270}{43},\ \frac{574}{1247} \leqslant t_1,\ 19t_2 - t_1 \leqslant \frac{161}{43},\ \frac{328}{43} \leqslant 12t_1 + 11t_2\right\},$$

$$U_{0105}(t_1,t_2) := \left\{2t_1+t_2 \leqslant \frac{446}{387},\ 58t_1 + 9t_2 \leqslant \frac{1264}{43},\ \frac{164}{387} \leqslant t_1,\ t_1 + 5t_2 \leqslant \frac{60}{43},\ \frac{615}{43} \leqslant 29t_1 + 10t_2\right\},$$

$$U_{0106}(t_1,t_2) := \left\{\frac{27}{43} \leqslant t_1+t_2 \leqslant \frac{219}{301},\ \frac{41}{215} \leqslant t_2,\ 6t_1 + t_2 \leqslant \frac{135}{43},\ \frac{205}{473} \leqslant t_1,\ 7t_2 - t_1 \leqslant \frac{55}{43},\ \frac{164}{43} \leqslant 6t_1 + 5t_2\right\},$$

$$U_{0107}(t_1,t_2) := \left\{t_1+t_2 \leqslant \frac{268}{473},\ 35t_1 + 23t_2 \leqslant \frac{767}{43},\ \frac{410}{473} \leqslant 2t_1 + t_2,\ 2t_1 + 13t_2 \leqslant \frac{130}{43},\ \frac{1476}{43} \leqslant 70t_1 + 59t_2\right\},$$

$$U_{0108}(t_1,t_2) := \left\{t_1+t_2 \leqslant \frac{313}{559},\ 41t_1 + 27t_2 \leqslant \frac{902}{43},\ \frac{492}{559} \leqslant 2t_1 + t_2,\ 2t_1 + 15t_2 \leqslant \frac{138}{43},\ \frac{1722}{43} \leqslant 82t_1 + 69t_2\right\},$$

$$U_{0109}(t_1,t_2) := \left\{2t_1+t_2 \leqslant \frac{536}{473},\ 70t_1 + 11t_2 \leqslant \frac{1534}{43},\ \frac{205}{473} \leqslant t_1,\ t_1 + 6t_2 \leqslant \frac{65}{43},\ \frac{738}{43} \leqslant 35t_1 + 12t_2\right\},$$

$$U_{0201}(t_1,t_2) := \left\{\frac{27267}{66994} \leqslant t_1+t_2 \leqslant \frac{231}{559},\ \frac{79}{817} \leqslant t_2 \leqslant -\frac{35}{23}(t_1+t_2) + \frac{767}{989}\right\},$$

$$U_{0202}(t_1,t_2) := \left\{\frac{231}{559} \leqslant t_1+t_2 \leqslant \frac{3275}{7826},\ \frac{79}{817} \leqslant t_2 \leqslant -(t_1+t_2) + \frac{313}{559}\right\},$$

$$U_{0203}(t_1,t_2) := \left\{\frac{3275}{7826} \leqslant t_1+t_2 \leqslant \frac{15005}{33497},\ \frac{79}{817} \leqslant t_2 \leqslant -\frac{41}{27}(t_1+t_2) + \frac{902}{1161}\right\},$$

$$U_{0204}(t_1,t_2) := \left\{\frac{13074}{28595} \leqslant t_1+t_2 \leqslant \frac{20}{43},\ \frac{79}{817} \leqslant t_2 \leqslant -\frac{1}{5}(t_1+t_2) + \frac{12}{43}\right\},$$

$$U_{0205}(t_1,t_2) := \left\{\frac{20}{43} \leqslant t_1+t_2 \leqslant \frac{41}{86},\ \frac{79}{817} \leqslant t_2 \leqslant \frac{1}{7}(t_1+t_2) + \frac{55}{301}\right\},$$

$$U_{0206}(t_1,t_2) := \left\{\frac{41}{86} \leqslant t_1+t_2 \leqslant \frac{227}{473},\ \frac{79}{817} \leqslant t_2 \leqslant -(t_1+t_2) + \frac{219}{301}\right\},$$

$$U_{0207}(t_1,t_2) := \left\{\frac{227}{473} \leqslant t_1+t_2 \leqslant \frac{2333}{4859},\ \frac{79}{817} \leqslant t_2 \leqslant -\frac{58}{9}(t_1+t_2) + \frac{1264}{387}\right\},$$

$$U_{0208}(t_1,t_2) := \left\{\frac{2333}{4859} \leqslant t_1+t_2 \leqslant \frac{2501}{5203},\ \frac{79}{817} \leqslant t_2 \leqslant -\frac{1}{6}(t_1+t_2) + \frac{65}{258}\right\},$$

$$U_{0209}(t_1,t_2) := \left\{\frac{2501}{5203} \leqslant t_1+t_2 \leqslant \frac{499}{1032},\ \frac{79}{817} \leqslant t_2 \leqslant -2(t_1+t_2) + \frac{536}{473}\right\},$$

$$U_{0210}(t_1,t_2) := \left\{\frac{499}{1032} \leqslant t_1+t_2 \leqslant \frac{28277}{57190},\ \frac{79}{817} \leqslant t_2 \leqslant -\frac{70}{11}(t_1+t_2) + \frac{1534}{473}\right\},$$

$$U_{0301}(t_1,t_2) := \Bigg\{(t_1+t_2)+t_2 \leqslant \frac{673}{1247},\ \frac{82}{473} \leqslant t_2,\ 29(t_1+t_2) - t_2 \leqslant \frac{427}{43},\ \frac{246}{817} \leqslant \left(t_1 + \frac{79}{817}\right),$$
$$29t_2 - \left(t_1 + \frac{79}{817}\right) \leqslant \frac{263}{43},\ \frac{246}{43} \leqslant 12\left(t_1 + \frac{79}{817}\right) + 11t_2\Bigg\},$$

$$U_{0302}(t_1,t_2) := \Bigg\{(t_1+t_2)+t_2 \leqslant \frac{223}{387},\ 29(t_1+t_2) + 19t_2 \leqslant \frac{632}{43},\ \frac{328}{387} \leqslant 2\left(t_1 + \frac{79}{817}\right) + t_2,$$

$$2(t_1+t_2)+11t_2 \leqslant \frac{120}{43},\ \frac{1230}{43} \leqslant 58\left(t_1+\frac{79}{817}\right)+49t_2\Big\},$$

$$U_{0303}(t_1,t_2) := \Big\{(t_1+t_2)+t_2 \leqslant \frac{268}{473},\ \frac{41}{215} \leqslant t_2,\ 6(t_1+t_2)+t_2 \leqslant \frac{94}{43},\ \frac{82}{301} \leqslant \left(t_1+\frac{79}{817}\right),$$
$$(t_1+t_2) \leqslant \frac{16}{43},\ (t_1+t_2)+8t_2 \leqslant \frac{98}{43},\ \frac{123}{43} \leqslant 6\left(t_1+\frac{79}{817}\right)+5t_2\Big\},$$

$$U_{0304}(t_1,t_2) := \Big\{(t_1+t_2)+t_2 \leqslant \frac{571}{817},\ \frac{82}{473} \leqslant t_2,\ 12(t_1+t_2)+t_2 \leqslant \frac{270}{43},\ \frac{574}{1247} \leqslant \left(t_1+\frac{79}{817}\right),$$
$$19t_2-\left(t_1+\frac{79}{817}\right) \leqslant \frac{161}{43},\ \frac{328}{43} \leqslant 12\left(t_1+\frac{79}{817}\right)+11t_2\Big\},$$

$$U_{0305}(t_1,t_2) := \Big\{2(t_1+t_2)+t_2 \leqslant \frac{446}{387},\ 58(t_1+t_2)+9t_2 \leqslant \frac{1264}{43},\ \frac{164}{387} \leqslant \left(t_1+\frac{79}{817}\right),$$
$$(t_1+t_2)+5t_2 \leqslant \frac{60}{43},\ \frac{615}{43} \leqslant 29\left(t_1+\frac{79}{817}\right)+10t_2\Big\},$$

$$U_{0306}(t_1,t_2) := \Big\{\frac{27}{43} \leqslant \left(t_1+\frac{79}{817}\right)+t_2,\ (t_1+t_2)+t_2 \leqslant \frac{219}{301},\ \frac{41}{215} \leqslant t_2,\ 6(t_1+t_2)+t_2 \leqslant \frac{135}{43},$$
$$\frac{205}{473} \leqslant \left(t_1+\frac{79}{817}\right),\ 7t_2-\left(t_1+\frac{79}{817}\right) \leqslant \frac{55}{43},\ \frac{164}{43} \leqslant 6\left(t_1+\frac{79}{817}\right)+5t_2\Big\},$$

$$U_{0307}(t_1,t_2) := \Big\{(t_1+t_2)+t_2 \leqslant \frac{268}{473},\ 35(t_1+t_2)+23t_2 \leqslant \frac{767}{43},\ \frac{410}{473} \leqslant 2\left(t_1+\frac{79}{817}\right)+t_2,$$
$$2(t_1+t_2)+13t_2 \leqslant \frac{130}{43},\ \frac{1476}{43} \leqslant 70\left(t_1+\frac{79}{817}\right)+59t_2\Big\},$$

$$U_{0308}(t_1,t_2) := \Big\{(t_1+t_2)+t_2 \leqslant \frac{313}{559},\ 41(t_1+t_2)+27t_2 \leqslant \frac{902}{43},\ \frac{492}{559} \leqslant 2\left(t_1+\frac{79}{817}\right)+t_2,$$
$$2(t_1+t_2)+15t_2 \leqslant \frac{138}{43},\ \frac{1722}{43} \leqslant 82\left(t_1+\frac{79}{817}\right)+69t_2\Big\},$$

$$U_{0309}(t_1,t_2) := \Big\{2(t_1+t_2)+t_2 \leqslant \frac{536}{473},\ 70(t_1+t_2)+11t_2 \leqslant \frac{1534}{43},\ \frac{205}{473} \leqslant \left(t_1+\frac{79}{817}\right),$$
$$(t_1+t_2)+6t_2 \leqslant \frac{65}{43},\ \frac{738}{43} \leqslant 35\left(t_1+\frac{79}{817}\right)+12t_2\Big\},$$

$$U_{0310}(t_1,t_2) := \Big\{(t_2+t_2)+t_1 \leqslant \frac{673}{1247},\ \frac{82}{473} \leqslant t_1,\ 29(t_2+t_2)-t_1 \leqslant \frac{427}{43},\ \frac{246}{817} \leqslant \left(t_2+\frac{79}{817}\right),$$
$$29t_1-\left(t_2+\frac{79}{817}\right) \leqslant \frac{263}{43},\ \frac{246}{43} \leqslant 12\left(t_2+\frac{79}{817}\right)+11t_1\Big\},$$

$$U_{0311}(t_1,t_2) := \Big\{(t_2+t_2)+t_1 \leqslant \frac{223}{387},\ 29(t_2+t_2)+19t_1 \leqslant \frac{632}{43},\ \frac{328}{387} \leqslant 2\left(t_2+\frac{79}{817}\right)+t_1,$$
$$2(t_2+t_2)+11t_1 \leqslant \frac{120}{43},\ \frac{1230}{43} \leqslant 58\left(t_2+\frac{79}{817}\right)+49t_1\Big\},$$

$$U_{0312}(t_1,t_2) := \Big\{(t_2+t_2)+t_1 \leqslant \frac{268}{473},\ \frac{41}{215} \leqslant t_1,\ 6(t_2+t_2)+t_1 \leqslant \frac{94}{43},\ \frac{82}{301} \leqslant \left(t_2+\frac{79}{817}\right),$$
$$(t_2+t_2) \leqslant \frac{16}{43},\ (t_2+t_2)+8t_1 \leqslant \frac{98}{43},\ \frac{123}{43} \leqslant 6\left(t_2+\frac{79}{817}\right)+5t_1\Big\},$$

$$U_{0313}(t_1,t_2) := \Big\{(t_2+t_2)+t_1 \leqslant \frac{571}{817},\ \frac{82}{473} \leqslant t_1,\ 12(t_2+t_2)+t_1 \leqslant \frac{270}{43},\ \frac{574}{1247} \leqslant \left(t_2+\frac{79}{817}\right),$$
$$19t_1-\left(t_2+\frac{79}{817}\right) \leqslant \frac{161}{43},\ \frac{328}{43} \leqslant 12\left(t_2+\frac{79}{817}\right)+11t_1\Big\},$$

$$U_{0314}(t_1,t_2) := \left\{2(t_2+t_2)+t_1 \leqslant \frac{446}{387},\ 58(t_2+t_2)+9t_1 \leqslant \frac{1264}{43},\ \frac{164}{387} \leqslant \left(t_2+\frac{79}{817}\right),\right.$$
$$\left.(t_2+t_2)+5t_1 \leqslant \frac{60}{43},\ \frac{615}{43} \leqslant 29\left(t_2+\frac{79}{817}\right)+10t_1\right\},$$
$$U_{0315}(t_1,t_2) := \left\{\frac{27}{43} \leqslant \left(t_2+\frac{79}{817}\right)+t_1,\ (t_2+t_2)+t_1 \leqslant \frac{219}{301},\ \frac{41}{215} \leqslant t_1,\ 6(t_2+t_2)+t_1 \leqslant \frac{135}{43},\right.$$
$$\left.\frac{205}{473} \leqslant \left(t_2+\frac{79}{817}\right),\ 7t_1-\left(t_2+\frac{79}{817}\right) \leqslant \frac{55}{43},\ \frac{164}{43} \leqslant 6\left(t_2+\frac{79}{817}\right)+5t_1\right\},$$
$$U_{0316}(t_1,t_2) := \left\{(t_2+t_2)+t_1 \leqslant \frac{268}{473},\ 35(t_2+t_2)+23t_1 \leqslant \frac{767}{43},\ \frac{410}{473} \leqslant 2\left(t_2+\frac{79}{817}\right)+t_1,\right.$$
$$\left.2(t_2+t_2)+13t_1 \leqslant \frac{130}{43},\ \frac{1476}{43} \leqslant 70\left(t_2+\frac{79}{817}\right)+59t_1\right\},$$
$$U_{0317}(t_1,t_2) := \left\{(t_2+t_2)+t_1 \leqslant \frac{313}{559},\ 41(t_2+t_2)+27t_1 \leqslant \frac{902}{43},\ \frac{492}{559} \leqslant 2\left(t_2+\frac{79}{817}\right)+t_1,\right.$$
$$\left.2(t_2+t_2)+15t_1 \leqslant \frac{138}{43},\ \frac{1722}{43} \leqslant 82\left(t_2+\frac{79}{817}\right)+69t_1\right\},$$
$$U_{0318}(t_1,t_2) := \left\{2(t_2+t_2)+t_1 \leqslant \frac{536}{473},\ 70(t_2+t_2)+11t_1 \leqslant \frac{1534}{43},\ \frac{205}{473} \leqslant \left(t_2+\frac{79}{817}\right),\right.$$
$$\left.(t_2+t_2)+6t_1 \leqslant \frac{65}{43},\ \frac{738}{43} \leqslant 35\left(t_2+\frac{79}{817}\right)+12t_1\right\},$$
$$U_{0319}(t_1,t_2) := \left\{t_1+(t_2+t_2) \leqslant \frac{673}{1247},\ \frac{82}{473} \leqslant \left(t_2+\frac{79}{817}\right),\ 29t_1-\left(t_2+\frac{79}{817}\right) \leqslant \frac{427}{43},\ \frac{246}{817} \leqslant t_1,\right.$$
$$\left.29(t_2+t_2)-t_1 \leqslant \frac{263}{43},\ \frac{246}{43} \leqslant 12t_1+11\left(t_2+\frac{79}{817}\right)\right\},$$
$$U_{0320}(t_1,t_2) := \left\{t_1+(t_2+t_2) \leqslant \frac{223}{387},\ 29t_1+19(t_2+t_2) \leqslant \frac{632}{43},\ \frac{328}{387} \leqslant 2t_1+\left(t_2+\frac{79}{817}\right),\right.$$
$$\left.2t_1+11(t_2+t_2) \leqslant \frac{120}{43},\ \frac{1230}{43} \leqslant 58t_1+49\left(t_2+\frac{79}{817}\right)\right\},$$
$$U_{0321}(t_1,t_2) := \left\{t_1+(t_2+t_2) \leqslant \frac{268}{473},\ \frac{41}{215} \leqslant \left(t_2+\frac{79}{817}\right),\ 6t_1+(t_2+t_2) \leqslant \frac{94}{43},\ \frac{82}{301} \leqslant t_1 \leqslant \frac{16}{43},\right.$$
$$\left.t_1+8(t_2+t_2) \leqslant \frac{98}{43},\ \frac{123}{43} \leqslant 6t_1+5\left(t_2+\frac{79}{817}\right)\right\},$$
$$U_{0322}(t_1,t_2) := \left\{t_1+(t_2+t_2) \leqslant \frac{571}{817},\ \frac{82}{473} \leqslant \left(t_2+\frac{79}{817}\right),\ 12t_1+(t_2+t_2) \leqslant \frac{270}{43},\ \frac{574}{1247} \leqslant t_1,\right.$$
$$\left.19(t_2+t_2)-t_1 \leqslant \frac{161}{43},\ \frac{328}{43} \leqslant 12t_1+11\left(t_2+\frac{79}{817}\right)\right\},$$
$$U_{0323}(t_1,t_2) := \left\{2t_1+(t_2+t_2) \leqslant \frac{446}{387},\ 58t_1+9(t_2+t_2) \leqslant \frac{1264}{43},\ \frac{164}{387} \leqslant t_1,\right.$$
$$\left.t_1+5(t_2+t_2) \leqslant \frac{60}{43},\ \frac{615}{43} \leqslant 29t_1+10\left(t_2+\frac{79}{817}\right)\right\},$$
$$U_{0324}(t_1,t_2) := \left\{\frac{27}{43} \leqslant t_1+\left(t_2+\frac{79}{817}\right),\ t_1+(t_2+t_2) \leqslant \frac{219}{301},\ \frac{41}{215} \leqslant \left(t_2+\frac{79}{817}\right),\ 6t_1+(t_2+t_2) \leqslant \frac{135}{43},\right.$$
$$\left.\frac{205}{473} \leqslant t_1,\ 7(t_2+t_2)-t_1 \leqslant \frac{55}{43},\ \frac{164}{43} \leqslant 6t_1+5\left(t_2+\frac{79}{817}\right)\right\},$$
$$U_{0325}(t_1,t_2) := \left\{t_1+(t_2+t_2) \leqslant \frac{268}{473},\ 35t_1+23(t_2+t_2) \leqslant \frac{767}{43},\ \frac{410}{473} \leqslant 2t_1+\left(t_2+\frac{79}{817}\right),\right.$$

$$2t_1 + 13(t_2 + t_2) \leqslant \frac{130}{43},\ \frac{1476}{43} \leqslant 70t_1 + 59\left(t_2 + \frac{79}{817}\right)\Big\},$$

$$U_{0326}(t_1,t_2) := \Big\{t_1 + (t_2 + t_2) \leqslant \frac{313}{559},\ 41t_1 + 27(t_2 + t_2) \leqslant \frac{902}{43},\ \frac{492}{559} \leqslant 2t_1 + \left(t_2 + \frac{79}{817}\right),$$

$$2t_1 + 15(t_2 + t_2) \leqslant \frac{138}{43},\ \frac{1722}{43} \leqslant 82t_1 + 69\left(t_2 + \frac{79}{817}\right)\Big\},$$

$$U_{0327}(t_1,t_2) := \Big\{2t_1 + (t_2 + t_2) \leqslant \frac{536}{473},\ 70t_1 + 11(t_2 + t_2) \leqslant \frac{1534}{43},\ \frac{205}{473} \leqslant t_1,$$

$$t_1 + 6(t_2 + t_2) \leqslant \frac{65}{43},\ \frac{738}{43} \leqslant 35t_1 + 12\left(t_2 + \frac{79}{817}\right)\Big\},$$

$$U_{0401}(t_1,t_2) := \left\{2(t_1 + t_2) + t_2 \leqslant 2 - \frac{20.5}{21.5},\ 4(t_1 + t_2) + 6t_2 \leqslant 4 - \frac{20.5}{21.5},\ 4t_2 \leqslant 4 - \frac{61.5}{21.5}\right\},$$

$$U_{0402}(t_1,t_2) := \left\{(t_1 + t_2) \leqslant \frac{45}{86},\ t_2 \leqslant \frac{41}{344}\right\},$$

$$U_{0403}(t_1,t_2) := \left\{\frac{55}{129} \leqslant (t_1 + t_2) \leqslant \frac{20}{43},\ t_2 \leqslant -\frac{1}{5}(t_1 + t_2) + \frac{12}{43}\right\},$$

$$U_{0404}(t_1,t_2) := \left\{\frac{20}{43} \leqslant (t_1 + t_2) \leqslant \frac{41}{86},\ t_2 \leqslant \frac{1}{7}(t_1 + t_2) + \frac{55}{301}\right\},$$

$$U_{0405}(t_1,t_2) := \left\{\frac{41}{86} \leqslant (t_1 + t_2) \leqslant \frac{227}{473},\ t_2 \leqslant -(t_1 + t_2) + \frac{219}{301}\right\},$$

$$U_{0406}(t_1,t_2) := \left\{\frac{227}{473} \leqslant (t_1 + t_2) \leqslant \frac{2333}{4859},\ t_2 \leqslant -\frac{58}{9}(t_1 + t_2) + \frac{1264}{387}\right\},$$

$$U_{0407}(t_1,t_2) := \left\{\frac{2333}{4859} \leqslant (t_1 + t_2) \leqslant \frac{2501}{5203},\ t_2 \leqslant -\frac{1}{6}(t_1 + t_2) + \frac{65}{258}\right\},$$

$$U_{0408}(t_1,t_2) := \left\{\frac{2501}{5203} \leqslant (t_1 + t_2) \leqslant \frac{499}{1032},\ t_2 \leqslant -2(t_1 + t_2) + \frac{536}{473}\right\},$$

$$U_{0409}(t_1,t_2) := \left\{\frac{499}{1032} \leqslant (t_1 + t_2) \leqslant \frac{28277}{57190},\ t_2 \leqslant -\frac{70}{11}(t_1 + t_2) + \frac{1534}{473}\right\},$$

$$U_{0501}(t_1,t_2) := \left\{2(t_2 + t_2) + t_1 \leqslant 2 - \frac{20.5}{21.5},\ 4(t_2 + t_2) + 6t_1 \leqslant 4 - \frac{20.5}{21.5},\ 4t_1 \leqslant 4 - \frac{61.5}{21.5},\right\},$$

$$U_{0502}(t_1,t_2) := \left\{(t_2 + t_2) \leqslant \frac{45}{86},\ t_1 \leqslant \frac{41}{344}\right\},$$

$$U_{0503}(t_1,t_2) := \left\{\frac{55}{129} \leqslant \left(t_2 + \frac{79}{817}\right),\ (t_2 + t_2) \leqslant \frac{20}{43},\ t_1 \leqslant -\frac{1}{5}(t_2 + t_2) + \frac{12}{43}\right\},$$

$$U_{0504}(t_1,t_2) := \left\{\frac{20}{43} \leqslant \left(t_2 + \frac{79}{817}\right),\ (t_2 + t_2) \leqslant \frac{41}{86},\ t_1 \leqslant \frac{1}{7}\left(t_2 + \frac{79}{817}\right) + \frac{55}{301}\right\},$$

$$U_{0505}(t_1,t_2) := \left\{\frac{41}{86} \leqslant \left(t_2 + \frac{79}{817}\right),\ (t_2 + t_2) \leqslant \frac{227}{473},\ t_1 \leqslant -(t_2 + t_2) + \frac{219}{301}\right\},$$

$$U_{0506}(t_1,t_2) := \left\{\frac{227}{473} \leqslant \left(t_2 + \frac{79}{817}\right),\ (t_2 + t_2) \leqslant \frac{2333}{4859},\ t_1 \leqslant -\frac{58}{9}(t_2 + t_2) + \frac{1264}{387}\right\},$$

$$U_{0507}(t_1,t_2) := \left\{\frac{2333}{4859} \leqslant \left(t_2 + \frac{79}{817}\right),\ (t_2 + t_2) \leqslant \frac{2501}{5203},\ t_1 \leqslant -\frac{1}{6}(t_2 + t_2) + \frac{65}{258}\right\},$$

$$U_{0508}(t_1,t_2) := \left\{\frac{2501}{5203} \leqslant \left(t_2 + \frac{79}{817}\right),\ (t_2 + t_2) \leqslant \frac{499}{1032},\ t_1 \leqslant -2(t_2 + t_2) + \frac{536}{473}\right\},$$

$$U_{0509}(t_1,t_2) := \left\{\frac{499}{1032} \leqslant \left(t_2 + \frac{79}{817}\right),\ (t_2 + t_2) \leqslant \frac{28277}{57190},\ t_1 \leqslant -\frac{70}{11}(t_2 + t_2) + \frac{1534}{473}\right\},$$

$$U_{0601}(t_1,t_2) := \left\{2t_1+t_2 \leqslant 2-\frac{20.5}{21.5},\ 4t_1+6t_2 \leqslant 4-\frac{20.5}{21.5},\ 4t_2 \leqslant 4-\frac{61.5}{21.5},\right.$$
$$\left. 2(1-t_1-t_2)+t_2 \leqslant 2-\frac{20.5}{21.5},\ 4(1-t_1-t_2)+6t_2 \leqslant 4-\frac{20.5}{21.5}\right\},$$
$$U_{0602}(t_1,t_2) := \left\{t_1 \leqslant \frac{45}{86},\ t_2 \leqslant \frac{41}{344},\ (1-t_1-t_2) \leqslant \frac{45}{86}\right\},$$

and let $U_0 = U_{01} \cup U_{02} \cup U_{03} \cup U_{04} \cup U_{05} \cup U_{06}$. The plot of the 6 regions can be found in Appendix. Then we have

$$\begin{aligned} S_3 =& \sum_{(t_1,t_2)\in U_{01}} S(\mathcal{A}_{p_1p_2},p_2) + \sum_{(t_1,t_2)\in U_{02}} S(\mathcal{A}_{p_1p_2},p_2) \\ &+ \sum_{(t_1,t_2)\in U_{03}} S(\mathcal{A}_{p_1p_2},p_2) + \sum_{(t_1,t_2)\in U_{04}} S(\mathcal{A}_{p_1p_2},p_2) \\ &+ \sum_{(t_1,t_2)\in U_{05}} S(\mathcal{A}_{p_1p_2},p_2) + \sum_{(t_1,t_2)\in U_{05}} S(\mathcal{A}_{p_1p_2},p_2) \\ &+ \sum_{(t_1,t_2)\notin U_{0}} S(\mathcal{A}_{p_1p_2},p_2) \\ =& S_{301}+S_{302}+S_{303}+\Sigma_4+\Sigma_5+\Sigma_6+\Sigma_0. \end{aligned} \tag{29}$$

For $S_{301}$, by Lemma 3.2 and Lemma 6.2 we have

$$\begin{aligned} S_{301} =& \sum_{(t_1,t_2)\in U_{01}} S(\mathcal{A}_{p_1p_2},p_2) \\ =& \eta \sum_{(t_1,t_2)\in U_{01}} S(\mathcal{B}_{p_1p_2},p_2) + O\left(\frac{\varepsilon\eta x}{\log x}\right) \\ =& \frac{\eta x}{\log x}\int_{\frac{79}{817}}^{\frac{1}{2}}\int_{\frac{79}{817}}^{\min\left(t_1,\frac{1-t_1}{2}\right)} \texttt{Boole}[(t_1,t_2)\in U_{01}] \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1t_2^2} dt_2dt_1 \\ \geqslant& 0.777974\frac{\eta x}{\log x}. \end{aligned} \tag{30}$$

For $S_{302}$, we can apply Buchstab's identity again to get

$$\begin{aligned} S_{302} =& \sum_{(t_1,t_2)\in U_{02}} S(\mathcal{A}_{p_1p_2},p_2) \\ =& \sum_{(t_1,t_2)\in U_{02}} S\left(\mathcal{A}_{p_1p_2},X^{\delta}\right) - \sum_{\substack{(t_1,t_2)\in U_{02}\\ \delta\leqslant t_3<\frac{79}{817}}} S(\mathcal{A}_{p_1p_2p_3},p_3) \\ &- \sum_{\substack{(t_1,t_2)\in U_{02}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S(\mathcal{A}_{p_1p_2p_3},p_3). \end{aligned} \tag{31}$$

By Lemma 5.1 and Iwaniec's linear sieve with $z=X^{\delta}$ and $D=\frac{X^{\frac{45}{86}}}{p_1p_2}$, we have

$$\sum_{(t_1,t_2)\in U_{02}} S\left(\mathcal{A}_{p_1p_2},X^{\delta}\right) = \eta \sum_{(t_1,t_2)\in U_{02}} S\left(\mathcal{B}_{p_1p_2},X^{\delta}\right) + O\left(\frac{\varepsilon\eta x}{\log x}\right). \tag{32}$$

By Lemma 6.1 we have

$$\sum_{\substack{(t_1,t_2)\in U_{02}\\ \delta\leqslant t_3<\frac{79}{817}}} S(\mathcal{A}_{p_1p_2p_3},p_3) = \eta \sum_{\substack{(t_1,t_2)\in U_{02}\\ \delta\leqslant t_3<\frac{79}{817}}} S(\mathcal{B}_{p_1p_2p_3},p_3) + O\left(\frac{\varepsilon\eta x}{\log x}\right). \tag{33}$$

By Lemma 3.3 and Lemma 6.2 with a small modification we have

$$\sum_{\substack{(t_1,t_2)\in U_{02}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)=\eta \sum_{\substack{(t_1,t_2)\in U_{02}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{B}_{p_1p_2p_3},p_3\right)+O\left(\frac{\varepsilon\eta x}{\log x}\right). \tag{34}$$

Combining together (31)–(34), we have

$$\begin{aligned}
S_{302}=&\sum_{(t_1,t_2)\in U_{02}} S\left(\mathcal{A}_{p_1p_2},p_2\right)\\
=&\sum_{(t_1,t_2)\in U_{02}} S\left(\mathcal{A}_{p_1p_2},X^{\delta}\right)-\sum_{\substack{(t_1,t_2)\in U_{02}\\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)\\
&-\sum_{\substack{(t_1,t_2)\in U_{02}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)\\
=&\ \eta\left(\sum_{(t_1,t_2)\in U_{02}} S\left(\mathcal{B}_{p_1p_2},X^{\delta}\right)-\sum_{\substack{(t_1,t_2)\in U_{02}\\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3},p_3\right)\right.\\
&\left.-\sum_{\substack{(t_1,t_2)\in U_{02}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{B}_{p_1p_2p_3},p_3\right)\right)+O\left(\frac{\varepsilon\eta x}{\log x}\right)\\
=&\ \eta\sum_{(t_1,t_2)\in U_{02}} S\left(\mathcal{B}_{p_1p_2},p_2\right)+O\left(\frac{\varepsilon\eta x}{\log x}\right)\\
=&\ \frac{\eta x}{\log x}\int_{\frac{79}{817}}^{\frac{1}{2}}\int_{\frac{79}{817}}^{\min\left(t_1,\frac{1-t_1}{2}\right)} \texttt{Boole}[(t_1,t_2)\in U_{02}]\frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1t_2^2}dt_2dt_1\\
\geqslant&\ 0.467498\frac{\eta x}{\log x}.
\end{aligned}\tag{35}$$

Similarly, by replacing Lemma 3.3 with Lemmas 3.4, 3.5 and 3.6 in above discussion, for $S_{303}$ we have

$$\begin{aligned}
S_{303}=&\sum_{(t_1,t_2)\in U_{03}} S\left(\mathcal{A}_{p_1p_2},p_2\right)\\
=&\sum_{(t_1,t_2)\in U_{03}} S\left(\mathcal{A}_{p_1p_2},X^{\delta}\right)-\sum_{\substack{(t_1,t_2)\in U_{03}\\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)\\
&-\sum_{\substack{(t_1,t_2)\in U_{03}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)\\
=&\ \eta\left(\sum_{(t_1,t_2)\in U_{03}} S\left(\mathcal{B}_{p_1p_2},X^{\delta}\right)-\sum_{\substack{(t_1,t_2)\in U_{03}\\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3},p_3\right)\right.\\
&\left.-\sum_{\substack{(t_1,t_2)\in U_{03}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{B}_{p_1p_2p_3},p_3\right)\right)+O\left(\frac{\varepsilon\eta x}{\log x}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \eta \sum_{(t_1,t_2)\in U_{03}} S\left(\mathcal{B}_{p_1p_2},p_2\right) + O\left(\frac{\varepsilon \eta x}{\log x}\right) \\
&= \frac{\eta x}{\log x} \int_{\frac{79}{817}}^{\frac{1}{2}} \int_{\frac{79}{817}}^{\min\left(t_1,\frac{1-t_1}{2}\right)} \texttt{Boole}[(t_1,t_2)\in U_{03}] \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 \\
&\geqslant 0.021012 \frac{\eta x}{\log x}. \qquad (36)
\end{aligned}
$$

We note that there is an extremely small region of $U_{03}$ with $t_1+t_2>\frac{45}{86}$. Fortunately we have $t_1<\frac{1}{2}$ and $t_2<\frac{41}{344}$ in this region. By this we can use Lemma 5.1 and Iwaniec's linear sieve with $z=X^{\delta}$ and $D=X^{\frac{1}{43}}$ to give asymptotic formulas.

For $\Sigma_4$ and $\Sigma_5$, we can apply Buchstab's identity two more times to get

$$
\begin{aligned}
\Sigma_4 =& \sum_{(t_1,t_2)\in U_{04}} S\left(\mathcal{A}_{p_1p_2},p_2\right) \\
=& \sum_{(t_1,t_2)\in U_{04}} S\left(\mathcal{A}_{p_1p_2},X^{\delta}\right) - \sum_{\substack{(t_1,t_2)\in U_{04} \\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right) \\
&- \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right) \\
=& \left( \sum_{(t_1,t_2)\in U_{04}} S\left(\mathcal{A}_{p_1p_2},X^{\delta}\right) - \sum_{\substack{(t_1,t_2)\in U_{04} \\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right) \right.\\
&- \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3},X^{\delta}\right) \\
&\left. + \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ \delta\leqslant t_4<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right) \right) \\
&+ \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ \frac{79}{817}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right) \\
=& S_{304} + \Sigma_{801} \qquad (37)
\end{aligned}
$$

and

$$
\begin{aligned}
\Sigma_5 =& \sum_{(t_1,t_2)\in U_{05}} S\left(\mathcal{A}_{p_1p_2},p_2\right) \\
=& \sum_{(t_1,t_2)\in U_{05}} S\left(\mathcal{A}_{p_1p_2},X^{\delta}\right) - \sum_{\substack{(t_1,t_2)\in U_{05} \\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right) \\
&- \sum_{\substack{(t_1,t_2)\in U_{05} \\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left( \sum_{(t_1,t_2)\in U_{05}} S\left(\mathcal{A}_{p_1p_2}, X^{\delta}\right) - \sum_{\substack{(t_1,t_2)\in U_{05}\\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right) \right.\\
&\quad - \sum_{\substack{(t_1,t_2)\in U_{05}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3}, X^{\delta}\right)\\
&\quad \left. + \sum_{\substack{(t_1,t_2)\in U_{05}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \delta\leqslant t_4<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \right)\\
&\quad + \sum_{\substack{(t_1,t_2)\in U_{05}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{79}{817}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right)\\
&= S_{305} + \Sigma_{802}.
\end{aligned} \tag{38}
$$

By Lemma 5.1 and Iwaniec's linear sieve with $z = X^{\delta}$ and $D = \frac{X^{\frac{45}{86}}}{p_1p_2}$, we have

$$
\sum_{(t_1,t_2)\in U_{04}} S\left(\mathcal{A}_{p_1p_2}, X^{\delta}\right) = \eta \sum_{(t_1,t_2)\in U_{04}} S\left(\mathcal{B}_{p_1p_2}, X^{\delta}\right) + O\left(\frac{\varepsilon\eta x}{\log x}\right). \tag{39}
$$

By shortening the length of the sum interval of $t_1$ in $\Sigma_4$ by $10^{-1000}$, using Iwaniec's linear sieve with $z = X^{\delta}$, $D = X^{10^{-1000}}$ and by Lemma 5.2, we have

$$
\sum_{\substack{(t_1,t_2)\in U_{04}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3}, X^{\delta}\right) = \eta \sum_{\substack{(t_1,t_2)\in U_{04}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{B}_{p_1p_2p_3}, X^{\delta}\right) + O\left(\frac{\varepsilon\eta x}{\log x}\right). \tag{40}
$$

By Lemma 6.1 we have

$$
\sum_{\substack{(t_1,t_2)\in U_{04}\\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right) = \eta \sum_{\substack{(t_1,t_2)\in U_{04}\\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3}, p_3\right) + O\left(\frac{\varepsilon\eta x}{\log x}\right) \tag{41}
$$

and

$$
\sum_{\substack{(t_1,t_2)\in U_{04}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \delta\leqslant t_4<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) = \eta \sum_{\substack{(t_1,t_2)\in U_{04}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \delta\leqslant t_4<\frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3p_4}, p_4\right) + O\left(\frac{\varepsilon\eta x}{\log x}\right). \tag{42}
$$

Combining together (39)–(42) and by (37), we have

$$
\begin{aligned}
S_{304} = &\sum_{(t_1,t_2)\in U_{04}} S\left(\mathcal{A}_{p_1p_2}, X^{\delta}\right) - \sum_{\substack{(t_1,t_2)\in U_{04}\\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right) - \sum_{\substack{(t_1,t_2)\in U_{04}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3}, X^{\delta}\right)\\
&+ \sum_{\substack{(t_1,t_2)\in U_{04}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \delta\leqslant t_4<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \eta \left( \sum_{(t_1,t_2)\in U_{04}} S\left(\mathcal{B}_{p_1p_2}, X^{\delta}\right) - \sum_{\substack{(t_1,t_2)\in U_{04} \\ \delta \leqslant t_3 < \frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3}, p_3\right) \quad \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{B}_{p_1p_2p_3}, X^{\delta}\right) \right. \\
&\quad \left. + \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \delta \leqslant t_4 < \frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3p_4}, p_4\right) \right) + O\left(\frac{\varepsilon \eta x}{\log x}\right) \\
&= \eta \left( \sum_{(t_1,t_2)\in U_{04}} S\left(\mathcal{B}_{p_1p_2}, X^{\frac{79}{817}}\right) - \sum_{\substack{(t_1,t_2)\in U_{04} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{B}_{p_1p_2p_3}, X^{\frac{79}{817}}\right) \right) + O\left(\frac{\varepsilon \eta x}{\log x}\right) \\
&\geqslant \frac{\eta x}{\log x} \left( \int_{\frac{79}{817}}^{\frac{1}{2}} \int_{\frac{79}{817}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \texttt{Boole}[(t_1,t_2)\in U_{04}] \frac{\frac{817}{79}\omega\left(\frac{817}{79}(1-t_1-t_2)\right)}{t_1 t_2} dt_2 dt_1 \right. \\
&\quad \left. - \int_{\frac{79}{817}}^{\frac{1}{2}} \int_{\frac{79}{817}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\frac{79}{817}}^{t_2} \texttt{Boole}[(t_1,t_2)\in U_{04}] \frac{\frac{817}{79}\omega\left(\frac{817}{79}(1-t_1-t_2-t_3)\right)}{t_1 t_2 t_3} dt_3 dt_2 dt_1 - 10^{-100} \right) \\
&\geqslant 2.292949 \frac{\eta x}{\log x} \qquad (43)
\end{aligned}
$$

where the total loss of shortening the sum interval of $t_1$ is less than $10^{-100}$. Similarly, for $S_{305}$ we have

$$
\begin{aligned}
S_{305} &= \sum_{(t_1,t_2)\in U_{05}} S\left(\mathcal{A}_{p_1p_2}, X^{\delta}\right) - \sum_{\substack{(t_1,t_2)\in U_{05} \\ \delta \leqslant t_3 < \frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3}, p_3\right) - \sum_{\substack{(t_1,t_2)\in U_{05} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1p_2p_3}, X^{\delta}\right) \\
&\quad + \sum_{\substack{(t_1,t_2)\in U_{05} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \delta \leqslant t_4 < \frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\
&= \eta \left( \sum_{(t_1,t_2)\in U_{05}} S\left(\mathcal{B}_{p_1p_2}, X^{\delta}\right) - \sum_{\substack{(t_1,t_2)\in U_{05} \\ \delta \leqslant t_3 < \frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3}, p_3\right) \quad \sum_{\substack{(t_1,t_2)\in U_{05} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{B}_{p_1p_2p_3}, X^{\delta}\right) \right. \\
&\quad \left. + \sum_{\substack{(t_1,t_2)\in U_{05} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \delta \leqslant t_4 < \frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3p_4}, p_4\right) \right) + O\left(\frac{\varepsilon \eta x}{\log x}\right) \\
&= \eta \left( \sum_{(t_1,t_2)\in U_{05}} S\left(\mathcal{B}_{p_1p_2}, X^{\frac{79}{817}}\right) - \sum_{\substack{(t_1,t_2)\in U_{05} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{B}_{p_1p_2p_3}, X^{\frac{79}{817}}\right) \right) + O\left(\frac{\varepsilon \eta x}{\log x}\right) \\
&\geqslant \frac{\eta x}{\log x} \left( \int_{\frac{79}{817}}^{\frac{1}{2}} \int_{\frac{79}{817}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \texttt{Boole}[(t_1,t_2)\in U_{05}] \frac{\frac{817}{79}\omega\left(\frac{817}{79}(1-t_1-t_2)\right)}{t_1 t_2} dt_2 dt_1 \right.
\end{aligned}
$$

$$- \int_{\frac{79}{817}}^{\frac{1}{2}} \int_{\frac{79}{817}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\frac{79}{817}}^{t_2} \texttt{Boole}[(t_1, t_2) \in U_{05}] \frac{\frac{817}{79} \omega\left(\frac{817}{79}(1 - t_1 - t_2 - t_3)\right)}{t_1 t_2 t_3} dt_3 dt_2 dt_1 - 10^{-100} \Bigg)$$
$$\geqslant 0.019050 \frac{\eta x}{\log x}. \tag{44}$$

For $\Sigma_6$, we can perform a role–reversal after using Buchstab's identity one time to get a small saving. For the definition of a role–reversal one can see [4] or [[14], Chapter 5], and we refer the readers to [17], [29] and [30] for more applications of role–reversals. By Buchstab's identity and similar arguments as in [[14], Chapter 9], we have

$$\begin{aligned}
\Sigma_6 = & \sum_{(t_1,t_2)\in U_{06}} S\left(\mathcal{A}_{p_1 p_2}, p_2\right) \\
= & \sum_{(t_1,t_2)\in U_{06}} S\left(\mathcal{A}_{p_1 p_2}, X^{\frac{79}{817}}\right) - \sum_{\substack{(t_1,t_2)\in U_{06} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
= & \sum_{(t_1,t_2)\in U_{06}} S\left(\mathcal{A}_{p_1 p_2}, X^{\frac{79}{817}}\right) - \sum_{\substack{(t_1,t_2)\in U_{06} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{\beta p_2 p_3}, \left(\frac{2X}{\beta p_2 p_3}\right)^{\frac{1}{2}}\right) \\
= & \sum_{(t_1,t_2)\in U_{06}} S\left(\mathcal{A}_{p_1 p_2}, X^{\frac{79}{817}}\right) - \sum_{\substack{(t_1,t_2)\in U_{06} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{\beta p_2 p_3}, X^{\frac{79}{817}}\right) \\
& + \sum_{\substack{(t_1,t_2)\in U_{06} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{79}{817} \leqslant t_4 < \frac{1}{2} t_1}} S\left(\mathcal{A}_{\beta p_2 p_3 p_4}, p_4\right) \\
= & \sum_{(t_1,t_2)\in U_{06}} S\left(\mathcal{A}_{p_1 p_2}, X^{\delta}\right) - \sum_{\substack{(t_1,t_2)\in U_{06} \\ \delta \leqslant t_3 < \frac{79}{817}}} S\left(\mathcal{A}_{p_1 p_2 p_3}, p_3\right) \\
& - \sum_{\substack{(t_1,t_2)\in U_{06} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{\beta p_2 p_3}, X^{\delta}\right) + \sum_{\substack{(t_1,t_2)\in U_{06} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \delta \leqslant t_4 < \frac{79}{817}}} S\left(\mathcal{A}_{\beta p_2 p_3 p_4}, p_4\right) \\
& + \sum_{\substack{(t_1,t_2)\in U_{06} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{79}{817} \leqslant t_4 < \frac{1}{2} t_1}} S\left(\mathcal{A}_{\beta p_2 p_3 p_4}, p_4\right) \\
= & \Sigma_{601} - \Sigma_{602} - \Sigma_{603} + \Sigma_{604} + \Sigma_{605},
\end{aligned} \tag{45}$$

where $\beta \sim X^{1-t_1-t_2-t_3}$ and $(\beta, P(p_3)) = 1$. By the definition of $U_{06}$, we know that Lemma 4.2 is valid for $(m,h) = (p_1, p_2)$ and $(m,h) = (\beta p_3, p_2)$. By shortening the length of the sum interval of $t_1$ by $10^{-1000}$, using Iwaniec's linear sieve with $z = X^{\delta}$, $D = X^{10^{-1000}}$ and by Lemma 5.2 with a small modification, we have

$$\Sigma_{601} = \sum_{(t_1,t_2)\in U_{06}} S\left(\mathcal{A}_{p_1 p_2}, X^{\delta}\right) = \eta \sum_{(t_1,t_2)\in U_{06}} S\left(\mathcal{B}_{p_1 p_2}, X^{\delta}\right) + O\left(\frac{\varepsilon \eta x}{\log x}\right) \tag{46}$$

and

$$\Sigma_{603} = \sum_{\substack{(t_1,t_2)\in U_{06} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{\beta p_2 p_3}, X^{\delta}\right) = \eta \sum_{\substack{(t_1,t_2)\in U_{06} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{B}_{\beta p_2 p_3}, X^{\delta}\right) + O\left(\frac{\varepsilon \eta x}{\log x}\right). \tag{47}$$

By Lemma 6.1 we have

$$\Sigma_{602}=\sum_{\substack{(t_1,t_2)\in U_{06}\\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)=\eta \sum_{\substack{(t_1,t_2)\in U_{06}\\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3},p_3\right)+O\left(\frac{\varepsilon\eta x}{\log x}\right) \tag{48}$$

and

$$\Sigma_{604}=\sum_{\substack{(t_1,t_2)\in U_{06}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \delta\leqslant t_4<\frac{79}{817}}} S\left(\mathcal{A}_{\beta p_2p_3p_4},p_4\right)=\eta \sum_{\substack{(t_1,t_2)\in U_{06}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \delta\leqslant t_4<\frac{79}{817}}} S\left(\mathcal{B}_{\beta p_2p_3p_4},p_4\right)+O\left(\frac{\varepsilon\eta x}{\log x}\right). \tag{49}$$

We can give asymptotic formulas for part of $\Sigma_{605}$ if we can group $\beta,p_2,p_3,p_4$ into $p_i,h$ or $h,p_i$ satisfy Lemma 3.2, where $i\in\{2,3,4\}$. Combining this part with (45)–(49) we have

$$\begin{aligned}
\Sigma_6 &= \sum_{(t_1,t_2)\in U_{06}} S\left(\mathcal{A}_{p_1p_2},p_2\right)\\
&= \sum_{(t_1,t_2)\in U_{06}} S\left(\mathcal{A}_{p_1p_2},X^{\delta}\right)-\sum_{\substack{(t_1,t_2)\in U_{06}\\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)\\
&\quad -\sum_{\substack{(t_1,t_2)\in U_{06}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{A}_{\beta p_2p_3},X^{\delta}\right)+\sum_{\substack{(t_1,t_2)\in U_{06}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \delta\leqslant t_4<\frac{79}{817}}} S\left(\mathcal{A}_{\beta p_2p_3p_4},p_4\right)\\
&\quad +\sum_{\substack{(t_1,t_2)\in U_{06}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{79}{817}\leqslant t_4<\frac{1}{2}t_1\\ (1-t_1-t_2-t_3,t_2,t_3,t_4)\text{ can be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}}} S\left(\mathcal{A}_{\beta p_2p_3p_4},p_4\right)\\
&\quad +\sum_{\substack{(t_1,t_2)\in U_{06}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{79}{817}\leqslant t_4<\frac{1}{2}t_1\\ (1-t_1-t_2-t_3,t_2,t_3,t_4)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}}} S\left(\mathcal{A}_{\beta p_2p_3p_4},p_4\right)\\
&\geqslant \eta\left(\sum_{(t_1,t_2)\in U_{06}} S\left(\mathcal{B}_{p_1p_2},X^{\delta}\right)-\sum_{\substack{(t_1,t_2)\in U_{06}\\ \delta\leqslant t_3<\frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3},p_3\right)\right.\\
&\quad -\sum_{\substack{(t_1,t_2)\in U_{06}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)}} S\left(\mathcal{B}_{\beta p_2p_3},X^{\delta}\right)+\sum_{\substack{(t_1,t_2)\in U_{06}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \delta\leqslant t_4<\frac{79}{817}}} S\left(\mathcal{B}_{\beta p_2p_3p_4},p_4\right)\\
&\quad \left.+\sum_{\substack{(t_1,t_2)\in U_{06}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{79}{817}\leqslant t_4<\frac{1}{2}t_1\\ (1-t_1-t_2-t_3,t_2,t_3,t_4)\text{ can be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}}} S\left(\mathcal{B}_{\beta p_2p_3p_4},p_4\right)\right)\\
&= \eta \sum_{(t_1,t_2)\in U_{06}} S\left(\mathcal{B}_{p_1p_2},p_2\right)
\end{aligned} \tag{50}$$

$$
- \eta \sum_{\substack{(t_1,t_2)\in U_{06} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{79}{817} \leqslant t_4 < \frac{1}{2} t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (t_i, h) \text{ or } (h, t_i) \in U_{01}}} S\left(\mathcal{B}_{\beta p_2 p_3 p_4}, p_4\right)
$$

$$
= S_{306} - S_{307}, \tag{51}
$$

where

$$
\begin{aligned}
S_{306} &= \eta \sum_{(t_1,t_2)\in U_{06}} S\left(\mathcal{B}_{p_1 p_2}, p_2\right) \\
&= \frac{\eta x}{\log x}\left(\int_{\frac{79}{817}}^{\frac{1}{2}} \int_{\frac{79}{817}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \texttt{Boole}[(t_1,t_2)\in U_{06}] \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1 - 10^{-100}\right) \\
&\geqslant 0.243521 \frac{\eta x}{\log x}
\end{aligned} \tag{52}
$$

and

$$
\begin{aligned}
S_{307} &= \eta \sum_{\substack{(t_1,t_2)\in U_{06} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{79}{817} \leqslant t_4 < \frac{1}{2} t_1 \\ (1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (t_i, h) \text{ or } (h, t_i) \in U_{01}}} S\left(\mathcal{B}_{\beta p_2 p_3 p_4}, p_4\right) \\
&\leqslant \frac{\eta x}{\log x}\left(\int_{\frac{79}{817}}^{\frac{1}{2}} \int_{\frac{79}{817}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\frac{79}{817}}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\frac{79}{817}}^{\frac{1}{2}t_1} \right. \\
&\qquad \left. \texttt{Boole}[(t_1,t_2,t_3,t_4)\in U_{07}] \frac{\omega_1\left(\frac{t_1-t_4}{t_4}\right)\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4^2} dt_4 dt_3 dt_2 dt_1 + 10^{-100}\right) \\
&\leqslant 0.229820 \frac{\eta x}{\log x},
\end{aligned} \tag{53}
$$

where $U_{07}(t_1,t_2,t_3,t_4)$ is defined as

$$
\begin{aligned}
U_{07}(t_1,t_2,t_3,t_4) := \Bigg\{ & (t_1,t_2)\in U_{06},\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
& \frac{79}{817} \leqslant t_4 < \frac{1}{2} t_1, \\
& (1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}, \\
& \frac{79}{817} \leqslant t_1 < \frac{1}{2},\ \frac{79}{817} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \Bigg\}.
\end{aligned}
$$

We remark that an extra use of Buchstab's identity in reverse on the last term of (50) leads to no more saving.

Let $\Sigma_8 = \Sigma_{801} + \Sigma_{802}$ where $\Sigma_{801}$ and $\Sigma_{802}$ are defined in (37) and (38). We can give asymptotic formulas for part of $\Sigma_8$ if we can group $p_1, p_2, p_3, p_4$ into $p_i, h$ or $h, p_i$ satisfy Lemma 3.2, where $i \in \{1,2,3,4\}$. We define this part as $S_{308}$. By Lemma 6.2 with a small modification, we get a saving on $S_{308}$:

$$
S_{308} = \sum_{\substack{(t_1,t_2)\in U_{04}\cup U_{05} \\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right) \\ \frac{79}{817} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4\right)
$$

$$= \eta \sum_{\substack{(t_1,t_2)\in U_{04}\cup U_{05} \\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right) \\ \frac{79}{817}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right) \\ (t_1,t_2,t_3,t_4) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{B}_{p_1p_2p_3p_4}, p_4\right) + O\left(\frac{\varepsilon\eta x}{\log x}\right)$$

$$\geqslant \frac{\eta x}{\log x}\left(\int_{\frac{79}{817}}^{\frac{1}{2}} \int_{\frac{79}{817}}^{\min\left(t_1,\frac{1-t_1}{2}\right)} \int_{\frac{79}{817}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)} \int_{\frac{79}{817}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)} \right.$$

$$\left. \texttt{Boole}[(t_1,t_2,t_3,t_4)\in U_{08}] \frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2} dt_4dt_3dt_2dt_1\right)$$

$$\geqslant 0.107081\frac{\eta x}{\log x}, \tag{54}$$

where $U_{08}(t_1,t_2,t_3,t_4)$ is defined as

$$\begin{aligned} U_{08}(t_1,t_2,t_3,t_4) := \Bigg\{ & (t_1,t_2)\in U_{04}\cup U_{05},\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right), \\ & \frac{79}{817}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right), \\ & (t_1,t_2,t_3,t_4) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}, \\ & \frac{79}{817}\leqslant t_1<\frac{1}{2},\ \frac{79}{817}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\Bigg\}. \end{aligned}$$

Let $\Sigma_9 = \Sigma_8 - S_{308}$. We can still get asymptotic formulas for some parts of $\Sigma_9$. Although we cannot group $p_1,p_2,p_3,p_4$ directly into $p_i,h$ or $h,p_i$ satisfy Lemma 3.2, we can group $p_1,p_2,p_3,p_4,p_5$ into $p_i,h$ or $h,p_i$ satisfy Lemma 3.2 for all $\frac{79}{817}\leqslant t_5<t_4$ in some parts. If we can also group $p_1,p_2,p_3,p_4$ into $m,h$ or $h,m$ satisfy Lemma 4.2 or Lemma 3.2 in these parts, then we can use Buchstab's identity to give asymptotic formulas. This can be done by constructing a four–dimensional region which is essentially the same as $U_{02}$. We define

$$S_{309} = \sum_{(t_1,t_2,t_3,t_4)\in U_{09}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right), \tag{55}$$

where

$$\begin{aligned} U_{09}(t_1,t_2,t_3,t_4) := \Bigg\{ & (t_1,t_2)\in U_{04}\cup U_{05},\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right), \\ & \frac{79}{817}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right), \\ & (t_1,t_2,t_3,t_4)\notin U_{08}, \\ & (t_1,t_2,t_3,t_4) \text{ can be partitioned into } (m,h) \text{ or } (h,m)\in U'_{0901}\cup U'_{0902}\cup U_{01}, \\ & (t_1,t_2,t_3,t_4)\in U_{0901}\cup U_{0902}\cup\cdots\cup U_{0910}, \\ & \frac{79}{817}\leqslant t_1<\frac{1}{2},\ \frac{79}{817}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\Bigg\}, \end{aligned}$$

where

$$U'_{0901}(t_1,t_2) := \left\{2t_1+t_2\leqslant 2-\frac{20.5}{21.5},\ 4t_1+6t_2\leqslant 4-\frac{20.5}{21.5},\ 4t_2\leqslant 4-\frac{61.5}{21.5}\right\},$$

$$U'_{0902}(t_1,t_2) := \left\{t_1\leqslant\frac{45}{86},\ t_2\leqslant\frac{41}{344}\right\},$$

$$U_{0901}(t_1,t_2,t_3,t_4) := \left\{\frac{27267}{66994}\leqslant t_1+t_2+t_3+t_4\leqslant\frac{231}{559},\ \frac{79}{817}\leqslant t_4\leqslant -\frac{35}{23}(t_1+t_2+t_3+t_4)+\frac{767}{989}\right\},$$

$$U_{0902}(t_1,t_2,t_3,t_4) := \left\{\frac{231}{559} \leqslant t_1+t_2+t_3+t_4 \leqslant \frac{3275}{7826},\ \frac{79}{817} \leqslant t_4 \leqslant -(t_1+t_2+t_3+t_4)+\frac{313}{559}\right\},$$

$$U_{0903}(t_1,t_2,t_3,t_4) := \left\{\frac{3275}{7826} \leqslant t_1+t_2+t_3+t_4 \leqslant \frac{15005}{33497},\ \frac{79}{817} \leqslant t_4 \leqslant -\frac{41}{27}(t_1+t_2+t_3+t_4)+\frac{902}{1161}\right\},$$

$$U_{0904}(t_1,t_2,t_3,t_4) := \left\{\frac{13074}{28595} \leqslant t_1+t_2+t_3+t_4 \leqslant \frac{20}{43},\ \frac{79}{817} \leqslant t_4 \leqslant -\frac{1}{5}(t_1+t_2+t_3+t_4)+\frac{12}{43}\right\},$$

$$U_{0905}(t_1,t_2,t_3,t_4) := \left\{\frac{20}{43} \leqslant t_1+t_2+t_3+t_4 \leqslant \frac{41}{86},\ \frac{79}{817} \leqslant t_4 \leqslant \frac{1}{7}(t_1+t_2+t_3+t_4)+\frac{55}{301}\right\},$$

$$U_{0906}(t_1,t_2,t_3,t_4) := \left\{\frac{41}{86} \leqslant t_1+t_2+t_3+t_4 \leqslant \frac{227}{473},\ \frac{79}{817} \leqslant t_4 \leqslant -(t_1+t_2+t_3+t_4)+\frac{219}{301}\right\},$$

$$U_{0907}(t_1,t_2,t_3,t_4) := \left\{\frac{227}{473} \leqslant t_1+t_2+t_3+t_4 \leqslant \frac{2333}{4859},\ \frac{79}{817} \leqslant t_4 \leqslant -\frac{58}{9}(t_1+t_2+t_3+t_4)+\frac{1264}{387}\right\},$$

$$U_{0908}(t_1,t_2,t_3,t_4) := \left\{\frac{2333}{4859} \leqslant t_1+t_2+t_3+t_4 \leqslant \frac{2501}{5203},\ \frac{79}{817} \leqslant t_4 \leqslant -\frac{1}{6}(t_1+t_2+t_3+t_4)+\frac{65}{258}\right\},$$

$$U_{0909}(t_1,t_2,t_3,t_4) := \left\{\frac{2501}{5203} \leqslant t_1+t_2+t_3+t_4 \leqslant \frac{499}{1032},\ \frac{79}{817} \leqslant t_4 \leqslant -2(t_1+t_2+t_3+t_4)+\frac{536}{473}\right\},$$

$$U_{0910}(t_1,t_2,t_3,t_4) := \left\{\frac{499}{1032} \leqslant t_1+t_2+t_3+t_4 \leqslant \frac{28277}{57190},\ \frac{79}{817} \leqslant t_4 \leqslant -\frac{70}{11}(t_1+t_2+t_3+t_4)+\frac{1534}{473}\right\}.$$

By Buchstab's identity, we have

$$\begin{aligned} S_{309} &= \sum_{(t_1,t_2,t_3,t_4)\in U_{09}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\ &= \sum_{(t_1,t_2,t_3,t_4)\in U_{09}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, X^{\delta}\right) \\ &\quad - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{09} \\ \delta\leqslant t_5<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{09} \\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right). \end{aligned} \tag{56}$$

By the definition of $U_{09}$, we know that Lemma 4.2 (or Lemma 3.2) is valid for at least one partition of $(t_1,t_2,t_3,t_4) \in U_{09}$. By shortening the length of the sum interval of $t_1$ by $10^{-1000}$, using Iwaniec's linear sieve with $z = X^{\delta}$, $D = X^{10^{-1000}}$ and by Lemma 5.2 with a small modification, we have

$$\sum_{(t_1,t_2,t_3,t_4)\in U_{09}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, X^{\delta}\right) = \eta \sum_{(t_1,t_2,t_3,t_4)\in U_{09}} S\left(\mathcal{B}_{p_1p_2p_3p_4}, X^{\delta}\right) + O\left(\frac{\varepsilon\eta x}{\log x}\right). \tag{57}$$

By Lemma 6.1 we have

$$\sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{09} \\ \delta\leqslant t_5<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) = \eta \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{09} \\ \delta\leqslant t_5<\frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5}, p_5\right) + O\left(\frac{\varepsilon\eta x}{\log x}\right). \tag{58}$$

By the definition of $U_{09}$ and Lemma 6.2 with a small modification we have

$$\begin{aligned} &\sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{09} \\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) \\ &= \eta \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{09} \\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5}, p_5\right) + O\left(\frac{\varepsilon\eta x}{\log x}\right). \end{aligned} \tag{59}$$

Combining together (56)–(59), we have

$$\begin{aligned}
S_{309} =& \sum_{(t_1,t_2,t_3,t_4)\in U_{09}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\
=& \sum_{(t_1,t_2,t_3,t_4)\in U_{09}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, X^{\delta}\right) \\
& - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{09} \\ \delta \leqslant t_5 < \frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{09} \\ \frac{79}{817} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) \\
=& \eta \left( \sum_{(t_1,t_2,t_3,t_4)\in U_{09}} S\left(\mathcal{B}_{p_1p_2p_3p_4}, X^{\delta}\right) \right. \\
& \left. - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{09} \\ \delta \leqslant t_5 < \frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5}, p_5\right) - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{09} \\ \frac{79}{817} \leqslant t_5 < \min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5}, p_5\right) \right) + O\left(\frac{\varepsilon \eta x}{\log x}\right) \\
=& \eta \sum_{(t_1,t_2,t_3,t_4)\in U_{09}} S\left(\mathcal{B}_{p_1p_2p_3p_4}, p_4\right) + O\left(\frac{\varepsilon \eta x}{\log x}\right) \\
\geqslant& \frac{\eta x}{\log x} \left( \int_{\frac{79}{817}}^{\frac{1}{2}} \int_{\frac{79}{817}}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\frac{79}{817}}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\frac{79}{817}}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \right. \\
& \left. \texttt{Boole}[(t_1,t_2,t_3,t_4) \in U_{09}] \frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1 - 10^{-100} \right) \\
\geqslant& 0.005862 \frac{\eta x}{\log x}. \qquad (60)
\end{aligned}$$

Let $\Sigma_{10} = \Sigma_9 - S_{309}$. We can perform Buchstab's identity two more times to get asymptotic formulas for some parts of $\Sigma_{10}$. For this, we need to be able to group $p_1, p_2, p_3, p_4, p_5$ into $m, h$ or $h, m$ satisfy Lemma 4.2 or Lemma 3.2 in these parts. One can easily see that the above condition is true if we can group $p_1, p_2, p_3, p_4, p_4$ into $m, h$ or $h, m$ lie in "most region" defined by Lemma 4.2 and Lemma 3.2. This can be done by constructing a four–dimensional region as the following:

$$\begin{aligned}
U_{10}(t_1,t_2,t_3,t_4) := \Bigg\{ & (t_1,t_2) \in U_{04} \cup U_{05},\ \frac{79}{817} \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
& \frac{79}{817} \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
& (t_1,t_2,t_3,t_4) \notin U_{08},\ (t_1,t_2,t_3,t_4) \notin U_{09}, \\
& (t_1,t_2,t_3,t_4) \text{ can be partitioned into } (m,h) \text{ or } (h,m) \in U'_{10}, \\
& \frac{79}{817} \leqslant t_1 < \frac{1}{2},\ \frac{79}{817} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \Bigg\},
\end{aligned}$$

where

$$\begin{aligned}
U'_{10}(t_1,t_2) :=& \left\{(t_1,t_2) \in U'_{1001} \cup U'_{1002} \cup \cdots \cup U'_{1007}\right\}, \\
U'_{1001}(t_1,t_2) :=& \left\{2t_1 + t_2 \leqslant 2 - \frac{20.5}{21.5},\ 4t_1 + 6t_2 \leqslant 4 - \frac{20.5}{21.5},\ 4t_2 \leqslant 4 - \frac{61.5}{21.5}\right\}, \\
U'_{1002}(t_1,t_2) :=& \left\{t_1 \leqslant \frac{45}{86},\ t_2 \leqslant \frac{41}{344}\right\},
\end{aligned}$$

$$U'_{1003}(t_1,t_2) := \left\{\frac{55}{129} \leqslant t_1 \leqslant \frac{227}{473},\ \frac{79}{817} \leqslant t_2 \leqslant -\frac{1}{5}t_1 + \frac{12}{43}\right\},$$

$$U'_{1004}(t_1,t_2) := \left\{\frac{227}{473} \leqslant t_1 \leqslant \frac{2333}{4859},\ \frac{79}{817} \leqslant t_2 \leqslant -\frac{58}{9}t_1 + \frac{1264}{387}\right\},$$

$$U'_{1005}(t_1,t_2) := \left\{\frac{2333}{4859} \leqslant t_1 \leqslant \frac{2501}{5203},\ \frac{79}{817} \leqslant t_2 \leqslant -\frac{1}{6}t_1 + \frac{65}{258}\right\},$$

$$U'_{1006}(t_1,t_2) := \left\{\frac{2501}{5203} \leqslant t_1 \leqslant \frac{499}{1032},\ \frac{79}{817} \leqslant t_2 \leqslant -2t_1 + \frac{536}{473}\right\},$$

$$U'_{1007}(t_1,t_2) := \left\{\frac{499}{1032} \leqslant t_1 \leqslant \frac{28277}{57190},\ \frac{79}{817} \leqslant t_2 \leqslant -\frac{70}{11}t_1 + \frac{1534}{473}\right\},$$

and we define

$$S_{310} = \sum_{(t_1,t_2,t_3,t_4)\in U_{10}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right). \tag{61}$$

By Buchstab's identity, we have

$$\begin{aligned}
S_{310} = & \sum_{(t_1,t_2,t_3,t_4)\in U_{10}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\
= & \sum_{(t_1,t_2,t_3,t_4)\in U_{10}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, X^{\delta}\right) - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \delta\leqslant t_5<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) \\
& - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) \\
= & \sum_{(t_1,t_2,t_3,t_4)\in U_{10}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, X^{\delta}\right) - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \delta\leqslant t_5<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) \\
& - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) \\
& - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) \\
= & \sum_{(t_1,t_2,t_3,t_4)\in U_{10}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, X^{\delta}\right) - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \delta\leqslant t_5<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) \\
& - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) \\
& - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, X^{\delta}\right) \\
& + \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}\\ \delta\leqslant t_6<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5p_6}, p_6\right)
\end{aligned}$$

$$+\sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}\\ \frac{79}{817}\leqslant t_6<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right)\\ (t_1,t_2,t_3,t_4,t_5,t_6)\text{ can be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5p_6},p_6\right)$$

$$+\sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}\\ \frac{79}{817}\leqslant t_6<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right)\\ (t_1,t_2,t_3,t_4,t_5,t_6)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5p_6},p_6\right)$$

$$=\Sigma_{1001}-\Sigma_{1002}-\Sigma_{1003}-\Sigma_{1004}+\Sigma_{1005}+\Sigma_{1006}+S_{311}. \tag{62}$$

By the definition of $U_{10}$, we know that a special case of Lemma 4.2 and Lemma 3.2 (defined as $U_{10}'$ above) is valid for at least one partition of $(t_1,t_2,t_3,t_4,t_4)$ for $(t_1,t_2,t_3,t_4)\in U_{10}$. By shortening the length of the sum interval of $t_1$ by $10^{-1000}$, using Iwaniec's linear sieve with $z=X^{\delta}$, $D=X^{10^{-1000}}$ and by Lemma 5.2 with a small modification, we have

$$\Sigma_{1001}=\sum_{(t_1,t_2,t_3,t_4)\in U_{10}} S\left(\mathcal{A}_{p_1p_2p_3p_4},X^{\delta}\right)=\eta\sum_{(t_1,t_2,t_3,t_4)\in U_{10}} S\left(\mathcal{B}_{p_1p_2p_3p_4},X^{\delta}\right)+O\left(\frac{\varepsilon\eta x}{\log x}\right) \tag{63}$$

and

$$\Sigma_{1004}=\sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5},X^{\delta}\right)$$

$$=\eta\sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5},X^{\delta}\right)+O\left(\frac{\varepsilon\eta x}{\log x}\right). \tag{64}$$

By Lemma 6.1 we have

$$\Sigma_{1002}=\sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \delta\leqslant t_5<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5},p_5\right)=\eta\sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \delta\leqslant t_5<\frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5},p_5\right)+O\left(\frac{\varepsilon\eta x}{\log x}\right) \tag{65}$$

and

$$\Sigma_{1005}=\sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}\\ \delta\leqslant t_6<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5p_6},p_6\right)$$

$$=\eta\sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}\\ \delta\leqslant t_6<\frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5p_6},p_6\right)+O\left(\frac{\varepsilon\eta x}{\log x}\right). \tag{66}$$

By the definition of $U_{10}$ and Lemma 6.2 with a small modification we have

$$\Sigma_{1003}=\sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ can be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5},p_5\right)$$

$$= \eta \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10} \\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right) \\ (t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5}, p_5\right) + O\left(\frac{\varepsilon\eta x}{\log x}\right) \tag{67}$$

and

$$\Sigma_{1006} = \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10} \\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right) \\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01} \\ \frac{79}{817}\leqslant t_6<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right) \\ (t_1,t_2,t_3,t_4,t_5,t_6) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5p_6}, p_6\right)$$

$$= \eta \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10} \\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right) \\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01} \\ \frac{79}{817}\leqslant t_6<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right) \\ (t_1,t_2,t_3,t_4,t_5,t_6) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5p_6}, p_6\right) + O\left(\frac{\varepsilon\eta x}{\log x}\right). \tag{68}$$

Combining together (62)–(68), we have

$$\begin{aligned}
S_{10} &\geqslant S_{310} - S_{311} \\
&= \sum_{(t_1,t_2,t_3,t_4)\in U_{10}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, p_4\right) \\
&\quad - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10} \\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right) \\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01} \\ \frac{79}{817}\leqslant t_6<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right) \\ (t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5p_6}, p_6\right) \\
&= \sum_{(t_1,t_2,t_3,t_4)\in U_{10}} S\left(\mathcal{A}_{p_1p_2p_3p_4}, X^{\delta}\right) - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10} \\ \delta\leqslant t_5<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) \\
&\quad - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10} \\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right) \\ (t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, p_5\right) \\
&\quad - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10} \\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right) \\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5}, X^{\delta}\right) \\
&\quad + \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10} \\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right) \\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01} \\ \delta\leqslant t_6<\frac{79}{817}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5p_6}, p_6\right) \\
&\quad + \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10} \\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right) \\ (t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01} \\ \frac{79}{817}\leqslant t_6<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right) \\ (t_1,t_2,t_3,t_4,t_5,t_6) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5p_6}, p_6\right)
\end{aligned}$$

$$= \eta \left( \sum_{(t_1,t_2,t_3,t_4)\in U_{10}} S\left(\mathcal{B}_{p_1p_2p_3p_4}, X^{\delta}\right) - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \delta\leqslant t_5<\frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5}, p_5\right) \right.$$

$$- \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ can be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5}, p_5\right)$$

$$- \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5}, X^{\delta}\right)$$

$$+ \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}\\ \delta\leqslant t_6<\frac{79}{817}}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5p_6}, p_6\right)$$

$$\left. + \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}\\ \frac{79}{817}\leqslant t_6<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right)\\ (t_1,t_2,t_3,t_4,t_5,t_6)\text{ can be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5p_6}, p_6\right) \right)$$

$$= \left( \sum_{(t_1,t_2,t_3,t_4)\in U_{10}} S\left(\mathcal{B}_{p_1p_2p_3p_4}, p_4\right) \right.$$

$$\left. - \sum_{\substack{(t_1,t_2,t_3,t_4)\in U_{10}\\ \frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right)\\ (t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}\\ \frac{79}{817}\leqslant t_6<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right)\\ (t_1,t_2,t_3,t_4,t_5,t_6)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}}} S\left(\mathcal{B}_{p_1p_2p_3p_4p_5p_6}, p_6\right) \right)$$

$$\geqslant \frac{\eta x}{\log x} \left( \int_{\frac{79}{817}}^{\frac{1}{2}} \int_{\frac{79}{817}}^{\min\left(t_1,\frac{1-t_1}{2}\right)} \int_{\frac{79}{817}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)} \int_{\frac{79}{817}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)} \right.$$

$$\left. \texttt{Boole}[(t_1,t_2,t_3,t_4)\in U_{10}] \frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2} dt_4dt_3dt_2dt_1 - 10^{-100} \right)$$

$$- \frac{\eta x}{\log x} \left( \int_{\frac{79}{817}}^{\frac{1}{2}} \int_{\frac{79}{817}}^{\min\left(t_1,\frac{1-t_1}{2}\right)} \int_{\frac{79}{817}}^{\min\left(t_2,\frac{1-t_1-t_2}{2}\right)} \int_{\frac{79}{817}}^{\min\left(t_3,\frac{1-t_1-t_2-t_3}{2}\right)} \right.$$

$$\int_{\frac{79}{817}}^{\min\left(t_4,\frac{1-t_1-t_2-t_3-t_4}{2}\right)} \int_{\frac{79}{817}}^{\min\left(t_5,\frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)}$$

$$\texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6)\in U_{11}]\frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1t_2t_3t_4t_5t_6^2}dt_6dt_5dt_4dt_3dt_2dt_1+10^{-100}\Bigg)$$

$$\geqslant 0.002980\frac{\eta x}{\log x}-0.000000648796\frac{\eta x}{\log x}, \tag{69}$$

where $U_{11}(t_1,t_2,t_3,t_4,t_5,t_6)$ is defined as

$$\begin{aligned}U_{11}(t_1,t_2,t_3,t_4,t_5,t_6):=\Bigg\{&(t_1,t_2)\in U_{04}\cup U_{05},\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right),\\ &\frac{79}{817}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right),\\ &(t_1,t_2,t_3,t_4)\notin U_{08},\ (t_1,t_2,t_3,t_4)\notin U_{09},\ (t_1,t_2,t_3,t_4)\in U_{10},\\ &\frac{79}{817}\leqslant t_5<\min\left(t_4,\frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\ &(t_1,t_2,t_3,t_4,t_5)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01},\\ &\frac{79}{817}\leqslant t_6<\min\left(t_5,\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right),\\ &(t_1,t_2,t_3,t_4,t_5,t_6)\text{ cannot be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01},\\ &\frac{79}{817}\leqslant t_1<\frac{1}{2},\ \frac{79}{817}\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\Bigg\}.\end{aligned}$$

Let $\Sigma_{12}=\Sigma_{10}-S_{310}$. We can use Buchstab's identity in reverse to get asymptotic formulas for some almost–primes counted in $\Sigma_{12}$. More specifically, when $t_4<\frac{1}{2}(1-t_1-t_2-t_3-t_4)$, we have

$$\begin{aligned}&\sum_{\substack{(t_1,t_2)\in U_{04}\cup U_{05}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{79}{817}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\notin U_{08}\cup U_{09}\cup U_{10}\\ t_4<\frac{1}{2}(1-t_1-t_2-t_3-t_4)}} S\left(\mathcal{A}_{p_1p_2p_3p_4},p_4\right)\\ =&\sum_{\substack{(t_1,t_2)\in U_{04}\cup U_{05}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{79}{817}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\notin U_{08}\cup U_{09}\cup U_{10}\\ t_4<\frac{1}{2}(1-t_1-t_2-t_3-t_4)}} S\left(\mathcal{A}_{p_1p_2p_3p_4},\left(\frac{2X}{p_1p_2p_3p_4}\right)^{\frac{1}{2}}\right)\\ &+\sum_{\substack{(t_1,t_2)\in U_{04}\cup U_{05}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{79}{817}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\notin U_{08}\cup U_{09}\cup U_{10}\\ t_4<t_5<\frac{1}{2}(1-t_1-t_2-t_3-t_4)}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5},p_5\right).\end{aligned}\tag{70}$$

We can give asymptotic formulas for part of the last term if we can group $p_1,p_2,p_3,p_4,p_5$ into $p_i,h$ or $h,p_i$ satisfy Lemma 3.2, where $i\in\{1,2,3,4,5\}$. We define this part as $S_{312}$. More specifically,

$$S_{312}=\sum_{\substack{(t_1,t_2)\in U_{04}\cup U_{05}\\ \frac{79}{817}\leqslant t_3<\min\left(t_2,\frac{1}{2}(1-t_1-t_2)\right)\\ \frac{79}{817}\leqslant t_4<\min\left(t_3,\frac{1}{2}(1-t_1-t_2-t_3)\right)\\ (t_1,t_2,t_3,t_4)\notin U_{08}\cup U_{09}\cup U_{10}\\ t_4<t_5<\frac{1}{2}(1-t_1-t_2-t_3-t_4)\\ (t_1,t_2,t_3,t_4,t_5)\text{ can be partitioned into }(t_i,h)\text{ or }(h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3p_4p_5},p_5\right).\tag{71}$$

In this way, by Lemma 6.2 with a small modification, we obtain a small saving on $S_{312}$:

$$S_{312} \geqslant \frac{\eta x}{\log x}\left(\int_{\frac{79}{817}}^{\frac{1}{2}}\int_{\frac{79}{817}}^{\min\left(t_1, \frac{1-t_1}{2}\right)}\int_{\frac{79}{817}}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)}\int_{\frac{79}{817}}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)}\int_{t_4}^{\frac{1-t_1-t_2-t_3-t_4}{2}} \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5)\in U_{12}]\frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1t_2t_3t_4t_5^2}dt_5dt_4dt_3dt_2dt_1\right)$$
$$\geqslant 0.0000649373\frac{\eta x}{\log x}, \tag{72}$$

where $U_{12}(t_1,t_2,t_3,t_4,t_5)$ is defined as

$$\begin{aligned}U_{12}(t_1,t_2,t_3,t_4,t_5) := \Bigg\{ & (t_1,t_2)\in U_{04}\cup U_{05},\ \frac{79}{817}\leqslant t_3<\min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\ & \frac{79}{817}\leqslant t_4<\min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\ & (t_1,t_2,t_3,t_4)\notin U_{08}\cup U_{09}\cup U_{10},\\ & t_4<t_5<\frac{1}{2}(1-t_1-t_2-t_3-t_4),\\ & (t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01},\\ & \frac{79}{817}\leqslant t_1<\frac{1}{2},\ \frac{79}{817}\leqslant t_2<\min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\}.\end{aligned}$$

We remark that applying Buchstab's identity in reverse again to the remaining sums of the last term in (70) leads to a saving less than $10^{-8}$.

For the remaining $\Sigma_0$, we can also use Buchstab's identity in reverse to get asymptotic formulas for some almost–primes counted. When $t_2<\frac{1}{2}(1-t_1-t_2)$, we have

$$\sum_{\substack{(t_1,t_2)\notin U_0\\ t_2<\frac{1}{2}(1-t_1-t_2)}} S\left(\mathcal{A}_{p_1p_2},p_2\right) = \sum_{\substack{(t_1,t_2)\notin U_0\\ t_2<\frac{1}{2}(1-t_1-t_2)}} S\left(\mathcal{A}_{p_1p_2},\left(\frac{2X}{p_1p_2}\right)^{\frac{1}{2}}\right) + \sum_{\substack{(t_1,t_2)\notin U_0\\ t_2<t_3<\frac{1}{2}(1-t_1-t_2)}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right). \tag{73}$$

Then, we can give asymptotic formulas for part of the last term if we can group $p_1,p_2,p_3$ into $p_i,h$ or $h,p_i$ satisfy Lemma 3.2, where $i\in\{1,2,3\}$. We define this part as $S_{313}$.

$$S_{313} = \sum_{\substack{(t_1,t_2)\notin U_0\\ t_2<t_3<\frac{1}{2}(1-t_1-t_2)\\ (t_1,t_2,t_3) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right). \tag{74}$$

Then by Lemma 6.2 with a small modification, we can get a small saving on $S_{313}$:

$$\begin{aligned}S_{313} &= \sum_{\substack{(t_1,t_2)\notin U_0\\ t_2<t_3<\frac{1}{2}(1-t_1-t_2)\\ (t_1,t_2,t_3) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{A}_{p_1p_2p_3},p_3\right)\\ &= \eta \sum_{\substack{(t_1,t_2)\notin U_0\\ t_2<t_3<\frac{1}{2}(1-t_1-t_2)\\ (t_1,t_2,t_3) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i)\in U_{01}}} S\left(\mathcal{B}_{p_1p_2p_3},p_3\right) + O\left(\frac{\varepsilon\eta x}{\log x}\right)\\ &= \frac{\eta x}{\log x}\int_{\frac{79}{817}}^{\frac{1}{2}}\int_{\frac{79}{817}}^{\min\left(t_1, \frac{1-t_1}{2}\right)}\int_{t_2}^{\frac{1-t_1-t_2}{2}} \texttt{Boole}[(t_1,t_2,t_3)\in U_{13}]\frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\\ &\geqslant 0.025686\frac{\eta x}{\log x},\end{aligned} \tag{75}$$

where $U_{13}(t_1,t_2,t_3)$ is defined as

$$U_{13}(t_1,t_2,t_3) := \Big\{ t_2 < t_3 < \frac{1}{2}(1-t_1-t_2),\ (t_1,t_2) \notin U_0,$$
$$(t_1,t_2,t_3) \text{ can be partitioned into } (t_i,h) \text{ or } (h,t_i) \in U_{01},$$
$$\frac{79}{817} \leqslant t_1 < \frac{1}{2},\ \frac{79}{817} \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right) \Big\}.$$

We remark that an extra use of Buchstab's identity in reverse on the remaining sums of the last term in (73) leads to no more saving.

Finally, by (1)–(2), (26), (28)–(30), (35)–(36), (43)–(44), (52)–(54), (60), (69), (72) and (75) we have

$$\begin{aligned}
&\pi(x+\eta x) - \pi(x) \\
=\ & S\left(\mathcal{A}, (2X)^{\frac{1}{2}}\right) \\
\geqslant\ & S_1 - S_2 + S_{301} + S_{302} + S_{303} + S_{304} + S_{305} + S_{306} - S_{307} + S_{308} + S_{309} + S_{310} - S_{311} + S_{312} + S_{313} \\
\geqslant\ & (5.806486 - 9.540312 + 0.777974 + 0.467498 + 0.021012 + 2.292949 + 0.019050 + 0.243521 \\
&- 0.229820 + 0.107081 + 0.005862 + 0.002980 - 0.000000648796 + 0.0000649373 + 0.025686)\frac{\eta x}{\log x} \\
\geqslant\ & 0.00003 \frac{\eta x}{\log x},
\end{aligned}$$

and the proof of Theorem 1.1 is completed. The exponent $\frac{1}{21.5}$ is rather near to the limit obtained by the method employed in this paper, and something like $\frac{1}{22}$ is far out of reach. With the exponent $\frac{1}{22}$, the above method only gives a lower bound constant less than $-0.1$, which is even worse than the trivial lower bound. We remark that the above method gives a lower bound constant $> 0.04$ when the exponent is $\frac{1}{21}$, which is useful for an application in the next section.

## 8. Applications of Theorem 1.1

Clearly our Theorem 1.1 has many interesting applications. The following application of our Theorem 1.1 is about Goldbach numbers (sum of two primes) in short intervals. By combining our Theorem 1.1 with the main theorem proved in [4], we can easily deduce the following theorem.

**Theorem 8.1.** *The interval* $[X, X + X^{\frac{21}{860}}]$ *contains Goldbach numbers. That is,*

$$g_{n+1} - g_n \ll g_n^{21/860},$$

*where* $g_n$ *denotes the* $n$*–th Goldbach number.*

Previous exponent $\frac{21}{800}$ (see either [41] or [43]) comes from Jia's exponent $\frac{1}{20}$ [27]. We remark that if we focus on Maillet numbers (difference of two primes) instead of Goldbach numbers in short intervals, Pintz [42] improved this exponent to any $\varepsilon > 0$.

Another application of our Theorem 1.1 is about prime values of the integer parts of real sequences, improving the previous result of Harman [13] by adding one more term on both of the sequences $[p^k \alpha]$ and $[(p\alpha)^k]$.

**Theorem 8.2.** *For almost all* $\alpha > 0$*, both of the following statements are true:*
*(1) Infinitely often* $p, [p\alpha], [p^2\alpha], \ldots, [p^{21}\alpha]$ *are all prime.*
*(2) Infinitely often* $p, [p\alpha], [(p\alpha)^2], \ldots, [(p\alpha)^{21}]$ *are all prime.*

The proof of Theorem 8.2 can be done by replacing [[13], Lemma 4] by a variant of our Theorem 1.1. In this way, the ratio $\frac{20}{19}$ in [[13], Theorem 4] can be improved to $\frac{21.5}{20.5}$. However, due to the lack of a positive lower bound when the exponent is $\frac{1}{22}$, we cannot add even one more term on our Theorem 8.2.

The last application of our Theorem 1.1 is about the Three Primes Theorem with small prime solutions, improving the previous result of Cai [5] by reducing the exponent $\frac{11}{400}$ to $\frac{11}{420}$.

**Theorem 8.3.** *Let $Y$ denote a sufficiently large odd integer. The equation*

$$Y = p_1 + p_2 + p_3, \quad p_1 \leqslant Y^{\frac{11}{420}}$$

*is solvable.*

The exponent $\frac{11}{420}$ comes from a variant of our Theorem 1.1 with exponent $\frac{1}{21}$. Note that we have $u_0 > 0.04$ in this case, and we can show that $u_1 < 2.88$ by similar arguments as in [5]. By using the vector sieve together with the same bounds for $v_0$ and $v_1$ as in [5], the proof of our Theorem 8.3 is essentially the same as the proof of [[5], Theorem]. However, our sieve machinery is not very numerically significant when the exponent drops to $\frac{1}{21.5}$ (that is, we only have $u_0 > 0.00003$ by our Theorem 1.1), we cannot use vector sieve to prove Theorem 8.3 with condition $p_1 \leqslant Y^{\frac{11}{430}}$.


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## Appendix: Mathematica code for the Theorem

**Code for the asymptotic regions.** The following Mathematica code shows the definition of the regions $U_{01}$–$U_{13}$. In above calculations, we can give asymptotic formulas for all sums $S(\mathcal{A}_{p_1p_2}, p_2)$ where $t_1$ and $t_2$ lie in $U_{01}$, $U_{02}$ or $U_{03}$. For the remaining sums which lie in $U_{04}$, $U_{05}$ or $U_{06}$, we can give asymptotic formulas for part of these sums.

```
(*Our sieve exponent is 79/817*)
SieveExponent = 79/817;

(*The definition of the region U01*)
U0101[t1_, t2_] := (t1 + t2 <= 673/1247) && (82/473 <= t2) && (29*t1 - t2 <= 427/43) &&
    (246/817 <= t1) && (29*t2 - t1 <= 263/43) && (246/43 <= 12*t1 + 11*t2)
U0102[t1_, t2_] := (t1 + t2 <= 223/387) && (29*t1 + 19*t2 <= 632/43) && (328/387 <= 2*t1 +
    t2) && (2*t1 + 11*t2 <= 120/43) && (1230/43 <= 58*t1 + 49*t2)
U0103[t1_, t2_] := (t1 + t2 <= 268/473) && (41/215 <= t2) && (6*t1 + t2 <= 94/43) && (82/301
     <= t1 <= 16/43) && (t1 + 8*t2 <= 98/43) && (123/43 <= 6*t1 + 5*t2)
U0104[t1_, t2_] := (t1 + t2 <= 571/817) && (82/473 <= t2) && (12*t1 + t2 <= 270/43) &&
    (574/1247 <= t1) && (19*t2 - t1 <= 161/43) && (328/43 <= 12*t1 + 11*t2)
U0105[t1_, t2_] := (2*t1 + t2 <= 446/387) && (58*t1 + 9*t2 <= 1264/43) && (164/387 <= t1) &&
     (t1 + 5*t2 <= 60/43) && (615/43 <= 29*t1 + 10*t2)
U0106[t1_, t2_] := (27/43 <= t1 + t2 <= 219/301) && (41/215 <= t2) && (6*t1 + t2 <= 135/43)
    && (205/473 <= t1) && (7*t2 - t1 <= 55/43) && (164/43 <= 6*t1 + 5*t2)
U0107[t1_, t2_] := (t1 + t2 <= 268/473) && (35*t1 + 23*t2 <= 767/43) && (410/473 <= 2*t1 +
    t2) && (2*t1 + 13*t2 <= 130/43) && (1476/43 <= 70*t1 + 59*t2)
U0108[t1_, t2_] := (t1 + t2 <= 313/559) && (41*t1 + 27*t2 <= 902/43) && (492/559 <= 2*t1 +
    t2) && (2*t1 + 15*t2 <= 138/43) && (1722/43 <= 82*t1 + 69*t2)
U0109[t1_, t2_] := (2*t1 + t2 <= 536/473) && (70*t1 + 11*t2 <= 1534/43) && (205/473 <= t1)
    && (t1 + 6*t2 <= 65/43) && (738/43 <= 35*t1 + 12*t2)
U01[t1_, t2_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2 (1 - t1)])
    && (U0101[t1, t2] || U0102[t1, t2] || U0103[t1, t2] || U0104[t1, t2] || U0105[t1, t2] ||
     U0106[t1, t2] || U0107[t1, t2] || U0108[t1, t2] || U0109[t1, t2])

(*The definition of the region U02*)
U0201[t1_, t2_] := (27267/66994 <= t1 + t2 <= 231/559) && (SieveExponent <= t2 <= -(35/23) (
    t1 + t2) + 767/989)
U0202[t1_, t2_] := (231/559 <= t1 + t2 <= 3275/7826) && (SieveExponent <= t2 <= -(t1 + t2) +
     313/559)
U0203[t1_, t2_] := (3275/7826 <= t1 + t2 <= 15005/33497) && (SieveExponent <= t2 <= -(41/27)
     (t1 + t2) + 902/1161)
U0204[t1_, t2_] := (13074/28595 <= t1 + t2 <= 20/43) && (SieveExponent <= t2 <= -(1/5) (t1 +
     t2) + 12/43)
U0205[t1_, t2_] := (20/43 <= t1 + t2 <= 41/86) && (SieveExponent <= t2 <= 1/7 (t1 + t2) +
    55/301)
U0206[t1_, t2_] := (41/86 <= t1 + t2 <= 227/473) && (SieveExponent <= t2 <= -(t1 + t2) +
    219/301)
U0207[t1_, t2_] := (227/473 <= t1 + t2 <= 2333/4859) && (SieveExponent <= t2 <= -(58/9) (t1
    + t2) + 1264/387)
U0208[t1_, t2_] := (2333/4859 <= t1 + t2 <= 2501/5203) && (SieveExponent <= t2 <= -(1/6) (t1
     + t2) + 65/258)
U0209[t1_, t2_] := (2501/5203 <= t1 + t2 <= 499/1032) && (SieveExponent <= t2 <= -2 (t1 + t2
    ) + 536/473)
U0210[t1_, t2_] := (499/1032 <= t1 + t2 <= 28277/57190) && (SieveExponent <= t2 <= -(70/11)
    (t1 + t2) + 1534/473)
U02[t1_, t2_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2 (1 - t1)])
    && (U0201[t1, t2] || U0202[t1, t2] || U0203[t1, t2] || U0204[t1, t2] || U0205[t1, t2] ||
     U0206[t1, t2] || U0207[t1, t2] || U0208[t1, t2] || U0209[t1, t2] || U0210[t1, t2]) &&
    (! U01[t1, t2])

(*The definition of the region U03*)
U0301[t1_, t2_] := ((t1 + t2) + t2 <= 673/1247) && (82/473 <= t2) && (29*(t1 + t2) - t2 <=
    427/43) && (246/817 <= (t1 + SieveExponent)) && (29*t2 - (t1 + SieveExponent) <= 263/43)
     && (246/43 <= 12*(t1 + SieveExponent) + 11*t2)
```

```
U0302[t1_, t2_] := ((t1 + t2) + t2 <= 223/387) && (29*(t1 + t2) + 19*t2 <= 632/43) &&
    (328/387 <= 2*(t1 + SieveExponent) + t2) && (2*(t1 + t2) + 11*t2 <= 120/43) && (1230/43
    <= 58*(t1 + SieveExponent) + 49*t2)
U0303[t1_, t2_] := ((t1 + t2) + t2 <= 268/473) && (41/215 <= t2) && (6*(t1 + t2) + t2 <=
    94/43) && (82/301 <= (t1 + SieveExponent)) && ((t1 + t2) <= 16/43) && ((t1 + t2) + 8*t2
    <= 98/43) && (123/43 <= 6*(t1 + SieveExponent) + 5*t2)
U0304[t1_, t2_] := ((t1 + t2) + t2 <= 571/817) && (82/473 <= t2) && (12*(t1 + t2) + t2 <=
    270/43) && (574/1247 <= (t1 + SieveExponent)) && (19*t2 - (t1 + SieveExponent) <=
    161/43) && (328/43 <= 12*(t1 + SieveExponent) + 11*t2)
U0305[t1_, t2_] := (2*(t1 + t2) + t2 <= 446/387) && (58*(t1 + t2) + 9*t2 <= 1264/43) &&
    (164/387 <= (t1 + SieveExponent)) && ((t1 + t2) + 5*t2 <= 60/43) && (615/43 <= 29*(t1 +
    SieveExponent) + 10*t2)
U0306[t1_, t2_] := (27/43 <= (t1 + SieveExponent) + t2) && ((t1 + t2) + t2 <= 219/301) &&
    (41/215 <= t2) && (6*(t1 + t2) + t2 <= 135/43) && (205/473 <= (t1 + SieveExponent)) &&
    (7*t2 - (t1 + SieveExponent) <= 55/43) && (164/43 <= 6*(t1 + SieveExponent) + 5*t2)
U0307[t1_, t2_] := ((t1 + t2) + t2 <= 268/473) && (35*(t1 + t2) + 23*t2 <= 767/43) &&
    (410/473 <= 2*(t1 + SieveExponent) + t2) && (2*(t1 + t2) + 13*t2 <= 130/43) && (1476/43
    <= 70*(t1 + SieveExponent) + 59*t2)
U0308[t1_, t2_] := ((t1 + t2) + t2 <= 313/559) && (41*(t1 + t2) + 27*t2 <= 902/43) &&
    (492/559 <= 2*(t1 + SieveExponent) + t2) && (2*(t1 + t2) + 15*t2 <= 138/43) && (1722/43
    <= 82*(t1 + SieveExponent) + 69*t2)
U0309[t1_, t2_] := (2*(t1 + t2) + t2 <= 536/473) && (70*(t1 + t2) + 11*t2 <= 1534/43) &&
    (205/473 <= (t1 + SieveExponent)) && ((t1 + t2) + 6*t2 <= 65/43) && (738/43 <= 35*(t1 +
    SieveExponent) + 12*t2)
U0310[t1_, t2_] := ((t2 + t2) + t1 <= 673/1247) && (82/473 <= t1) && (29*(t2 + t2) - t1 <=
    427/43) && (246/817 <= (t2 + SieveExponent)) && (29*t1 - (t2 + SieveExponent) <= 263/43)
     && (246/43 <= 12*(t2 + SieveExponent) + 11*t1)
U0311[t1_, t2_] := ((t2 + t2) + t1 <= 223/387) && (29*(t2 + t2) + 19*t1 <= 632/43) &&
    (328/387 <= 2*(t2 + SieveExponent) + t1) && (2*(t2 + t2) + 11*t1 <= 120/43) && (1230/43
    <= 58*(t2 + SieveExponent) + 49*t1)
U0312[t1_, t2_] := ((t2 + t2) + t1 <= 268/473) && (41/215 <= t1) && (6*(t2 + t2) + t1 <=
    94/43) && (82/301 <= (t2 + SieveExponent)) && ((t2 + t2) <= 16/43) && ((t2 + t2) + 8*t1
    <= 98/43) && (123/43 <= 6*(t2 + SieveExponent) + 5*t1)
U0313[t1_, t2_] := ((t2 + t2) + t1 <= 571/817) && (82/473 <= t1) && (12*(t2 + t2) + t1 <=
    270/43) && (574/1247 <= (t2 + SieveExponent)) && (19*t1 - (t2 + SieveExponent) <=
    161/43) && (328/43 <= 12*(t2 + SieveExponent) + 11*t1)
U0314[t1_, t2_] := (2*(t2 + t2) + t1 <= 446/387) && (58*(t2 + t2) + 9*t1 <= 1264/43) &&
    (164/387 <= (t2 + SieveExponent)) && ((t2 + t2) + 5*t1 <= 60/43) && (615/43 <= 29*(t2 +
    SieveExponent) + 10*t1)
U0315[t1_, t2_] := (27/43 <= (t2 + SieveExponent) + t1) && ((t2 + t2) + t1 <= 219/301) &&
    (41/215 <= t1) && (6*(t2 + t2) + t1 <= 135/43) && (205/473 <= (t2 + SieveExponent)) &&
    (7*t1 - (t2 + SieveExponent) <= 55/43) && (164/43 <= 6*(t2 + SieveExponent) + 5*t1)
U0316[t1_, t2_] := ((t2 + t2) + t1 <= 268/473) && (35*(t2 + t2) + 23*t1 <= 767/43) &&
    (410/473 <= 2*(t2 + SieveExponent) + t1) && (2*(t2 + t2) + 13*t1 <= 130/43) && (1476/43
    <= 70*(t2 + SieveExponent) + 59*t1)
U0317[t1_, t2_] := ((t2 + t2) + t1 <= 313/559) && (41*(t2 + t2) + 27*t1 <= 902/43) &&
    (492/559 <= 2*(t2 + SieveExponent) + t1) && (2*(t2 + t2) + 15*t1 <= 138/43) && (1722/43
    <= 82*(t2 + SieveExponent) + 69*t1)
U0318[t1_, t2_] := (2*(t2 + t2) + t1 <= 536/473) && (70*(t2 + t2) + 11*t1 <= 1534/43) &&
    (205/473 <= (t2 + SieveExponent)) && ((t2 + t2) + 6*t1 <= 65/43) && (738/43 <= 35*(t2 +
    SieveExponent) + 12*t1)
U0319[t1_, t2_] := (t1 + (t2 + t2) <= 673/1247) && (82/473 <= (t2 + SieveExponent)) && (29*
    t1 - (t2 + SieveExponent) <= 427/43) && (246/817 <= t1) && (29*(t2 + t2) - t1 <= 263/43)
     && (246/43 <= 12*t1 + 11*(t2 + SieveExponent))
U0320[t1_, t2_] := (t1 + (t2 + t2) <= 223/387) && (29*t1 + 19*(t2 + t2) <= 632/43) &&
    (328/387 <= 2*t1 + (t2 + SieveExponent)) && (2*t1 + 11*(t2 + t2) <= 120/43) && (1230/43
    <= 58*t1 + 49*(t2 + SieveExponent))
U0321[t1_, t2_] := (t1 + (t2 + t2) <= 268/473) && (41/215 <= (t2 + SieveExponent)) && (6*t1
    + (t2 + t2) <= 94/43) && (82/301 <= t1 <= 16/43) && (t1 + 8*(t2 + t2) <= 98/43) &&
    (123/43 <= 6*t1 + 5*(t2 + SieveExponent))
U0322[t1_, t2_] := (t1 + (t2 + t2) <= 571/817) && (82/473 <= (t2 + SieveExponent)) && (12*t1
     + (t2 + t2) <= 270/43) && (574/1247 <= t1) && (19*(t2 + t2) - t1 <= 161/43) && (328/43
    <= 12*t1 + 11*(t2 + SieveExponent))
```

```
U0323[t1_, t2_] := (2*t1 + (t2 + t2) <= 446/387) && (58*t1 + 9*(t2 + t2) <= 1264/43) &&
    (164/387 <= t1) && (t1 + 5*(t2 + t2) <= 60/43) && (615/43 <= 29*t1 + 10*(t2 +
    SieveExponent))
U0324[t1_, t2_] := (27/43 <= t1 + (t2 + SieveExponent)) && (t1 + (t2 + t2) <= 219/301) &&
    (41/215 <= (t2 + SieveExponent)) && (6*t1 + (t2 + t2) <= 135/43) && (205/473 <= t1) &&
    (7*(t2 + t2) - t1 <= 55/43) && (164/43 <= 6*t1 + 5*(t2 + SieveExponent))
U0325[t1_, t2_] := (t1 + (t2 + t2) <= 268/473) && (35*t1 + 23*(t2 + t2) <= 767/43) &&
    (410/473 <= 2*t1 + (t2 + SieveExponent)) && (2*t1 + 13*(t2 + t2) <= 130/43) && (1476/43
    <= 70*t1 + 59*(t2 + SieveExponent))
U0326[t1_, t2_] := (t1 + (t2 + t2) <= 313/559) && (41*t1 + 27*(t2 + t2) <= 902/43) &&
    (492/559 <= 2*t1 + (t2 + SieveExponent)) && (2*t1 + 15*(t2 + t2) <= 138/43) && (1722/43
    <= 82*t1 + 69*(t2 + SieveExponent))
U0327[t1_, t2_] := (2*t1 + (t2 + t2) <= 536/473) && (70*t1 + 11*(t2 + t2) <= 1534/43) &&
    (205/473 <= t1) && (t1 + 6*(t2 + t2) <= 65/43) && (738/43 <= 35*t1 + 12*(t2 +
    SieveExponent))
U03[t1_, t2_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2 (1 - t1)])
    && (U0301[t1, t2] || U0302[t1, t2] || U0303[t1, t2] || U0304[t1, t2] || U0305[t1, t2] ||
     U0306[t1, t2] || U0307[t1, t2] || U0308[t1, t2] || U0309[t1, t2] || U0310[t1, t2] ||
    U0311[t1, t2] || U0312[t1, t2] || U0313[t1, t2] || U0314[t1, t2] || U0315[t1, t2] ||
    U0316[t1, t2] || U0317[t1, t2] || U0318[t1, t2] || U0319[t1, t2] || U0320[t1, t2] ||
    U0321[t1, t2] || U0322[t1, t2] || U0323[t1, t2] || U0324[t1, t2] || U0325[t1, t2] ||
    U0326[t1, t2] || U0327[t1, t2]) && (! U01[t1, t2]) && (! U02[t1, t2])

(*The definition of the region U04*)
U0401[t1_, t2_] := (2*(t1 + t2) + t2 <= 2 - 20.5/21.5) && (4*(t1 + t2) + 6*t2 <= 4 -
    20.5/21.5) && (4*t2 <= 4 - 61.5/21.5)
U0402[t1_, t2_] := ((t1 + t2) <= 45/86) && (t2 <= 41/344)
U0403[t1_, t2_] := (55/129 <= (t1 + t2) <= 20/43) && (t2 <= -(1/5) (t1 + t2) + 12/43)
U0404[t1_, t2_] := (20/43 <= (t1 + t2) <= 41/86) && (t2 <= 1/7 (t1 + t2) + 55/301)
U0405[t1_, t2_] := (41/86 <= (t1 + t2) <= 227/473) && (t2 <= -(t1 + t2) + 219/301)
U0406[t1_, t2_] := (227/473 <= (t1 + t2) <= 2333/4859) && (t2 <= -(58/9) (t1 + t2) +
    1264/387)
U0407[t1_, t2_] := (2333/4859 <= (t1 + t2) <= 2501/5203) && (t2 <= -(1/6) (t1 + t2) +
    65/258)
U0408[t1_, t2_] := (2501/5203 <= (t1 + t2) <= 499/1032) && (t2 <= -2 (t1 + t2) + 536/473)
U0409[t1_, t2_] := (499/1032 <= (t1 + t2) <= 28277/57190) && (t2 <= -(70/11) (t1 + t2) +
    1534/473)
U04[t1_, t2_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2 (1 - t1)])
    && (U0401[t1, t2] || U0402[t1, t2] || U0403[t1, t2] || U0404[t1, t2] || U0405[t1, t2] ||
     U0406[t1, t2] || U0407[t1, t2] || U0408[t1, t2] || U0409[t1, t2]) && (! U01[t1, t2]) &&
     (! U02[t1, t2]) && (! U03[t1, t2])

(*The definition of the region U05*)
U0501[t1_, t2_] := (2*(t2 + t2) + t1 <= 2 - 20.5/21.5) && (4*(t2 + t2) + 6*t1 <= 4 -
    20.5/21.5) && (4*t1 <= 4 - 61.5/21.5)
U0502[t1_, t2_] := ((t2 + t2) <= 45/86) && (t1 <= 41/344)
U0503[t1_, t2_] := (55/129 <= (t2 + SieveExponent)) && ((t2 + t2) <= 20/43) && (t1 <= -(1/5)
     (t2 + t2) + 12/43)
U0504[t1_, t2_] := (20/43 <= (t2 + SieveExponent)) && ((t2 + t2) <= 41/86) && (t1 <= 1/7 (t2
     + SieveExponent) + 55/301)
U0505[t1_, t2_] := (41/86 <= (t2 + SieveExponent)) && ((t2 + t2) <= 227/473) && (t1 <= -(t2
    + t2) + 219/301)
U0506[t1_, t2_] := (227/473 <= (t2 + SieveExponent)) && ((t2 + t2) <= 2333/4859) && (t1 <=
    -(58/9) (t2 + t2) + 1264/387)
U0507[t1_, t2_] := (2333/4859 <= (t2 + SieveExponent)) && ((t2 + t2) <= 2501/5203) && (t1 <=
     -(1/6) (t2 + t2) + 65/258)
U0508[t1_, t2_] := (2501/5203 <= (t2 + SieveExponent)) && ((t2 + t2) <= 499/1032) && (t1 <=
    -2 (t2 + t2) + 536/473)
U0509[t1_, t2_] := (499/1032 <= (t2 + SieveExponent)) && ((t2 + t2) <= 28277/57190) && (t1
    <= -(70/11) (t2 + t2) + 1534/473)
```

```
U05[t1_, t2_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2 (1 - t1)])
    && (U0501[t1, t2] || U0502[t1, t2] || U0503[t1, t2] || U0504[t1, t2] || U0505[t1, t2] ||
     U0506[t1, t2] || U0507[t1, t2] || U0508[t1, t2] || U0509[t1, t2]) && (! U01[t1, t2]) &&
     (! U02[t1, t2]) && (! U03[t1, t2]) && (! U04[t1, t2])

(*The definition of the region U06*)
U0601[t1_, t2_] := (2*t1 + t2 <= 2 - 20.5/21.5) && (4*t1 + 6*t2 <= 4 - 20.5/21.5) && (4*t2
    <= 4 - 61.5/21.5) && (2*(1 - t1 - t2) + t2 <= 2 - 20.5/21.5) && (4*(1 - t1 - t2) + 6*t2
    <= 4 - 20.5/21.5)
U0602[t1_, t2_] := (t1 <= 45/86) && (t2 <= 41/344) && ((1 - t1 - t2) <= 45/86)
U06[t1_, t2_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2 (1 - t1)])
    && (U0601[t1, t2] || U0602[t1, t2]) && (! U01[t1, t2]) && (! U02[t1, t2]) && (! U03[t1,
    t2]) && (! U04[t1, t2]) && (! U05[t1, t2])

(*The definition of the region U07*)
U07[t1_, t2_, t3_, t4_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2
    (1 - t1)]) && (U06[t1, t2]) && (SieveExponent <= t3 < Min[t2, 1/2 (1 - t1 - t2)]) && (
    SieveExponent <= t4 < 1/2 t1) && (! (U01[t2, (1 - t1 - t2 - t3) + t3 + t4] || U01[t3, (1
     - t1 - t2 - t3) + t2 + t4] || U01[t4, (1 - t1 - t2 - t3) + t2 + t3] || U01[(1 - t1 - t2
     - t3) + t2 + t3, t4] || U01[(1 - t1 - t2 - t3) + t2 + t4, t3] || U01[(1 - t1 - t2 - t3)
     + t3 + t4, t2]))

(*The definition of the region U08*)
U08[t1_, t2_, t3_, t4_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2
    (1 - t1)]) && (U04[t1, t2] || U05[t1, t2]) && (SieveExponent <= t3 < Min[t2, 1/2 (1 - t1
     - t2)]) && (SieveExponent <= t4 < Min[t3, 1/2 (1 - t1 - t2 - t3)]) && (U01[t1, t2 + t3
    + t4] || U01[t2, t1 + t3 + t4] || U01[t3, t1 + t2 + t4] || U01[t4, t1 + t2 + t3] || U01[
    t1 + t2 + t3, t4] || U01[t1 + t2 + t4, t3] || U01[t1 + t3 + t4, t2] || U01[t2 + t3 + t4,
     t1])

(*The definition of the region U09*)
TypeIforU0901[t1_, t2_] := (2*t1 + t2 <= 2 - 20.5/21.5) && (4*t1 + 6*t2 <= 4 - 20.5/21.5) &&
     (4*t2 <= 4 - 61.5/21.5)
TypeIforU0902[t1_, t2_] := (t1 <= 45/86) && (t2 <= 41/344)
TypeIforU09[t1_, t2_] := TypeIforU0901[t1, t2] || TypeIforU0902[t1, t2] || U01[t1, t2]
U0901[t1_, t2_, t3_, t4_] := (27267/66994 <= t1 + t2 + t3 + t4 <= 231/559) && (SieveExponent
     <= t4 <= -(35/23) (t1 + t2 + t3 + t4) + 767/989)
U0902[t1_, t2_, t3_, t4_] := (231/559 <= t1 + t2 + t3 + t4 <= 3275/7826) && (SieveExponent
    <= t4 <= -(t1 + t2 + t3 + t4) + 313/559)
U0903[t1_, t2_, t3_, t4_] := (3275/7826 <= t1 + t2 + t3 + t4 <= 15005/33497) && (
    SieveExponent <= t4 <= -(41/27) (t1 + t2 + t3 + t4) + 902/1161)
U0904[t1_, t2_, t3_, t4_] := (13074/28595 <= t1 + t2 + t3 + t4 <= 20/43) && (SieveExponent
    <= t4 <= -(1/5) (t1 + t2 + t3 + t4) + 12/43)
U0905[t1_, t2_, t3_, t4_] := (20/43 <= t1 + t2 + t3 + t4 <= 41/86) && (SieveExponent <= t4
    <= 1/7 (t1 + t2 + t3 + t4) + 55/301)
U0906[t1_, t2_, t3_, t4_] := (41/86 <= t1 + t2 + t3 + t4 <= 227/473) && (SieveExponent <= t4
     <= -(t1 + t2 + t3 + t4) + 219/301)
U0907[t1_, t2_, t3_, t4_] := (227/473 <= t1 + t2 + t3 + t4 <= 2333/4859) && (SieveExponent
    <= t4 <= -(58/9) (t1 + t2 + t3 + t4) + 1264/387)
U0908[t1_, t2_, t3_, t4_] := (2333/4859 <= t1 + t2 + t3 + t4 <= 2501/5203) && (SieveExponent
     <= t4 <= -(1/6) (t1 + t2 + t3 + t4) + 65/258)
U0909[t1_, t2_, t3_, t4_] := (2501/5203 <= t1 + t2 + t3 + t4 <= 499/1032) && (SieveExponent
    <= t4 <= -2 (t1 + t2 + t3 + t4) + 536/473)
U0910[t1_, t2_, t3_, t4_] := (499/1032 <= t1 + t2 + t3 + t4 <= 28277/57190) && (
    SieveExponent <= t4 <= -(70/11) (t1 + t2 + t3 + t4) + 1534/473)
```

```
U09[t1_, t2_, t3_, t4_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2
    (1 - t1)]) && (U04[t1, t2] || U05[t1, t2]) && (SieveExponent <= t3 < Min[t2, 1/2 (1 - t1
     - t2)]) && (SieveExponent <= t4 < Min[t3, 1/2 (1 - t1 - t2 - t3)]) && (! U08[t1, t2, t3
    , t4]) && (TypeIforU09[t1, t2 + t3 + t4] || TypeIforU09[t2, t1 + t3 + t4] || TypeIforU09
    [t3, t1 + t2 + t4] || TypeIforU09[t4, t1 + t2 + t3] || TypeIforU09[t1 + t2, t3 + t4] ||
    TypeIforU09[t1 + t3, t2 + t4] || TypeIforU09[t1 + t4, t2 + t3] || TypeIforU09[t2 + t3,
    t1 + t4] || TypeIforU09[t2 + t4, t1 + t3] || TypeIforU09[t3 + t4, t1 + t2] ||
    TypeIforU09[t1 + t2 + t3, t4] || TypeIforU09[t1 + t2 + t4, t3] || TypeIforU09[t1 + t3 +
    t4, t2] || TypeIforU09[t2 + t3 + t4, t1]) && (U0901[t1, t2, t3, t4] || U0902[t1, t2, t3,
     t4] || U0903[t1, t2, t3, t4] || U0904[t1, t2, t3, t4] || U0905[t1, t2, t3, t4] || U0906
    [t1, t2, t3, t4] || U0907[t1, t2, t3, t4] || U0908[t1, t2, t3, t4] || U0909[t1, t2, t3,
    t4] || U0910[t1, t2, t3, t4])

(*The definition of the regions U10 and U11*)
TypeIforU1001[t1_, t2_] := (2*t1 + t2 <= 2 - 20.5/21.5) && (4*t1 + 6*t2 <= 4 - 20.5/21.5) &&
     (4*t2 <= 4 - 61.5/21.5)
TypeIforU1002[t1_, t2_] := (t1 <= 45/86) && (t2 <= 41/344)
TypeIforU1003[t1_, t2_] := (55/129 <= t1 <= 227/473) && (t2 <= -(1/5) t1 + 12/43)
TypeIforU1004[t1_, t2_] := (227/473 <= t1 <= 2333/4859) && (t2 <= -(58/9) t1 + 1264/387)
TypeIforU1005[t1_, t2_] := (2333/4859 <= t1 <= 2501/5203) && (t2 <= -(1/6) t1 + 65/258)
TypeIforU1006[t1_, t2_] := (2501/5203 <= t1 <= 499/1032) && (t2 <= -2 t1 + 536/473)
TypeIforU1007[t1_, t2_] := (499/1032 <= t1 <= 28277/57190) && (t2 <= -(70/11) t1 + 1534/473)
TypeIforU10[t1_, t2_] := TypeIforU1001[t1, t2] || TypeIforU1002[t1, t2] || TypeIforU1003[t1,
     t2] || TypeIforU1004[t1, t2] || TypeIforU1005[t1, t2] || TypeIforU1006[t1, t2] ||
    TypeIforU1007[t1, t2]

U10[t1_, t2_, t3_, t4_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2
    (1 - t1)]) && (U04[t1, t2] || U05[t1, t2]) && (SieveExponent <= t3 < Min[t2, 1/2 (1 - t1
     - t2)]) && (SieveExponent <= t4 < Min[t3, 1/2 (1 - t1 - t2 - t3)]) && (! U08[t1, t2, t3
    , t4]) && (! U09[t1, t2, t3, t4]) && (TypeIforU10[t1, t2 + t3 + t4 + t4] || TypeIforU10[
    t2, t1 + t3 + t4 + t4] || TypeIforU10[t3, t1 + t2 + t4 + t4] || TypeIforU10[t4, t1 + t2
    + t3 + t4] || TypeIforU10[t1 + t2, t3 + t4 + t4] || TypeIforU10[t1 + t3, t2 + t4 + t4]
    || TypeIforU10[t1 + t4, t2 + t3 + t4] || TypeIforU10[t2 + t3, t1 + t4 + t4] ||
    TypeIforU10[t2 + t4, t1 + t3 + t4] || TypeIforU10[t3 + t4, t1 + t2 + t4] || TypeIforU10[
    t4 + t4, t1 + t2 + t3] || TypeIforU10[t1 + t2 + t3, t4 + t4] || TypeIforU10[t1 + t2 + t4
    , t3 + t4] || TypeIforU10[t1 + t3 + t4, t2 + t4] || TypeIforU10[t1 + t4 + t4, t2 + t3]
    || TypeIforU10[t2 + t3 + t4, t1 + t4] || TypeIforU10[t2 + t4 + t4, t1 + t3] ||
    TypeIforU10[t3 + t4 + t4, t1 + t2] || TypeIforU10[t1 + t2 + t3 + t4, t4] || TypeIforU10[
    t1 + t2 + t4 + t4, t3] || TypeIforU10[t1 + t3 + t4 + t4, t2] || TypeIforU10[t2 + t3 + t4
     + t4, t1])

U11[t1_, t2_, t3_, t4_, t5_, t6_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min
    [t1, 1/2 (1 - t1)]) && (U04[t1, t2] || U05[t1, t2]) && (SieveExponent <= t3 < Min[t2,
    1/2 (1 - t1 - t2)]) && (SieveExponent <= t4 < Min[t3, 1/2 (1 - t1 - t2 - t3)]) && (! U08
    [t1, t2, t3, t4]) && (! U09[t1, t2, t3, t4]) && (U10[t1, t2, t3, t4]) && (SieveExponent
    <= t5 < Min[t4, 1/2 (1 - t1 - t2 - t3 - t4)]) && (! (U01[t1, t2 + t3 + t4 + t5] || U01[
    t2, t1 + t3 + t4 + t5] || U01[t3, t1 + t2 + t4 + t5] || U01[t4, t1 + t2 + t3 + t5] ||
    U01[t5, t1 + t2 + t3 + t4] || U01[t1 + t2 + t3 + t4, t5] || U01[t1 + t2 + t3 + t5, t4]
    || U01[t1 + t2 + t4 + t5, t3] || U01[t1 + t3 + t4 + t5, t2] || U01[t2 + t3 + t4 + t5, t1
    ])) && (SieveExponent <= t6 < Min[t5, 1/2 (1 - t1 - t2 - t3 - t4 - t5)]) && (! (U01[t1,
     t2 + t3 + t4 + t5 + t6] || U01[t2, t1 + t3 + t4 + t5 + t6] || U01[t3, t1 + t2 + t4 + t5
     + t6] || U01[t4, t1 + t2 + t3 + t5 + t6] || U01[t5, t1 + t2 + t3 + t4 + t6] || U01[t6,
    t1 + t2 + t3 + t4 + t5] || U01[t1 + t2 + t3 + t4 + t5, t6] || U01[t1 + t2 + t3 + t4 + t6
    , t5] || U01[t1 + t2 + t3 + t5 + t6, t4] || U01[t1 + t2 + t4 + t5 + t6, t3] || U01[t1 +
    t3 + t4 + t5 + t6, t2] || U01[t2 + t3 + t4 + t5 + t6, t1]))

(*The definition of the region U12*)
```

```
U12[t1_, t2_, t3_, t4_, t5_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1,
    1/2 (1 - t1)]) && (U04[t1, t2] || U05[t1, t2]) && (SieveExponent <= t3 < Min[t2, 1/2 (1
    - t1 - t2)]) && (SieveExponent <= t4 < Min[t3, 1/2 (1 - t1 - t2 - t3)]) && (! U08[t1, t2
    , t3, t4]) && (! U09[t1, t2, t3, t4]) && (! U10[t1, t2, t3, t4]) && (t4 < 1/2 (1 - t1 -
    t2 - t3 - t4)) && (t4 < t5 < 1/2 (1 - t1 - t2 - t3 - t4)) && (U01[t1, t2 + t3 + t4 + t5]
     || U01[t2, t1 + t3 + t4 + t5] || U01[t3, t1 + t2 + t4 + t5] || U01[t4, t1 + t2 + t3 +
    t5] || U01[t5, t1 + t2 + t3 + t4] || U01[t1 + t2 + t3 + t4, t5] || U01[t1 + t2 + t3 + t5
    , t4] || U01[t1 + t2 + t4 + t5, t3] || U01[t1 + t3 + t4 + t5, t2] || U01[t2 + t3 + t4 +
    t5, t1])

(*The definition of the region U13*)
U13[t1_, t2_, t3_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2 (1 -
    t1)]) && (t2 < 1/2 (1 - t1 - t2)) && (t2 < t3 < 1/2 (1 - t1 - t2)) && (! U01[t1, t2]) &&
     (! U02[t1, t2]) && (! U03[t1, t2]) && (! U04[t1, t2]) && (! U05[t1, t2]) && (! U06[t1,
    t2]) && (U01[t1, t2 + t3] || U01[t2, t1 + t3] || U01[t3, t1 + t2] || U01[t1 + t2, t3] ||
     U01[t1 + t3, t2] || U01[t2 + t3, t1])
```

**Code for the numerical calculations.** The following Mathematica code shows all the numerical calculations in the present paper. We use an Intel(R) Xeon(R) Platinum 8383C CPU with 80 Wolfram kernels to run this code, and it took several days to give numerical values.

```
(*The linear sieve functions F[u] and f[u], in Galway's SieveFunctions package*)
FF[u_] := UpperSieveFunc[1, u]
ff[u_] := LowerSieveFunc[1, u]

(*The Buchstab-Selberg function \[Omega][u]*)
\[Omega][u_] := (FF[u] + ff[u])/(2 Exp[EulerGamma])

(*Numerical upper and lower bounds for \[Omega][u] which will be used in calculating the
    value of sums related to regions U07, U08, U09, U10, U11 and U12*)
\[Omega]0[u_] := Piecewise[{{1/u, 1 <= u < 2}, {(1 + Log[u - 1])/u, 2 <= u < 3}, {(1 + Log[u
     - 1])/u + (\[Pi]^2/12 + Log[-2 + u] Log[-1 + u] + PolyLog[2, 2 - u])/u, 3 <= u < 4},
    {0.5612, u >= 4}}];
\[Omega]1[u_] := Piecewise[{{1/u, 1 <= u < 2}, {(1 + Log[u - 1])/u, 2 <= u < 3}, {(1 + Log[u
     - 1])/u + (\[Pi]^2/12 + Log[-2 + u] Log[-1 + u] + PolyLog[2, 2 - u])/u, 3 <= u < 4},
    {0.5617, u >= 4}}];

(*Numerical values of S1 and S2*)
S1 = 817/79 \[Omega][817/79]
S2 = 817/79 NIntegrate[\[Omega][817/79 (1 - t)]/t, {t, SieveExponent, 1/2}]
(*S1 = 5.806486152249589*)
(*S2 = 9.54031171046251*)

(*Numerical value of S301*)
S301 = NIntegrate[Boole[U01[t1, t2]]*\[Omega][(1 - t1 - t2)/t2]/(t1 t2^2), {t1,
    SieveExponent, 1/2}, {t2, SieveExponent, Min[t1, 1/2 (1 - t1)]}, Method -> {"
    GlobalAdaptive", "MaxErrorIncreases" -> 30000}, MinRecursion -> 9]
(*S301 = 0.7780429879766562 with error less than 0.00006891062655672592*)

(*Numerical value of S302*)
S302 = NIntegrate[Boole[U02[t1, t2]]*\[Omega][(1 - t1 - t2)/t2]/(t1 t2^2), {t1,
    SieveExponent, 1/2}, {t2, SieveExponent, Min[t1, 1/2 (1 - t1)]}, Method -> {"
    GlobalAdaptive", "MaxErrorIncreases" -> 30000}, MinRecursion -> 9]
(*S302 = 0.46761474180894713 with error less than 0.0001162439926380884*)

(*Numerical value of S303*)
S303 = NIntegrate[Boole[U03[t1, t2]]*\[Omega][(1 - t1 - t2)/t2]/(t1 t2^2), {t1,
    SieveExponent, 1/2}, {t2, SieveExponent, Min[t1, 1/2 (1 - t1)]}, Method -> {"
    GlobalAdaptive", "MaxErrorIncreases" -> 1000}, MinRecursion -> 9]
(*S303 = 0.021051326757928233 with error less than 0.000038695977220248286*)

(*Numerical value of S304*)
S304 = NIntegrate[Boole[U04[t1, t2]]*(817/79 \[Omega][817/79 (1 - t1 - t2)])/(t1 t2), {t1,
    SieveExponent, 1/2}, {t2, SieveExponent, Min[t1, 1/2 (1 - t1)]}, Method -> {"
    GlobalAdaptive", "MaxErrorIncreases" -> 10000}, MinRecursion -> 9] - NIntegrate[Boole[
    U04[t1, t2]]*(817/79 \[Omega][817/79 (1 - t1 - t2 - t3)])/(t1 t2 t3), {t1, SieveExponent
    , 1/2}, {t2, SieveExponent, Min[t1, 1/2 (1 - t1)]}, {t3, SieveExponent, t2}, Method -> {
    "GlobalAdaptive", "MaxErrorIncreases" -> 30000}, MinRecursion -> 9]
(*S304 = 2.293755844235633 with error less than 0.000806587867538698*)

(*Numerical value of S305*)
S305 = NIntegrate[Boole[U05[t1, t2]]*(817/79 \[Omega][817/79 (1 - t1 - t2)])/(t1 t2), {t1,
    SieveExponent, 1/2}, {t2, SieveExponent, Min[t1, 1/2 (1 - t1)]}, Method -> {"
    GlobalAdaptive", "MaxErrorIncreases" -> 30000}, MinRecursion -> 9] - NIntegrate[Boole[
    U05[t1, t2]]*(817/79 \[Omega][817/79 (1 - t1 - t2 - t3)])/(t1 t2 t3), {t1, SieveExponent
    , 1/2}, {t2, SieveExponent, Min[t1, 1/2 (1 - t1)]}, {t3, SieveExponent, t2}, Method -> {
    "GlobalAdaptive", "MaxErrorIncreases" -> 30000}, MinRecursion -> 9]
(*S305 = 0.01911550408428917 with error less than 0.00006500005875860931*)

(*Numerical value of S306*)
```

```
S306 = NIntegrate[Boole[U06[t1, t2]]*\[Omega][(1 - t1 - t2)/t2]/(t1 t2^2), {t1,
    SieveExponent, 1/2}, {t2, SieveExponent, Min[t1, 1/2 (1 - t1)]}, Method -> {"
    GlobalAdaptive", "MaxErrorIncreases" -> 10000}, MinRecursion -> 9]
(*S306 = 0.2435717173450804 with error less than 0.00004985272810300297*)

(*Numerical value of S307, using ParallelSum with 80 Wolfram kernels*)
SubIntervals = Table[{SieveExponent + (i - 1)*(1/2 - SieveExponent)/80, SieveExponent + i
    *(1/2 - SieveExponent)/80}, {i, 1, 80}];

S307 = Total[ParallelSum[NIntegrate[Boole[U07[t1, t2, t3, t4]]*(\[Omega]1[(t1 - t4)/t4]*\[
    Omega]1[(1 - t1 - t2 - t3)/t3])/(t2 t3^2 t4^2), {t1, interval[[1]], interval[[2]]}, {t2,
     SieveExponent, Min[t1, 1/2 (1 - t1)]}, {t3, SieveExponent, Min[t2, 1/2 (1 - t1 - t2)]},
     {t4, SieveExponent, 1/2 t1}, Method -> {"GlobalAdaptive", "MaxErrorIncreases" ->
    30000}, MinRecursion -> 9], {interval, SubIntervals}]]
(*S307 = 0.2286383981286717 with total error less than 0.001181290534696173*)

(*Numerical value of S308, using ParallelSum with 80 Wolfram kernels*)
SubIntervals = Table[{SieveExponent + (i - 1)*(1/2 - SieveExponent)/80, SieveExponent + i
    *(1/2 - SieveExponent)/80}, {i, 1, 80}];

S308 = Total[ParallelSum[NIntegrate[Boole[U08[t1, t2, t3, t4]]*\[Omega]0[(1 - t1 - t2 - t3 -
     t4)/t4]/(t1 t2 t3 t4^2), {t1, interval[[1]], interval[[2]]}, {t2, SieveExponent, Min[t1
    , 1/2 (1 - t1)]}, {t3, SieveExponent, Min[t2, 1/2 (1 - t1 - t2)]}, {t4, SieveExponent,
    Min[t3, 1/2 (1 - t1 - t2 - t3)]}, Method -> {"GlobalAdaptive", "MaxErrorIncreases" ->
    30000}, MinRecursion -> 9], {interval, SubIntervals}]]
(*S308 = 0.1080288361639965 with total error less than 0.000947703841604724*)

(*Numerical value of S309, using ParallelSum with 80 Wolfram kernels*)
SubIntervals = Table[{SieveExponent + (i - 1)*(1/2 - SieveExponent)/80, SieveExponent + i
    *(1/2 - SieveExponent)/80}, {i, 1, 80}];

S309 = Total[ParallelSum[NIntegrate[Boole[U09[t1, t2, t3, t4]]*\[Omega]0[(1 - t1 - t2 - t3 -
     t4)/t4]/(t1 t2 t3 t4^2), {t1, interval[[1]], interval[[2]]}, {t2, SieveExponent, Min[t1
    , 1/2 (1 - t1)]}, {t3, SieveExponent, Min[t2, 1/2 (1 - t1 - t2)]}, {t4, SieveExponent,
    Min[t3, 1/2 (1 - t1 - t2 - t3)]}, Method -> {"GlobalAdaptive", "MaxErrorIncreases" ->
    30000}, MinRecursion -> 9], {interval, SubIntervals}]]
(*S309 = 0.005917808762848026 with total error less than 0.00005522428549700662*)

(*Numerical value of S310, using ParallelSum with 80 Wolfram kernels*)
SubIntervals = Table[{SieveExponent + (i - 1)*(1/2 - SieveExponent)/80, SieveExponent + i
    *(1/2 - SieveExponent)/80}, {i, 1, 80}];

S310 = Total[ParallelSum[NIntegrate[Boole[U10[t1, t2, t3, t4]]*\[Omega]0[(1 - t1 - t2 - t3 -
     t4)/t4]/(t1 t2 t3 t4^2), {t1, interval[[1]], interval[[2]]}, {t2, SieveExponent, Min[t1
    , 1/2 (1 - t1)]}, {t3, SieveExponent, Min[t2, 1/2 (1 - t1 - t2)]}, {t4, SieveExponent,
    Min[t3, 1/2 (1 - t1 - t2 - t3)]}, Method -> {"GlobalAdaptive", "MaxErrorIncreases" ->
    30000}, MinRecursion -> 9], {interval, SubIntervals}]]
(*S310 = 0.003108593198969999 with total error less than 0.0001285051481890434*)

(*Numerical value of S311, using ParallelSum with 80 Wolfram kernels*)
SubIntervals = Table[{SieveExponent + (i - 1)*(1/2 - SieveExponent)/80, SieveExponent + i
    *(1/2 - SieveExponent)/80}, {i, 1, 80}];

S311 = Total[ParallelSum[NIntegrate[Boole[U11[t1, t2, t3, t4, t5, t6]]*\[Omega]1[(1 - t1 -
    t2 - t3 - t4 - t5 - t6)/t6]/(t1 t2 t3 t4 t5 t6^2), {t1, interval[[1]], interval[[2]]}, {
    t2, SieveExponent, Min[t1, 1/2 (1 - t1)]}, {t3, SieveExponent, Min[t2, 1/2 (1 - t1 - t2)
    ]}, {t4, SieveExponent, Min[t3, 1/2 (1 - t1 - t2 - t3)]}, {t5, SieveExponent, Min[t4,
    1/2 (1 - t1 - t2 - t3 - t4)]}, {t6, SieveExponent, Min[t5, 1/2 (1 - t1 - t2 - t3 - t4 -
    t5)]}, Method -> {"GlobalAdaptive", "MaxErrorIncreases" -> 30000}, MinRecursion -> 9], {
    interval, SubIntervals}]]
(*S311 = 0.0000004339697200875225 with total error less than 0.0000002148256005728281*)

(*Numerical value of S312, using ParallelSum with 80 Wolfram kernels*)
```

```
SubIntervals = Table[{SieveExponent + (i - 1)*(1/2 - SieveExponent)/80, SieveExponent + i
    *(1/2 - SieveExponent)/80}, {i, 1, 80}];

S312 = Total[ParallelSum[NIntegrate[Boole[U12[t1, t2, t3, t4, t5]]*\[Omega]0[(1 - t1 - t2 -
    t3 - t4 - t5)/t5]/(t1 t2 t3 t4 t5^2), {t1, interval[[1]], interval[[2]]}, {t2,
    SieveExponent, Min[t1, 1/2 (1 - t1)]}, {t3, SieveExponent, Min[t2, 1/2 (1 - t1 - t2)]},
    {t4, SieveExponent, Min[t3, 1/2 (1 - t1 - t2 - t3)]}, {t5, t4, 1/2 (1 - t1 - t2 - t3 -
    t4)}, Method -> {"GlobalAdaptive", "MaxErrorIncreases" -> 30000}, MinRecursion -> 9], {
    interval, SubIntervals}]]
(*S312 = 0.0001098743536826065 with total error less than 0.00004493697265011205*)

(*Numerical value of S313*)
S313 = NIntegrate[Boole[U13[t1, t2, t3]]*\[Omega][(1 - t1 - t2 - t3)/t3]/(t1 t2 t3^2), {t1,
    SieveExponent, 1/2}, {t2, SieveExponent, Min[t1, 1/2 (1 - t1)]}, {t3, t2, 1/2 (1 - t1 -
    t2)}, Method -> {"GlobalAdaptive", "MaxErrorIncreases" -> 30000}, MinRecursion -> 9]
(*S313 = 0.025995780981324837 with error less than 0.00030888857052706895*)
```

**Code for the plot of regions.** The following Mathematica code gives the plot of regions $U_{01}$–$U_{06}$, the part of $U_{01}$ which produces $U_{02}$, the two dimensional Type–I information region produced by $U_{01}$ and the three–dimensional plot of $U_{13}$.

```
RegionPlot[{{SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2 (1 - t1)]}, {U01
    [t1, t2]}, {U02[t1, t2]}, {U03[t1, t2]}, {U04[t1, t2]}, {U05[t1, t2]}, {U06[t1, t2]}}, {
    t1, SieveExponent, 1/2}, {t2, SieveExponent, 1/2}, PlotLegends -> {"TotalRegion", "U01",
     "U02", "U03", "U04", "U05", "U06"}, MaxRecursion -> 13]
```

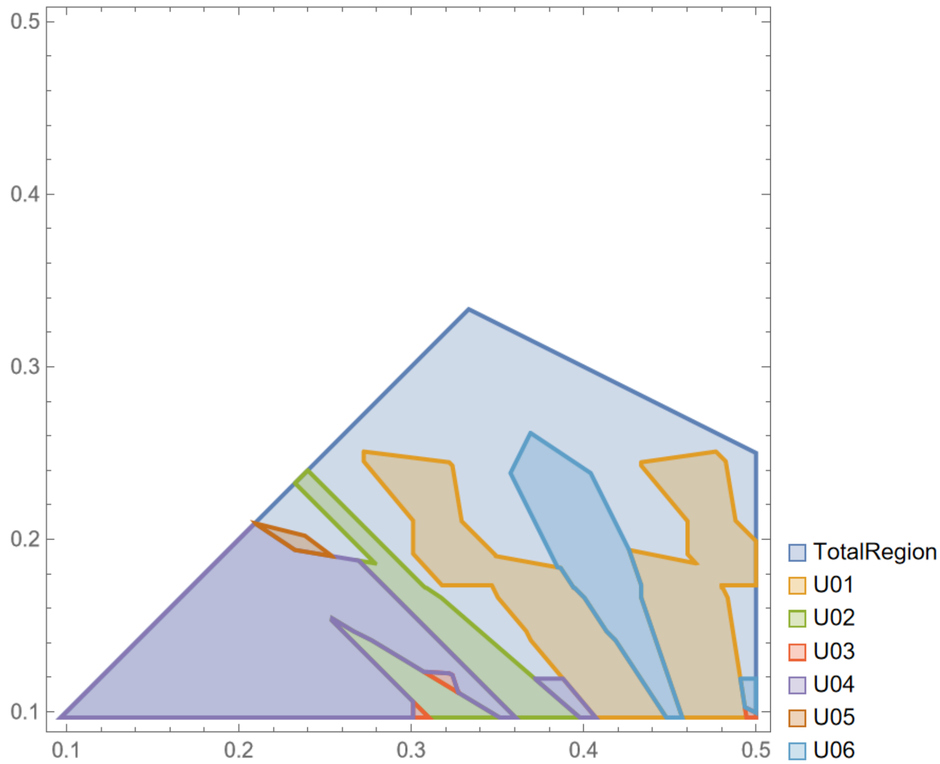


**Figure 1: Plot of regions $U_{01}$–$U_{06}$**

```
U02V201[t1_, t2_] := (27267/66994 <= t1 <= 231/559) && (SieveExponent <= t2 <= -(35/23) t1 +
     767/989)
U02V202[t1_, t2_] := (231/559 <= t1 <= 3275/7826) && (SieveExponent <= t2 <= -t1 + 313/559)
U02V203[t1_, t2_] := (3275/7826 <= t1 <= 15005/33497) && (SieveExponent <= t2 <= -(41/27) t1
     + 902/1161)
U02V204[t1_, t2_] := (13074/28595 <= t1 <= 20/43) && (SieveExponent <= t2 <= -(1/5) t1 +
    12/43)
U02V205[t1_, t2_] := (20/43 <= t1 <= 41/86) && (SieveExponent <= t2 <= 1/7 t1 + 55/301)
U02V206[t1_, t2_] := (41/86 <= t1 <= 227/473) && (SieveExponent <= t2 <= -t1 + 219/301)
U02V207[t1_, t2_] := (227/473 <= t1 <= 2333/4859) && (SieveExponent <= t2 <= -(58/9) t1 +
    1264/387)
U02V208[t1_, t2_] := (2333/4859 <= t1 <= 2501/5203) && (SieveExponent <= t2 <= -(1/6) t1 +
    65/258)
U02V209[t1_, t2_] := (2501/5203 <= t1 <= 499/1032) && (SieveExponent <= t2 <= -2 t1 +
    536/473)
U02V210[t1_, t2_] := (499/1032 <= t1 <= 28277/57190) && (SieveExponent <= t2 <= -(70/11) t1
    + 1534/473)
U02in2Variables[t1_, t2_] := U02V201[t1, t2] || U02V202[t1, t2] || U02V203[t1, t2] ||
    U02V204[t1, t2] || U02V205[t1, t2] || U02V206[t1, t2] || U02V207[t1, t2] || U02V208[t1,
    t2] || U02V209[t1, t2] || U02V210[t1, t2]

RegionPlot[{{SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2 (1 - t1)]}, {U01
    [t1, t2]}, {U02in2Variables[t1, t2]}}, {t1, SieveExponent, 1/2}, {t2, SieveExponent,
    1/2}, PlotLegends -> {"TotalRegion", "U01", "U02 in 2 variables"}, MaxRecursion -> 9]
```

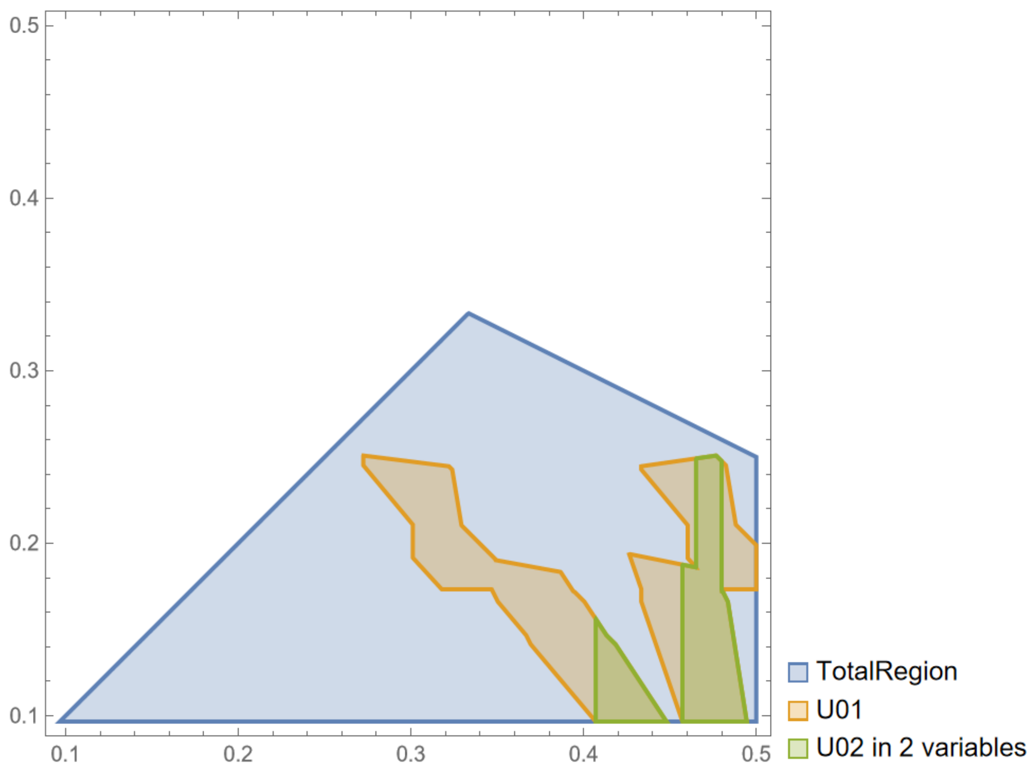


**Figure 2: Plot of the part of region $U_{01}$ which produces $U_{02}$**

```
WattTypeI1[t1_, t2_] := (2*t1 + t2 <= 2 - 20.5/21.5) && (4*t1 + 6*t2 <= 4 - 20.5/21.5) &&
    (4*t2 <= 4 - 61.5/21.5)
WattTypeI2[t1_, t2_] := (t1 <= 45/86) && (t2 <= 41/344)
WattTypeI[t1_, t2_] := (SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2 (1 -
    t1)]) && (WattTypeI1[t1, t2] || WattTypeI2[t1, t2])
ExtraTypeI1[t1_, t2_] := (55/129 <= t1 <= 20/43) && (t2 <= -(1/5) t1 + 12/43)
ExtraTypeI2[t1_, t2_] := (20/43 <= t1 <= 41/86) && (t2 <= 1/7 t1 + 55/301)
ExtraTypeI3[t1_, t2_] := (41/86 <= t1 <= 227/473) && (t2 <= -t1 + 219/301)
ExtraTypeI4[t1_, t2_] := (227/473 <= t1 <= 2333/4859) && (t2 <= -(58/9) t1 + 1264/387)
ExtraTypeI5[t1_, t2_] := (2333/4859 <= t1 <= 2501/5203) && (t2 <= -(1/6) t1 + 65/258)
ExtraTypeI6[t1_, t2_] := (2501/5203 <= t1 <= 499/1032) && (t2 <= -2 t1 + 536/473)
ExtraTypeI7[t1_, t2_] := (499/1032 <= t1 <= 28277/57190) && (t2 <= -(70/11) t1 + 1534/473)
ExtraTypeI[t1_, t2_] := (ExtraTypeI1[t1, t2] || ExtraTypeI2[t1, t2] || ExtraTypeI3[t1, t2]
    || ExtraTypeI4[t1, t2] || ExtraTypeI5[t1, t2] || ExtraTypeI6[t1, t2] || ExtraTypeI7[t1,
    t2]) && (! WattTypeI[t1, t2])

RegionPlot[{{SieveExponent <= t1 < 1/2 && SieveExponent <= t2 < Min[t1, 1/2 (1 - t1)]}, {U01
    [t1, t2]}, {WattTypeI[t1, t2]}, {ExtraTypeI[t1, t2]}}, {t1, SieveExponent, 1/2}, {t2,
    SieveExponent, 1/2}, PlotLegends -> {"TotalRegion", "U01", "Watt Type-I", "Extra Type-I"
    }, MaxRecursion -> 9]
```

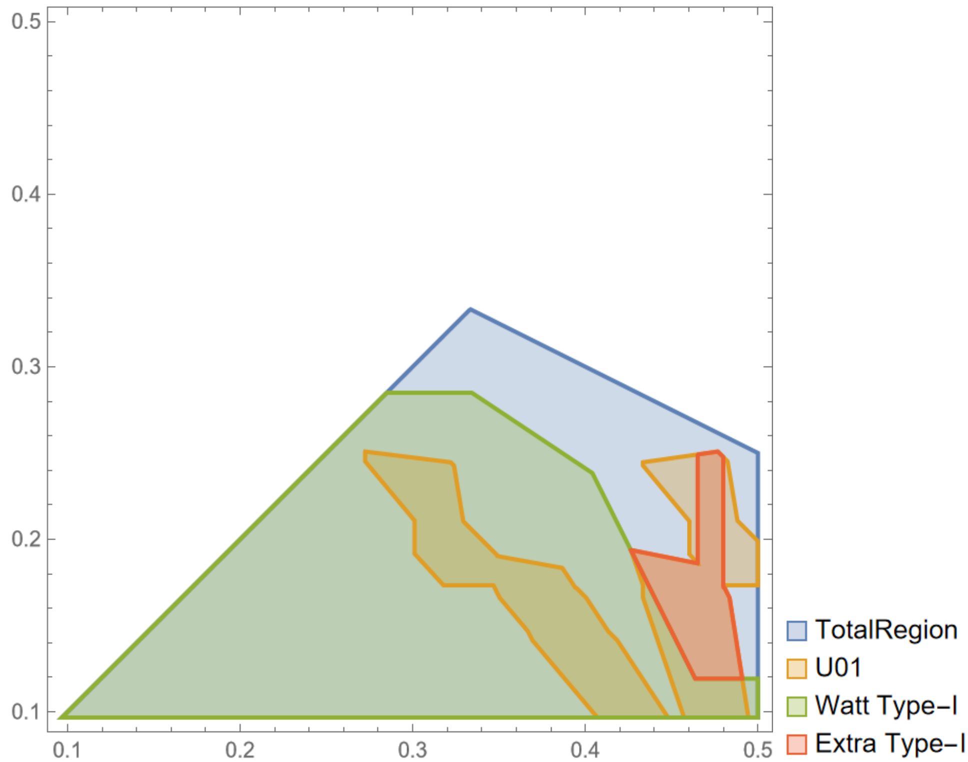


**Figure 3: Plot of the two–dimensional Type–I information**

```
RegionPlot3D[{{U13[t1, t2, t3]}}, {t1, 0.2, 0.4}, {t2, SieveExponent, 0.3}, {t3,
    SieveExponent, 0.3}, PlotLegends -> {"U13"}, MaxRecursion -> 12]

RegionPlot3D[{{U13[t1, t2, t3]}, {U01[t1, t2]}, {U02[t1, t2]}, {U03[t1, t2]}, {U04[t1, t2]},
     {U05[t1, t2]}, {U06[t1, t2]}}, {t1, 0.2, 0.4}, {t2, SieveExponent, 0.3}, {t3,
    SieveExponent, 0.3}, PlotLegends -> {"U13", "U01", "U02", "U03", "U04", "U05", "U06"},
    MaxRecursion -> 12]
```

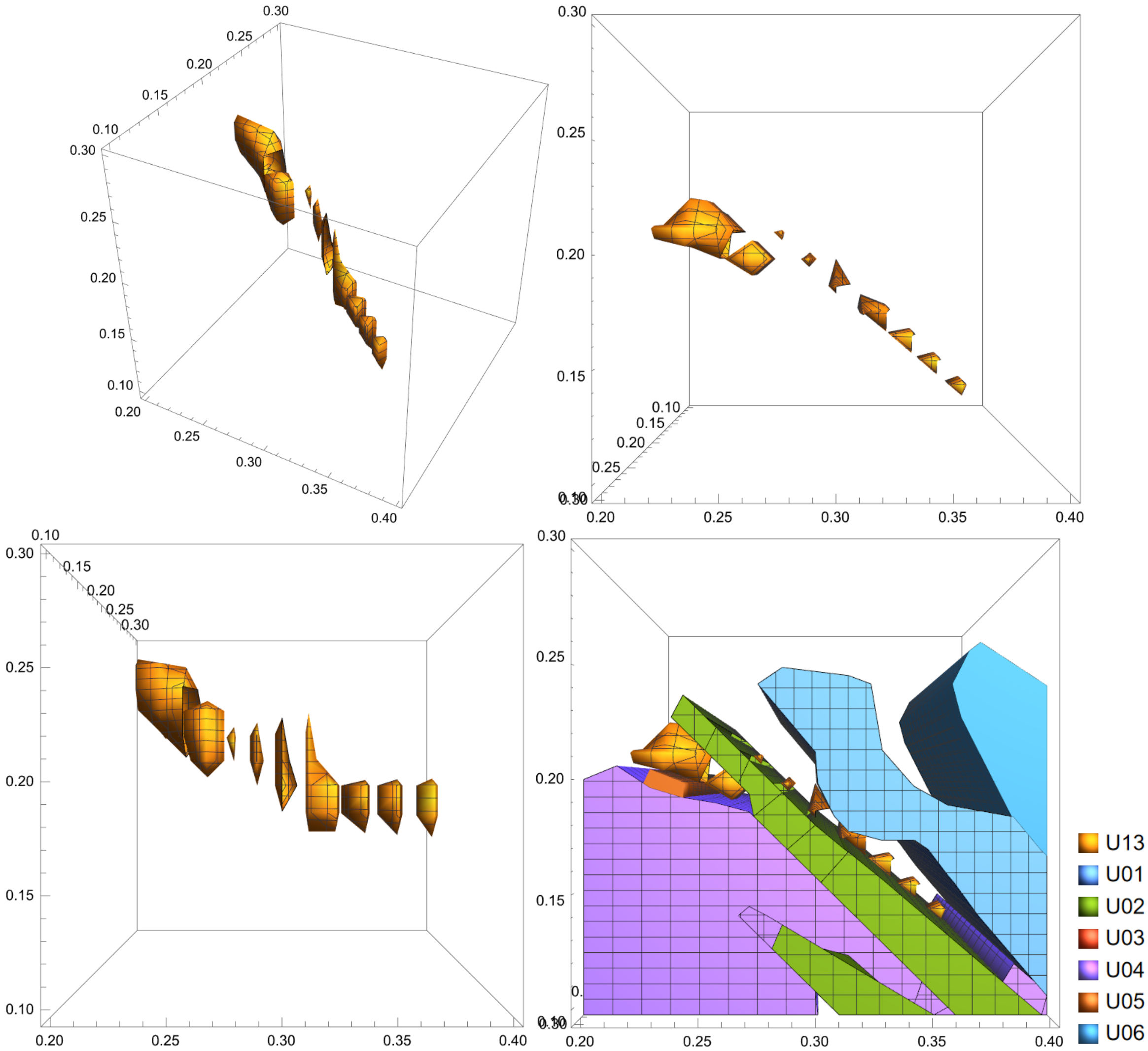


**Figure 4: Three–dimensional plot of region $U_{13}$ (Default view, Top view, Front view, With other regions)**

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