ON THE WEIGHTED AM–GM INEQUALITY AND REFINED INEQUALITIES BETWEEN ARITHMETIC FUNCTIONS

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Abstract. Let $\varphi(n)$, $\psi(n)$ and $\sigma(n)$ denote the Euler totient function and the Dedekind function respectively. Using improved versions of the weighted AM–GM inequality, we obtain a series of sharp upper bounds for

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} \quad \text{and} \quad \varphi(n)^{\psi(n)}\psi(n)^{\varphi(n)},$$

improving previous bounds showed by Sándor and Atanassov.

Seems horrible but actually trivial. —*EthanWYX2009*

Contents

1. Introduction

Let n be a positive integer. Let $\varphi(n)$, $\psi(n)$ and $\sigma(n)$ denote the Euler totient function, Dedekind function and sum–of–divisors function respectively. For $n > 1$, we have

$$\varphi(n) = n \prod_{p \leqslant n} \frac{p-1}{p} \quad \text{and} \quad \psi(n) = n \prod_{p \leqslant n} \frac{p+1}{p}. \tag{1}$$

These arithmetic functions satisfy many important properties. For example, the following inequality is well–known:

$$\varphi(n) \leqslant \psi(n) \leqslant \sigma(n). \tag{2}$$

In this paper we are looking for bounds for quantities

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} \quad \text{and} \quad \varphi(n)^{\psi(n)}\psi(n)^{\varphi(n)}.$$

In 2011, Atanassov [6] first obtained a lower bound for $\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)}$. He showed that for any $n > 1$, we have

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} > n^{2n}. \tag{3}$$

In 2013, Kannan and Srikanth [13] sharpened (3) by showing that

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} > n^{\varphi(n)+\psi(n)}. \tag{4}$$

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Finally, Sándor and Atanassov [14] in 2019 proved the following refined estimates using the weighted AM–GM inequality.

$$
\begin{aligned}
n^{\varphi(n)+\psi(n)} < \left(\frac{\varphi(n)+\psi(n)}{2}\right)^{\varphi(n)+\psi(n)} &< \varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} < \left(\frac{\varphi(n)^2+\psi(n)^2}{2}\right)^{\frac{\varphi(n)+\psi(n)}{2}} \\
&< \psi(n)^{\varphi(n)+\psi(n)}, \qquad (5)\\
\left(\frac{\varphi(n)\psi(n)(\varphi(n)+\psi(n))}{\varphi(n)^2+\psi(n)^2}\right)^{\varphi(n)+\psi(n)} &< \varphi(n)^{\psi(n)}\psi(n)^{\varphi(n)} < \left(\frac{2\varphi(n)\psi(n)}{\varphi(n)+\psi(n)}\right)^{\varphi(n)+\psi(n)} \\
&< \left(\varphi(n)\psi(n)\right)^{\frac{\varphi(n)+\psi(n)}{2}} < n^{\varphi(n)+\psi(n)}. \qquad (6)
\end{aligned}
$$

For other types of inequalities between arithmetic functions, we refer the readers to [10] and its references. In this paper, we shall use some refined inequalities to improve the upper bounds proved by Sándor and Atanassov [14].

Theorem 1.1. *For any integer $n > 1$, we have the following inequalities:*

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} \leqslant \left(\frac{\varphi(n)^3+4\varphi(n)^2\psi(n)+\varphi(n)\psi(n)^2+2\psi(n)^3}{2(\varphi(n)+\psi(n))^2}\right)^{\varphi(n)+\psi(n)}, \qquad \textbf{(A1)}$$

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} \leqslant \left(\frac{2\varphi(n)^2\psi(n)+2\psi(n)^3-\varphi(n)\left(\varphi(n)-\varphi(n)^{\frac{\varphi(n)}{\varphi(n)+\psi(n)}}-\psi(n)^{\frac{\psi(n)}{\varphi(n)+\psi(n)}}\right)^2}{2\psi(n)(\varphi(n)+\psi(n))} \right.$$
$$\left. \frac{-\psi(n)\left(\psi(n)-\varphi(n)^{\frac{\varphi(n)}{\varphi(n)+\psi(n)}}-\psi(n)^{\frac{\psi(n)}{\varphi(n)+\psi(n)}}\right)^2}{2\psi(n)(\varphi(n)+\psi(n))}\right)^{\varphi(n)+\psi(n)}, \qquad \textbf{(B1)}$$

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} \leqslant \left(\frac{\varphi(n)^{\frac{3}{2}}+\psi(n)^{\frac{3}{2}}}{\varphi(n)+\psi(n)}\right)^{2(\varphi(n)+\psi(n))}, \qquad \textbf{(C1)}$$

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} \leqslant \left(\frac{2\varphi(n)^{\frac{3}{2}}\psi(n)^{\frac{1}{2}}-\varphi(n)\psi(n)+\psi(n)^2}{\varphi(n)+\psi(n)}\right)^{\varphi(n)+\psi(n)}, \qquad \textbf{(D1)}$$

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} \leqslant \left(\frac{\varphi(n)^2+\psi(n)^2}{(\varphi(n)+\psi(n))\exp\left(2-2\frac{\varphi(n)^{\frac{3}{2}}+\psi(n)^{\frac{3}{2}}}{((\varphi(n)^2+\psi(n)^2)(\varphi(n)+\psi(n)))^{\frac{1}{2}}}\right)}\right)^{\varphi(n)+\psi(n)}, \qquad \textbf{(E1)}$$

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} \leqslant \left(\frac{\varphi(n)^2+\psi(n)^2-\frac{3\varphi(n)\psi(n)(\psi(n)-\varphi(n))^2}{2\varphi(n)^2+8\varphi(n)\psi(n)+2\psi(n)^2}}{\varphi(n)+\psi(n)}\right)^{\varphi(n)+\psi(n)}, \qquad \textbf{(F1)}$$

$$\varphi(n)^{\psi(n)}\psi(n)^{\varphi(n)} \leqslant \left(\frac{-\varphi(n)^3+6\varphi(n)^2\psi(n)+3\varphi(n)\psi(n)^2}{2(\varphi(n)+\psi(n))^2}\right)^{\varphi(n)+\psi(n)}, \qquad \textbf{(A2)}$$

$$\varphi(n)^{\psi(n)}\psi(n)^{\varphi(n)} \leqslant \left(\frac{4\varphi(n)\psi(n)^2-\varphi(n)\left(\psi(n)-\psi(n)^{\frac{\varphi(n)}{\varphi(n)+\psi(n)}}-\varphi(n)^{\frac{\psi(n)}{\varphi(n)+\psi(n)}}\right)^2}{2\psi(n)(\varphi(n)+\psi(n))}\right.$$
$$\left. \frac{-\psi(n)\left(\varphi(n)-\psi(n)^{\frac{\varphi(n)}{\varphi(n)+\psi(n)}}-\varphi(n)^{\frac{\psi(n)}{\varphi(n)+\psi(n)}}\right)^2}{2\psi(n)(\varphi(n)+\psi(n))}\right)^{\varphi(n)+\psi(n)}, \qquad \textbf{(B2)}$$

$$\varphi(n)^{\psi(n)}\psi(n)^{\varphi(n)} \leqslant \left(\frac{\varphi(n)^{\frac{1}{2}}\psi(n)^{\frac{1}{2}}\left(\varphi(n)^{\frac{1}{2}}+\psi(n)^{\frac{1}{2}}\right)}{\varphi(n)+\psi(n)}\right)^{2(\varphi(n)+\psi(n))}, \tag{C2}$$

$$\varphi(n)^{\psi(n)}\psi(n)^{\varphi(n)} \leqslant \left(\frac{2\varphi(n)^{\frac{3}{2}}\psi(n)^{\frac{1}{2}}+\varphi(n)\psi(n)-\varphi(n)^2}{\varphi(n)+\psi(n)}\right)^{\varphi(n)+\psi(n)}, \tag{D2}$$

$$\varphi(n)^{\psi(n)}\psi(n)^{\varphi(n)} \leqslant \left(\frac{2\varphi(n)\psi(n)}{(\varphi(n)+\psi(n))\exp\left(2-2\frac{\varphi(n)^{\frac{1}{2}}\psi(n)^{\frac{1}{2}}\left(\varphi(n)^{\frac{1}{2}}+\psi(n)^{\frac{1}{2}}\right)}{(2\varphi(n)\psi(n)(\varphi(n)+\psi(n)))^{\frac{1}{2}}}\right)}\right)^{\varphi(n)+\psi(n)} \tag{E2}$$

and

$$\varphi(n)^{\psi(n)}\psi(n)^{\varphi(n)} \leqslant \left(\frac{2\varphi(n)\psi(n)-\frac{3\varphi(n)\psi(n)(\psi(n)-\varphi(n))^2}{4(\varphi(n)^2+\varphi(n)\psi(n)+\psi(n)^2)}}{\varphi(n)+\psi(n)}\right)^{\varphi(n)+\psi(n)}. \tag{F2}$$

All inequalities above may be replaced with function $\sigma(n)$ instead of function $\psi(n)$.

Note that we have

$$\textbf{(B1)} \implies \textbf{(A1)}, \quad \textbf{(B2)} \implies \textbf{(A2)},$$

$$\textbf{(D1)} \implies \textbf{(C1)}, \quad \textbf{(D2)} \implies \textbf{(C2)},$$

where $\mathbf{X} \implies \mathbf{Y}$ means that $\mathbf{X}$ implies $\mathbf{Y}$.

2. Refinements of the weighted AM–GM inequality

In this section we shall list several variants of the weighted AM–GM inequality. First, recall that the classical weighted AM–GM inequality states that

Lemma 2.1. *If real numbers $\alpha_1,\dots,\alpha_n>0$ satisfy $\alpha_1+\cdots+\alpha_n=1$, then for $x_1,\dots,x_n \geqslant 0$, we have*

$$x_1^{\alpha_1}\cdots x_n^{\alpha_n} \leqslant \alpha_1 x_1+\cdots+\alpha_n x_n.$$

Moreover, for $n=2$ this becomes

$$\frac{1}{\frac{\alpha_1}{x_1}+\frac{\alpha_2}{x_2}} \leqslant x_1^{\alpha_1}x_2^{\alpha_2} \leqslant \alpha_1 x_1+\alpha_2 x_2.$$

In Lemma 2.1, put $x_1=a$, $x_2=b$, $\alpha_1=\frac{a}{a+b}$ and $\alpha_2=\frac{b}{a+b}$ we get

Lemma 2.2. *([14], Proposition 1]). For any $a,b>0$, we have*

$$\left(\frac{a+b}{2}\right)^{a+b} \leqslant a^a b^b \leqslant \left(\frac{a^2+b^2}{a+b}\right)^{a+b}.$$

Again, put $x_1=b$, $x_2=a$, $\alpha_1=\frac{a}{a+b}$ and $\alpha_2=\frac{b}{a+b}$ we get

Lemma 2.3. *([14], Proposition 2]). For any $a,b>0$, we have*

$$\left(\frac{ab(a+b)}{a^2+b^2}\right)^{a+b} \leqslant a^b b^a \leqslant \left(\frac{2ab}{a+b}\right)^{a+b}.$$

We remark that Sándor and Atanassov [14] used Lemmas 2.1–2.3 to prove their bounds.

Next, we mention some results that yield improvements on Lemma 2.1. We will use them to prove our Theorem 1.1 in the next section.

Lemma 2.4. *([7], Theorem], [2], Remark 3]). Let $n \geqslant 2$, $x_1, \dots, x_n \geqslant 0$ and $\alpha_1, \dots, \alpha_n > 0$. Suppose that $\alpha_1 + \cdots + \alpha_n = 1$. Then we have*

$$\begin{aligned}
&\frac{1}{2\max(x_1, \cdots, x_n)} \sum_{1 \leqslant i \leqslant n} \alpha_i \left(x_i - \sum_{1 \leqslant j \leqslant n} \alpha_j x_j \right)^2 \\
\leqslant\ & \sum_{1 \leqslant i \leqslant n} \alpha_i x_i - \prod_{1 \leqslant i \leqslant n} x_i^{\alpha_i} \\
\leqslant\ & \frac{1}{2\min(x_1, \cdots, x_n)} \sum_{1 \leqslant i \leqslant n} \alpha_i \left(x_i - \sum_{1 \leqslant j \leqslant n} \alpha_j x_j \right)^2 .
\end{aligned}$$

Moreover, for $n = 2$ this becomes

$$\begin{aligned}
&\frac{1}{2\max(x_1, x_2)} \left(\alpha_1 \left(x_1 - \alpha_1 x_1 - \alpha_2 x_2\right)^2 + \alpha_2 \left(x_2 - \alpha_1 x_1 - \alpha_2 x_2\right)^2 \right) \\
\leqslant\ & \left(\alpha_1 x_1 + \alpha_2 x_2 - x_1^{\alpha_1} x_2^{\alpha_2}\right) \\
\leqslant\ & \frac{1}{2\min(x_1, x_2)} \left(\alpha_1 \left(x_1 - \alpha_1 x_1 - \alpha_2 x_2\right)^2 + \alpha_2 \left(x_2 - \alpha_1 x_1 - \alpha_2 x_2\right)^2 \right).
\end{aligned}$$

Lemma 2.5. *([5], Theorem]). Let $n \geqslant 2$, $x_1, \dots, x_n \geqslant 0$ and $\alpha_1, \dots, \alpha_n > 0$. Suppose that $\alpha_1 + \cdots + \alpha_n = 1$. Then we have*

$$\frac{1}{2\max(x_1, \cdots, x_n)} \sum_{1 \leqslant i \leqslant n} \alpha_i \left(x_i - \prod_{1 \leqslant j \leqslant n} x_j^{\alpha_j} \right)^2 \leqslant \sum_{1 \leqslant i \leqslant n} \alpha_i x_i - \prod_{1 \leqslant i \leqslant n} x_i^{\alpha_i}.$$

Moreover, for $n = 2$ this becomes

$$\frac{1}{2\max(x_1, x_2)} \left(\alpha_1 \left(x_1 - x_1^{\alpha_1} x_2^{\alpha_2}\right)^2 + \alpha_2 \left(x_2 - x_1^{\alpha_1} x_2^{\alpha_2}\right)^2 \right) \leqslant \left(\alpha_1 x_1 + \alpha_2 x_2 - x_1^{\alpha_1} x_2^{\alpha_2}\right).$$

Lemma 2.6. *([2], Theorem 1]). Let $n \geqslant 2$, $x_1, \dots, x_n \geqslant 0$ and $\alpha_1, \dots, \alpha_n > 0$. Suppose that $\alpha_1 + \cdots + \alpha_n = 1$. Then we have*

$$\sum_{1 \leqslant i \leqslant n} \alpha_i \left(x_i^{\frac{1}{2}} - \sum_{1 \leqslant j \leqslant n} \alpha_j x_j^{\frac{1}{2}} \right)^2 \leqslant \sum_{1 \leqslant i \leqslant n} \alpha_i x_i - \prod_{1 \leqslant i \leqslant n} x_i^{\alpha_i}.$$

Moreover, for $n = 2$ this becomes

$$\alpha_1 \left(x_1^{\frac{1}{2}} - \alpha_1 x_1^{\frac{1}{2}} - \alpha_2 x_2^{\frac{1}{2}} \right)^2 + \alpha_2 \left(x_2^{\frac{1}{2}} - \alpha_1 x_1^{\frac{1}{2}} - \alpha_2 x_2^{\frac{1}{2}} \right)^2 \leqslant \left(\alpha_1 x_1 + \alpha_2 x_2 - x_1^{\alpha_1} x_2^{\alpha_2}\right).$$

Lemma 2.7. *([4], Theorem 2.2]). Let $n \geqslant 2$, $x_1, \dots, x_n \geqslant 0$ and $\alpha_1, \dots, \alpha_n > 0$. Suppose that $\alpha_1 + \cdots + \alpha_n = 1$. Then we have*

$$\begin{aligned}
&\frac{1}{1 - \min(\alpha_1, \cdots, \alpha_n)} \sum_{1 \leqslant i \leqslant n} \alpha_i \left(x_i^{\frac{1}{2}} - \sum_{1 \leqslant j \leqslant n} \alpha_j x_j^{\frac{1}{2}} \right)^2 \\
\leqslant\ & \sum_{1 \leqslant i \leqslant n} \alpha_i x_i - \prod_{1 \leqslant i \leqslant n} x_i^{\alpha_i} \\
\leqslant\ & \frac{1}{\min(\alpha_1, \cdots, \alpha_n)} \sum_{1 \leqslant i \leqslant n} \alpha_i \left(x_i^{\frac{1}{2}} - \sum_{1 \leqslant j \leqslant n} \alpha_j x_j^{\frac{1}{2}} \right)^2 .
\end{aligned}$$

Moreover, for $n = 2$ this becomes

$$\begin{aligned}
&\frac{1}{1 - \min(\alpha_1, \alpha_2)} \left(\alpha_1 \left(x_1^{\frac{1}{2}} - \alpha_1 x_1^{\frac{1}{2}} - \alpha_2 x_2^{\frac{1}{2}} \right)^2 + \alpha_2 \left(x_2^{\frac{1}{2}} - \alpha_1 x_1^{\frac{1}{2}} - \alpha_2 x_2^{\frac{1}{2}} \right)^2 \right) \\
\leqslant\ & \left(\alpha_1 x_1 + \alpha_2 x_2 - x_1^{\alpha_1} x_2^{\alpha_2}\right)
\end{aligned}$$

$$\leqslant \frac{1}{\min(\alpha_1,\alpha_2)}\left(\alpha_1\left(x_1^{\frac{1}{2}}-\alpha_1 x_1^{\frac{1}{2}}-\alpha_2 x_2^{\frac{1}{2}}\right)^2+\alpha_2\left(x_2^{\frac{1}{2}}-\alpha_1 x_1^{\frac{1}{2}}-\alpha_2 x_2^{\frac{1}{2}}\right)^2\right).$$

Lemma 2.8. *([3], Corollary 2.3]). Let $n\geqslant 2$, $x_1,\ldots,x_n\geqslant 0$ and $\alpha_1,\ldots,\alpha_n>0$. Suppose that $\alpha_1+\cdots+\alpha_n=1$. Then we have*

$$\begin{aligned}&n\min(\alpha_1,\cdots,\alpha_n)\left(\frac{1}{n}\sum_{1\leqslant i\leqslant n}x_i-\prod_{1\leqslant i\leqslant n}x_i^{\frac{1}{n}}\right)\\ &\leqslant \sum_{1\leqslant i\leqslant n}\alpha_i x_i-\prod_{1\leqslant i\leqslant n}x_i^{\alpha_i}\\ &\leqslant n\max(\alpha_1,\cdots,\alpha_n)\left(\frac{1}{n}\sum_{1\leqslant i\leqslant n}x_i-\prod_{1\leqslant i\leqslant n}x_i^{\frac{1}{n}}\right).\end{aligned}$$

Moreover, for $n=2$ this becomes

$$2\min(\alpha_1,\alpha_2)\left(\frac{1}{2}(x_1+x_2)-x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}\right)\leqslant(\alpha_1x_1+\alpha_2x_2-x_1^{\alpha_1}x_2^{\alpha_2})\leqslant 2\max(\alpha_1,\alpha_2)\left(\frac{1}{2}(x_1+x_2)-x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}\right).$$

Note that the left–hand side is just [[12], Proposition 5.1].

Lemma 2.9. *([1], Theorem 1]). Let $n\geqslant 2$, $x_1,\ldots,x_n\geqslant 0$ and $\alpha_1,\ldots,\alpha_n>0$. Suppose that $\alpha_1+\cdots+\alpha_n=1$. Then we have*

$$\exp\left(2-2\frac{\sum_{1\leqslant i\leqslant n}\alpha_i x_i^{\frac{1}{2}}}{\left(\sum_{1\leqslant i\leqslant n}\alpha_i x_i\right)^{\frac{1}{2}}}\right)\prod_{1\leqslant i\leqslant n}x_i^{\alpha_i}\leqslant\sum_{1\leqslant i\leqslant n}\alpha_i x_i.$$

Moreover, for $n=2$ this becomes

$$\exp\left(2-2\frac{\alpha_1x_1^{\frac{1}{2}}+\alpha_2x_2^{\frac{1}{2}}}{(\alpha_1x_1+\alpha_2x_2)^{\frac{1}{2}}}\right)x_1^{\alpha_1}x_2^{\alpha_2}\leqslant\alpha_1x_1+\alpha_2x_2.$$

Lemma 2.10. *([8], Proposition 2.7]). Let $n\geqslant 2$, $0\leqslant x_1\leqslant\ldots\leqslant x_j\leqslant\ldots\leqslant x_k\leqslant\ldots\leqslant x_n$ and $\alpha_1,\ldots,\alpha_n>0$. Suppose that $\alpha_1+\cdots+\alpha_n=1$. Then we have*

$$\frac{3(\alpha_1+\cdots+\alpha_j)(\alpha_k+\cdots+\alpha_n)(x_k-x_j)^2}{(4(\alpha_1+\cdots+\alpha_j)+2(\alpha_k+\cdots+\alpha_n))x_k+(4(\alpha_k+\cdots+\alpha_n)+2(\alpha_1+\cdots+\alpha_j))x_j}\leqslant\sum_{1\leqslant i\leqslant n}\alpha_ix_i-\prod_{1\leqslant i\leqslant n}x_i^{\alpha_i}.$$

Moreover, for $n=2$, $j=1$ and $k=2$, this becomes

$$\frac{3\alpha_1\alpha_2(x_2-x_1)^2}{(4\alpha_1+2\alpha_2)x_2+(4\alpha_2+2\alpha_1)x_1}\leqslant(\alpha_1x_1+\alpha_2x_2-x_1^{\alpha_1}x_2^{\alpha_2}).$$

3. Proof of Theorem 1.1

3.1. **Proof of (A1) and (A2).** By Lemma 2.4 we know that for $x_1,x_2\geqslant 0$, $\alpha_1,\alpha_2>0$ and $\alpha_1+\alpha_2=1$, we have

$$x_1^{\alpha_1}x_2^{\alpha_2}\leqslant\alpha_1x_1+\alpha_2x_2-\frac{1}{2\max(x_1,x_2)}\left(\alpha_1(x_1-\alpha_1x_1-\alpha_2x_2)^2+\alpha_2(x_2-\alpha_1x_1-\alpha_2x_2)^2\right).\tag{7}$$

For **(A1)**, put $x_1=a$, $x_2=b$, $\alpha_1=\frac{a}{a+b}$ and $\alpha_2=\frac{b}{a+b}$, (7) becomes

$$\begin{aligned}&a^{\frac{a}{a+b}}b^{\frac{b}{a+b}}\\ &\leqslant\frac{a}{a+b}a+\frac{b}{a+b}b-\frac{1}{2\max(a,b)}\left(\frac{a}{a+b}\left(a-\frac{a}{a+b}a-\frac{b}{a+b}b\right)^2+\frac{b}{a+b}\left(b-\frac{a}{a+b}a-\frac{b}{a+b}b\right)^2\right)\\ &=\frac{a^2+b^2}{a+b}-\frac{1}{2\max(a,b)}\left(\frac{a}{a+b}\left(\frac{a(a+b)-a^2-b^2}{a+b}\right)^2+\frac{b}{a+b}\left(\frac{b(a+b)-a^2-b^2}{a+b}\right)^2\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{a^2+b^2}{a+b} - \frac{1}{2\max(a,b)}\left(\frac{a}{a+b}\left(\frac{ab-b^2}{a+b}\right)^2 + \frac{b}{a+b}\left(\frac{ab-a^2}{a+b}\right)^2\right)\\
&= \frac{a^2+b^2}{a+b} - \frac{1}{2\max(a,b)}\left(\frac{a(ab-b^2)^2+b(ab-a^2)^2}{(a+b)^3}\right)\\
&= \frac{a^2+b^2}{a+b} - \frac{1}{2\max(a,b)}\left(\frac{ab(a-b)^2}{(a+b)^2}\right). \qquad (8)
\end{aligned}
$$

Let $a=\varphi(n)$ and $b=\psi(n)$. By (2) we have $a\leqslant b$, hence $\max(a,b)=b$. Putting this into (8), we have

$$
\begin{aligned}
a^{\frac{a}{a+b}}b^{\frac{b}{a+b}} &\leqslant \frac{a^2+b^2}{a+b} - \frac{1}{2\max(a,b)}\left(\frac{ab(a-b)^2}{(a+b)^2}\right)\\
&= \frac{a^2+b^2}{a+b} - \frac{1}{2b}\left(\frac{ab(a-b)^2}{(a+b)^2}\right)\\
&= \frac{(a^2+b^2)(a+b)-\frac{1}{2}a(a-b)^2}{(a+b)^2}\\
&= \frac{a^3+4a^2b+ab^2+2b^3}{2(a+b)^2}, \qquad (9)
\end{aligned}
$$

$$
a^ab^b \leqslant \left(\frac{a^3+4a^2b+ab^2+2b^3}{2(a+b)^2}\right)^{a+b}. \qquad (10)
$$

Now **(A1)** is proved. For **(A2)**, put $x_1=b=\psi(n)$, $x_2=a=\varphi(n)$, $\alpha_1=\frac{a}{a+b}$ and $\alpha_2=\frac{b}{a+b}$, (7) becomes

$$
\begin{aligned}
&b^{\frac{a}{a+b}}a^{\frac{b}{a+b}}\\
&\leqslant \frac{a}{a+b}b+\frac{b}{a+b}a-\frac{1}{2b}\left(\frac{a}{a+b}\left(b-\frac{a}{a+b}b-\frac{b}{a+b}a\right)^2+\frac{b}{a+b}\left(a-\frac{a}{a+b}b-\frac{b}{a+b}a\right)^2\right)\\
&= \frac{2ab}{a+b}-\frac{1}{2b}\left(\frac{a}{a+b}\left(\frac{b(a+b)-2ab}{a+b}\right)^2+\frac{b}{a+b}\left(\frac{a(a+b)-2ab}{a+b}\right)^2\right)\\
&= \frac{2ab}{a+b}-\frac{1}{2b}\left(\frac{a}{a+b}\left(\frac{b^2-ab}{a+b}\right)^2+\frac{b}{a+b}\left(\frac{a^2-ab}{a+b}\right)^2\right)\\
&= \frac{2ab}{a+b}-\frac{1}{2b}\left(\frac{ab(a-b)^2}{(a+b)^2}\right)\\
&= \frac{2ab(a+b)-\frac{1}{2}a(a-b)^2}{(a+b)^2}\\
&= \frac{-a^3+6a^2b+3ab^2}{2(a+b)^2}, \qquad (11)
\end{aligned}
$$

$$
b^aa^b \leqslant \left(\frac{-a^3+6a^2b+3ab^2}{2(a+b)^2}\right)^{a+b}. \qquad (12)
$$

Now **(A2)** is proved.

3.2. **Proof of (B1) and (B2).** By Lemma 2.5 we know that for $x_1,x_2\geqslant 0$, $\alpha_1,\alpha_2>0$ and $\alpha_1+\alpha_2=1$, we have

$$
x_1^{\alpha_1}x_2^{\alpha_2}\leqslant \alpha_1x_1+\alpha_2x_2-\frac{1}{2\max(x_1,x_2)}\left(\alpha_1\left(x_1-x_1^{\alpha_1}x_2^{\alpha_2}\right)^2+\alpha_2\left(x_2-x_1^{\alpha_1}x_2^{\alpha_2}\right)^2\right). \qquad (13)
$$

For **(B1)**, put $x_1=a=\varphi(n)$, $x_2=b=\psi(n)$, $\alpha_1=\frac{a}{a+b}$ and $\alpha_2=\frac{b}{a+b}$, (13) becomes

$$
a^{\frac{a}{a+b}}b^{\frac{b}{a+b}}\leqslant \frac{a}{a+b}a+\frac{b}{a+b}b-\frac{1}{2b}\left(\frac{a}{a+b}\left(a-a^{\frac{a}{a+b}}-b^{\frac{b}{a+b}}\right)^2+\frac{b}{a+b}\left(b-a^{\frac{a}{a+b}}-b^{\frac{b}{a+b}}\right)^2\right)
$$

$$= \frac{a^2+b^2}{a+b} - \frac{a\left(a - a^{\frac{a}{a+b}} - b^{\frac{b}{a+b}}\right)^2 + b\left(b - a^{\frac{a}{a+b}} - b^{\frac{b}{a+b}}\right)^2}{2b(a+b)}, \tag{14}$$

$$a^a b^b \leqslant \left(\frac{2a^2b + 2b^3 - a\left(a - a^{\frac{a}{a+b}} - b^{\frac{b}{a+b}}\right)^2 - b\left(b - a^{\frac{a}{a+b}} - b^{\frac{b}{a+b}}\right)^2}{2b(a+b)}\right)^{a+b}. \tag{15}$$

Now **(B1)** is proved. For **(B2)**, put $x_1 = b = \psi(n)$, $x_2 = a = \varphi(n)$, $\alpha_1 = \frac{a}{a+b}$ and $\alpha_2 = \frac{b}{a+b}$, (13) becomes

$$b^{\frac{a}{a+b}} a^{\frac{b}{a+b}} \leqslant \frac{a}{a+b} b + \frac{b}{a+b} a - \frac{1}{2b}\left(\frac{a}{a+b}\left(b - b^{\frac{a}{a+b}} - a^{\frac{b}{a+b}}\right)^2 + \frac{b}{a+b}\left(a - b^{\frac{a}{a+b}} - a^{\frac{b}{a+b}}\right)^2\right)$$

$$= \frac{2ab}{a+b} - \frac{a\left(b - b^{\frac{a}{a+b}} - a^{\frac{b}{a+b}}\right)^2 + b\left(a - b^{\frac{a}{a+b}} - a^{\frac{b}{a+b}}\right)^2}{2b(a+b)}, \tag{16}$$

$$b^a a^b \leqslant \left(\frac{4ab^2 - a\left(b - b^{\frac{a}{a+b}} - a^{\frac{b}{a+b}}\right)^2 - b\left(a - b^{\frac{a}{a+b}} - a^{\frac{b}{a+b}}\right)^2}{2b(a+b)}\right)^{a+b}. \tag{17}$$

Now **(B2)** is proved.

3.3. **Proof of (C1) and (C2).** By Lemma 2.6 we know that for $x_1, x_2 \geqslant 0$, $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$, we have

$$x_1^{\alpha_1} x_2^{\alpha_2} \leqslant \alpha_1 x_1 + \alpha_2 x_2 - \left(\alpha_1\left(x_1^{\frac{1}{2}} - \alpha_1 x_1^{\frac{1}{2}} - \alpha_2 x_2^{\frac{1}{2}}\right)^2 + \alpha_2\left(x_2^{\frac{1}{2}} - \alpha_1 x_1^{\frac{1}{2}} - \alpha_2 x_2^{\frac{1}{2}}\right)^2\right). \tag{18}$$

For **(C1)**, put $x_1 = a = \varphi(n)$, $x_2 = b = \psi(n)$, $\alpha_1 = \frac{a}{a+b}$ and $\alpha_2 = \frac{b}{a+b}$, (18) becomes

$$a^{\frac{a}{a+b}} b^{\frac{b}{a+b}} \leqslant \frac{a}{a+b} a + \frac{b}{a+b} b - \left(\frac{a}{a+b}\left(a^{\frac{1}{2}} - \frac{a}{a+b} a^{\frac{1}{2}} - \frac{b}{a+b} b^{\frac{1}{2}}\right)^2 + \frac{b}{a+b}\left(b^{\frac{1}{2}} - \frac{a}{a+b} a^{\frac{1}{2}} - \frac{b}{a+b} b^{\frac{1}{2}}\right)^2\right)$$

$$= \frac{a^2+b^2}{a+b} - \frac{ab\left(a + b - 2a^{\frac{1}{2}} b^{\frac{1}{2}}\right)}{(a+b)^2}$$

$$= \left(\frac{a^{\frac{3}{2}} + b^{\frac{3}{2}}}{a+b}\right)^2, \tag{19}$$

$$a^a b^b \leqslant \left(\frac{a^{\frac{3}{2}} + b^{\frac{3}{2}}}{a+b}\right)^{2(a+b)}. \tag{20}$$

Now **(C1)** is proved. For **(C2)**, put $x_1 = b = \psi(n)$, $x_2 = a = \varphi(n)$, $\alpha_1 = \frac{a}{a+b}$ and $\alpha_2 = \frac{b}{a+b}$, (18) becomes

$$b^{\frac{a}{a+b}} a^{\frac{b}{a+b}} \leqslant \frac{a}{a+b} b + \frac{b}{a+b} a - \left(\frac{a}{a+b}\left(b^{\frac{1}{2}} - \frac{a}{a+b} b^{\frac{1}{2}} - \frac{b}{a+b} a^{\frac{1}{2}}\right)^2 + \frac{b}{a+b}\left(a^{\frac{1}{2}} - \frac{a}{a+b} b^{\frac{1}{2}} - \frac{b}{a+b} a^{\frac{1}{2}}\right)^2\right)$$

$$= \frac{2ab}{a+b} - \frac{ab\left(a + b - 2a^{\frac{1}{2}} b^{\frac{1}{2}}\right)}{(a+b)^2}$$

$$= \left(\frac{a^{\frac{1}{2}} b^{\frac{1}{2}} \left(a^{\frac{1}{2}} + b^{\frac{1}{2}} \right)}{a+b} \right)^2, \tag{21}$$

$$b^a a^b \leqslant \left(\frac{a^{\frac{1}{2}} b^{\frac{1}{2}} \left(a^{\frac{1}{2}} + b^{\frac{1}{2}} \right)}{a+b} \right)^{2(a+b)}. \tag{22}$$

Now **(C2)** is proved.

3.4. **Proof of (D1) and (D2).** By Lemma 2.7 we know that for $x_1, x_2 \geqslant 0$, $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$, we have

$$x_1^{\alpha_1} x_2^{\alpha_2} \leqslant \alpha_1 x_1 + \alpha_2 x_2 - \frac{1}{1 - \min(\alpha_1, \alpha_2)} \left(\alpha_1 \left(x_1^{\frac{1}{2}} - \alpha_1 x_1^{\frac{1}{2}} - \alpha_2 x_2^{\frac{1}{2}} \right)^2 + \alpha_2 \left(x_2^{\frac{1}{2}} - \alpha_1 x_1^{\frac{1}{2}} - \alpha_2 x_2^{\frac{1}{2}} \right)^2 \right). \tag{23}$$

For **(D1)**, we put $x_1 = a = \varphi(n)$, $x_2 = b = \psi(n)$, $\alpha_1 = \frac{a}{a+b}$ and $\alpha_2 = \frac{b}{a+b}$. Since $a + b > 0$ and $a \leqslant b$, we have

$$\alpha_1 = \frac{a}{a+b} \leqslant \frac{b}{a+b} = \alpha_2,$$

hence $1 - \min(\alpha_1, \alpha_2) = \max(\alpha_1, \alpha_2) = \alpha_2 = \frac{b}{a+b}$. Then (23) becomes

$$\begin{aligned} a^{\frac{a}{a+b}} b^{\frac{b}{a+b}} &\leqslant \frac{a}{a+b} a + \frac{b}{a+b} b - \frac{1}{\frac{b}{a+b}} \left(\frac{a}{a+b} \left(a^{\frac{1}{2}} - \frac{a}{a+b} a^{\frac{1}{2}} - \frac{b}{a+b} b^{\frac{1}{2}} \right)^2 \right. \\ &\qquad \left. + \frac{b}{a+b} \left(b^{\frac{1}{2}} - \frac{a}{a+b} a^{\frac{1}{2}} - \frac{b}{a+b} b^{\frac{1}{2}} \right)^2 \right) \\ &= \frac{a^2 + b^2}{a+b} - \frac{a+b}{b} \cdot \frac{ab \left(a + b - 2a^{\frac{1}{2}} b^{\frac{1}{2}} \right)}{(a+b)^2} \\ &= \frac{a^2 + b^2}{a+b} - \frac{a \left(a + b - 2a^{\frac{1}{2}} b^{\frac{1}{2}} \right)}{a+b} \\ &= \frac{2a^{\frac{3}{2}} b^{\frac{1}{2}} - ab + b^2}{a+b}, \end{aligned} \tag{24}$$

$$a^a b^b \leqslant \left(\frac{2a^{\frac{3}{2}} b^{\frac{1}{2}} - ab + b^2}{a+b} \right)^{a+b}. \tag{25}$$

Now **(D1)** is proved. For **(D2)**, put $x_1 = b = \psi(n)$, $x_2 = a = \varphi(n)$, $\alpha_1 = \frac{a}{a+b}$ and $\alpha_2 = \frac{b}{a+b}$, (23) becomes

$$\begin{aligned} b^{\frac{a}{a+b}} a^{\frac{b}{a+b}} &\leqslant \frac{a}{a+b} b + \frac{b}{a+b} a - \frac{1}{\frac{b}{a+b}} \left(\frac{a}{a+b} \left(b^{\frac{1}{2}} - \frac{a}{a+b} b^{\frac{1}{2}} - \frac{b}{a+b} a^{\frac{1}{2}} \right)^2 \right. \\ &\qquad \left. + \frac{b}{a+b} \left(a^{\frac{1}{2}} - \frac{a}{a+b} b^{\frac{1}{2}} - \frac{b}{a+b} a^{\frac{1}{2}} \right)^2 \right) \\ &= \frac{2ab}{a+b} - \frac{a \left(a + b - 2a^{\frac{1}{2}} b^{\frac{1}{2}} \right)}{a+b} \\ &= \frac{2a^{\frac{3}{2}} b^{\frac{1}{2}} + ab - a^2}{a+b}, \end{aligned} \tag{26}$$

$$b^a a^b \leqslant \left(\frac{2a^{\frac{3}{2}} b^{\frac{1}{2}} + ab - a^2}{a+b} \right)^{a+b}. \tag{27}$$

Now **(D2)** is proved.

We note that Lemma 2.8 and Lemma 2.7 actually yield the same result here. By Lemma 2.8 we know that for $x_1, x_2 \geqslant 0$, $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$, we have

$$x_1^{\alpha_1} x_2^{\alpha_2} \leqslant \alpha_1 x_1 + \alpha_2 x_2 - \left(2 \min(\alpha_1, \alpha_2) \left(\frac{1}{2}(x_1 + x_2) - x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}\right)\right). \tag{28}$$

For **(D1)**, we put $x_1 = a = \varphi(n)$, $x_2 = b = \psi(n)$, $\alpha_1 = \frac{a}{a+b}$ and $\alpha_2 = \frac{b}{a+b}$. Again, we have $\min(\alpha_1, \alpha_2) = \alpha_1 = \frac{a}{a+b}$. Then (28) becomes

$$\begin{aligned} a^{\frac{a}{a+b}} b^{\frac{b}{a+b}} &\leqslant \frac{a}{a+b} a + \frac{b}{a+b} b - 2\frac{a}{a+b}\left(\frac{1}{2}a + \frac{1}{2}b - a^{\frac{1}{2}} b^{\frac{1}{2}}\right) \\ &= \frac{a^2 + b^2 - 2a\left(\frac{1}{2}a + \frac{1}{2}b - a^{\frac{1}{2}} b^{\frac{1}{2}}\right)}{a+b} \\ &= \frac{2a^{\frac{3}{2}} b^{\frac{1}{2}} - ab + b^2}{a+b}, \qquad (29) \\ a^a b^b &\leqslant \left(\frac{2a^{\frac{3}{2}} b^{\frac{1}{2}} - ab + b^2}{a+b}\right)^{a+b}. \qquad (30) \end{aligned}$$

Now **(D1)** is proved. For **(D2)**, put $x_1 = b = \psi(n)$, $x_2 = a = \varphi(n)$, $\alpha_1 = \frac{a}{a+b}$ and $\alpha_2 = \frac{b}{a+b}$, (28) becomes

$$\begin{aligned} a^{\frac{a}{a+b}} b^{\frac{b}{a+b}} &\leqslant \frac{a}{a+b} b + \frac{b}{a+b} a - 2\frac{a}{a+b}\left(\frac{1}{2}b + \frac{1}{2}a - b^{\frac{1}{2}} a^{\frac{1}{2}}\right) \\ &= \frac{2ab - 2a\left(\frac{1}{2}a + \frac{1}{2}b - a^{\frac{1}{2}} b^{\frac{1}{2}}\right)}{a+b} \\ &= \frac{2a^{\frac{3}{2}} b^{\frac{1}{2}} + ab - a^2}{a+b}, \qquad (31) \\ b^a a^b &\leqslant \left(\frac{2a^{\frac{3}{2}} b^{\frac{1}{2}} + ab - a^2}{a+b}\right)^{a+b}. \qquad (32) \end{aligned}$$

Now **(D2)** is proved.

3.5. **Proof of (E1) and (E2).** By Lemma 2.9 we know that for $x_1, x_2 \geqslant 0$, $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$, we have

$$x_1^{\alpha_1} x_2^{\alpha_2} \leqslant \frac{\alpha_1 x_1 + \alpha_2 x_2}{\exp\left(2 - 2\frac{\alpha_1 x_1^{\frac{1}{2}} + \alpha_2 x_2^{\frac{1}{2}}}{(\alpha_1 x_1 + \alpha_2 x_2)^{\frac{1}{2}}}\right)}. \tag{33}$$

For **(E1)**, put $x_1 = a = \varphi(n)$, $x_2 = b = \psi(n)$, $\alpha_1 = \frac{a}{a+b}$ and $\alpha_2 = \frac{b}{a+b}$, (33) becomes

$$\begin{aligned} a^{\frac{a}{a+b}} b^{\frac{b}{a+b}} &\leqslant \frac{\frac{a}{a+b} a + \frac{b}{a+b} b}{\exp\left(2 - 2\frac{\frac{a}{a+b} a^{\frac{1}{2}} + \frac{b}{a+b} b^{\frac{1}{2}}}{(\frac{a}{a+b} a + \frac{b}{a+b} b)^{\frac{1}{2}}}\right)} \\ &= \frac{a^2 + b^2}{(a+b)\exp\left(2 - 2\frac{a^{\frac{3}{2}} + b^{\frac{3}{2}}}{((a^2+b^2)(a+b))^{\frac{1}{2}}}\right)}, \qquad (34) \\ a^a b^b &\leqslant \left(\frac{a^2 + b^2}{(a+b)\exp\left(2 - 2\frac{a^{\frac{3}{2}} + b^{\frac{3}{2}}}{((a^2+b^2)(a+b))^{\frac{1}{2}}}\right)}\right)^{a+b}. \qquad (35) \end{aligned}$$

Now **(E1)** is proved. For **(E2)**, put $x_1 = b = \psi(n)$, $x_2 = a = \varphi(n)$, $\alpha_1 = \frac{a}{a+b}$ and $\alpha_2 = \frac{b}{a+b}$, (33) becomes

$$b^{\frac{a}{a+b}} a^{\frac{b}{a+b}} \leqslant \frac{\frac{a}{a+b} b + \frac{b}{a+b} a}{\exp\left(2 - 2\frac{\frac{a}{a+b} b^{\frac{1}{2}} + \frac{b}{a+b} a^{\frac{1}{2}}}{\left(\frac{a}{a+b} b + \frac{b}{a+b} a\right)^{\frac{1}{2}}}\right)}$$

$$= \frac{2ab}{(a+b)\exp\left(2 - 2\frac{a^{\frac{1}{2}} b^{\frac{1}{2}}\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)}{(2ab(a+b))^{\frac{1}{2}}}\right)}, \tag{36}$$

$$b^a a^b \leqslant \left(\frac{2ab}{(a+b)\exp\left(2 - 2\frac{a^{\frac{1}{2}} b^{\frac{1}{2}}\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)}{(2ab(a+b))^{\frac{1}{2}}}\right)}\right)^{a+b}. \tag{37}$$

Now **(E2)** is proved.

3.6. **Proof of (F1) and (F2).** By Lemma 2.10 we know that for $0 \leqslant x_1 \leqslant x_2$, $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$, we have

$$x_1^{\alpha_1} x_2^{\alpha_2} \leqslant \alpha_1 x_1 + \alpha_2 x_2 - \frac{3\alpha_1\alpha_2(x_2 - x_1)^2}{(4\alpha_1 + 2\alpha_2)x_2 + (4\alpha_2 + 2\alpha_1)x_1}. \tag{38}$$

For **(F1)**, we put $x_1 = a = \varphi(n)$, $x_2 = b = \psi(n)$, $\alpha_1 = \frac{a}{a+b}$ and $\alpha_2 = \frac{b}{a+b}$. Since $a \leqslant b$, we can write (38) as

$$a^{\frac{a}{a+b}} b^{\frac{b}{a+b}} \leqslant \frac{a}{a+b} a + \frac{b}{a+b} b - \frac{3\frac{a}{a+b} \cdot \frac{b}{a+b}(b-a)^2}{(4\frac{a}{a+b} + 2\frac{b}{a+b})b + (4\frac{b}{a+b} + 2\frac{a}{a+b})a}$$

$$= \frac{a^2 + b^2}{a+b} - \frac{\frac{3ab(b-a)^2}{(a+b)^2}}{\frac{2a^2+8ab+2b^2}{a+b}}$$

$$= \frac{a^2 + b^2 - \frac{3ab(b-a)^2}{2a^2+8ab+2b^2}}{a+b}, \tag{39}$$

$$a^a b^b \leqslant \left(\frac{a^2 + b^2 - \frac{3ab(b-a)^2}{2a^2+8ab+2b^2}}{a+b}\right)^{a+b}. \tag{40}$$

Now **(F1)** is proved. For **(F2)**, we put $x_1 = a = \varphi(n)$, $x_2 = b = \psi(n)$, $\alpha_1 = \frac{b}{a+b}$ and $\alpha_2 = \frac{a}{a+b}$. Since $a \leqslant b$, (38) becomes

$$a^{\frac{b}{a+b}} b^{\frac{a}{a+b}} \leqslant \frac{b}{a+b} a + \frac{a}{a+b} b - \frac{3\frac{b}{a+b} \cdot \frac{a}{a+b}(b-a)^2}{(4\frac{b}{a+b} + 2\frac{a}{a+b})b + (4\frac{a}{a+b} + 2\frac{b}{a+b})a}$$

$$= \frac{2ab}{a+b} - \frac{\frac{3ab(b-a)^2}{(a+b)^2}}{\frac{4a^2+4ab+4b^2}{a+b}}$$

$$= \frac{2ab - \frac{3ab(b-a)^2}{4(a^2+ab+b^2)}}{a+b}, \tag{41}$$

$$a^b b^a \leqslant \left(\frac{2ab - \frac{3ab(b-a)^2}{4(a^2+ab+b^2)}}{a+b}\right)^{a+b}. \tag{42}$$

Now **(F2)** is proved.

4. Appendix: an application of Karamata's inequality

By the definition of $\sigma(n)$, we can easily show that $\sigma(n) \geqslant n+1$. By [[6], Lemma], we also know that

$$\varphi(n)+\psi(n) \geqslant 2n. \tag{43}$$

Thus,

$$\varphi(n)+\psi(n)+\sigma(n) \geqslant 3n+1=(n-1)+2(n+1). \tag{44}$$

In 2023, Dimitrov [[9], Theorem 1] proved the quadratic case of (44):

$$\varphi^2(n)+\psi^2(n)+\sigma^2(n) \geqslant (3n^2+2n+3)=(n-1)^2+2(n+1)^2, \tag{45}$$

and he [[11], Theorems 1 and 2] proved the cubic and quartic cases in 2024:

$$\varphi^3(n)+\psi^3(n)+\sigma^3(n) \geqslant (3n^3+3n^2+9n+1)=(n-1)^3+2(n+1)^3, \tag{46}$$

$$\varphi^4(n)+\psi^4(n)+\sigma^4(n) \geqslant (3n^4+4n^3+18n^2+4n+3)=(n-1)^4+2(n+1)^4. \tag{47}$$

By (44)–(47), one can naturally conjecture that for any integer $k>0$, we have

$$\varphi^k(n)+\psi^k(n)+\sigma^k(n) \geqslant (n-1)^k+2(n+1)^k. \tag{48}$$

In 2024, user EthanWYX2009 on AoPS gave a simple but amazing proof of (48). His proof is much shorter than Dimitrov's proof of cases $k \leqslant 4$. In this appendix, we shall rewrite his remarkable proof.

Theorem 4.1. *(EthanWYX2009). For any integer $k>0$, we have*

$$\varphi^k(n)+\psi^k(n)+\sigma^k(n) \geqslant (n-1)^k+2(n+1)^k.$$

Proof. We first make the definition of *Weakly Majorization* in an elementary manner.

Definition 4.2. A sequence $a_1,\ldots,a_n$ *weakly majorizes* a sequence $b_1,\ldots,b_n$ if and only if $a_1 \geqslant \cdots \geqslant a_n$, $b_1 \geqslant \cdots \geqslant b_n$ and

$$\begin{aligned} a_1 &\geqslant b_1, \\ a_1+a_2 &\geqslant b_1+b_2, \\ a_1+a_2+a_3 &\geqslant b_1+b_2+b_3, \\ &\vdots \\ a_1+\cdots+a_{n-1} &\geqslant b_1+\cdots+b_{n-1}, \\ a_1+\cdots+a_n &\geqslant b_1+\cdots+b_n. \end{aligned}$$

Moreover, if we also have

$$a_1+\cdots+a_n=b_1+\cdots+b_n,$$

then $a_1,\ldots,a_n$ *majorizes* $b_1,\ldots,b_n$.

By the definition, we know that

$$(\psi(n),\varphi(n)) \text{ weakly majorizes } (n+1,n-1).$$

Next we shall provide the famous Karamata's inequality, which plays a crucial role in the proof.

Lemma 4.3. *(Karamata's inequality). Let $f: I \to \mathbb{R}$ be an increasing function on an interval $I \subset \mathbb{R}$, and let $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ be two sequences of real numbers in I. Suppose that $a_1,\ldots,a_n$ weakly majorizes $b_1,\ldots,b_n$. Then*

$$f(a_1)+\cdots+f(a_n) \geqslant f(b_1)+\cdots+f(b_n).$$

Now, let $a_1=\psi(n)$, $a_2=\varphi(n)$, $b_1=n+1$, $b_2=n-1$ and $f(x)=x^k$. By Lemma 4.3, the bound $\sigma(n) \geqslant n+1$ and (43), the proof of Theorem 4.1 is completed. □

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