ON THE UPPER AND LOWER BOUND ORDERS OF ALMOST PRIME TRIPLES

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ABSTRACT. A Hardy-Littlewood triple is a 3-tuple of integers with the form $(n, n+2, n+6)$. In this paper, we study Hardy-Littlewood triples of the form (p, P_a, P_b) and improve the upper and lower bound orders of it, where p is a prime and P_r has at most r prime factors. Our new results generalize and improve the previous results.

CONTENTS

1. INTRODUCTION

Let x be a sufficiently large integer, N be a sufficiently large even integer, p be a prime, and let P_r denote an integer with at most r prime factors counted with multiplicity. For each $N \geqslant 4$ and $r \geqslant 2$, we define

$$\pi_{1,r}(x) := |\{p : p \leqslant x, p+2 = P_r\}| \tag{1}$$

and

$$D_{1,r}(N) := |\{p : p \leqslant N, N-p = P_r\}|. \tag{2}$$

In 1966 Jingrun Chen [3] proved his remarkable Chen's theorem: let x be a sufficiently large integer and N be a sufficiently large even integer, then

$$\pi_{1,2}(x) \gg \frac{C_2 x}{(\log x)^2} \quad \text{and} \quad D_{1,2}(N) \gg \frac{C(N)N}{(\log N)^2}, \tag{3}$$

where

$$C_2 := 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right) \quad \text{and} \quad C(N) := \prod_{\substack{p\mid N\\ p>2}}\frac{p-1}{p-2}\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right) \tag{4}$$

and the detail was published in [4]. In 1990, Wu [14] generalized Chen's theorem and proved that

$$D_{1,3}(N) \gg \frac{C(N)N}{(\log N)^2}\log\log N \tag{5}$$

and

$$D_{1,r}(N) \gg \frac{C(N)N}{(\log N)^2}(\log\log N)^{r-2}. \tag{6}$$

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and Kan [10] proved the similar result in 1991. Kan [11] also proved the more generalized theorem in 1992:

$$D_{s,r}(N) \gg \frac{C(N)N}{(\log N)^2}(\log\log N)^{s+r-3}, \tag{7}$$

where $s \geqslant 1$,

$$D_{s,r}(N) := |\{P_s : P_s \leqslant N, N - P_s = P_r\}|. \tag{8}$$

Clearly their methods can be modified to get a similar lower bound order on the twin prime version. For this, we refer the interested readers to [9].

Now we focus on the Hardy-Littlewood triples $(n, n+2, n+6)$ with almost-prime values. In fact, if we define

$$\pi_{1,a,b}(x) := |\{p : p \leqslant x, p+2 = P_a, p+6 = P_b\}| \tag{9}$$

and

$$D_{1,a,b}(N) := |\{p : p \leqslant N, N - p = P_a, p+6 = P_b\}|, \tag{10}$$

then a special case of Hardy-Littlewood conjecture states that $\pi_{1,1,1}(x)$ should be asymptotic to $\frac{C_3 x}{(\log x)^3}$, where

$$C_3 = \frac{9}{2}\prod_{p>3}\left(1 - \frac{3p-1}{(p-1)^3}\right) \approx 2.86259. \tag{11}$$

In 2015, Heath-Brown and Li [7] proved that $\pi_{1,2,76}(x) \gg \frac{C_3 x}{(\log x)^3}$ and Cai [2] improved this result to $\pi_{1,2,14}(x) \gg \frac{C_3 x}{(\log x)^3}$ by using a delicate sieve process later. Their results refined Chen's theorem. Like Wu's generalization of Chen's theorem, we may conjecture that $\pi_{1,3,r}(x)$ should be asymptotic to $\frac{C_3 x \log\log x}{(\log x)^3}$ for some large r. Very recently, Li and Liu [12] proved $\pi_{1,3,6}(x) \gg \frac{C_3 x}{(\log x)^3}$. They also got $\pi_{1,3,3}(x) \gg \frac{C_3 x}{(\log x)^3}$ by assuming GEH(0.99). In this paper, we improve their asymptotic estimates of $\pi_{1,3,r}(x)$ on the orders by fixing some small prime factors q and prove that:

Theorem 1.1. *For every integer $a \geqslant 2$ and $b \geqslant 14$, we have*

$$\pi_{1,a,b}(x) \gg \frac{C_3 x}{(\log x)^3}(\log\log x)^{a-2} \quad \text{and} \quad D_{1,a,b}(N) \gg \frac{N}{(\log N)^3}(\log\log N)^{a-2},$$

where $\pi_{1,a,b}(x)$ and $D_{1,a,b}(N)$ are defined above.

By similar arguments, we also obtain those theorems:

Theorem 1.2. *For every integer $a \geqslant 3$ and $b \geqslant 6$, we have*

$$\pi_{1,a,b}(x) \gg \frac{C_3 x}{(\log x)^3}(\log\log x)^{a-3} \quad \text{and} \quad D_{1,a,b}(N) \gg \frac{N}{(\log N)^3}(\log\log N)^{a-3}.$$

Theorem 1.3. *For every integer $a \geqslant 4$ and $b \geqslant 5$, we have*

$$\pi_{1,a,b}(x) \gg \frac{C_3 x}{(\log x)^3}(\log\log x)^{a-4} \quad \text{and} \quad D_{1,a,b}(N) \gg \frac{N}{(\log N)^3}(\log\log N)^{a-4}.$$

We also prove some conditional results:

Theorem 1.4. *For every integer $a \geqslant 2$ and $b \geqslant 4$, assuming GEH(0.99), we have*

$$\pi_{1,a,b}(x) \gg \frac{C_3 x}{(\log x)^3}(\log\log x)^{a-2} \quad \text{and} \quad D_{1,a,b}(N) \gg \frac{N}{(\log N)^3}(\log\log N)^{a-2}.$$

Theorem 1.5. *For every integer $a \geqslant 3$ and $b \geqslant 3$, assuming GEH(0.99), we have*

$$\pi_{1,a,b}(x) \gg \frac{C_3 x}{(\log x)^3}(\log\log x)^{a-3} \quad \text{and} \quad D_{1,a,b}(N) \gg \frac{N}{(\log N)^3}(\log\log N)^{a-3}.$$

While calculating, we find that it seems hard to improve our Theorems 1.1–1.5. (For example, you need to get an improvement about 45 percent on the sieve process to replace the condition $b \geqslant 14$ by $b \geqslant 13$ in our Theorem 1.)

In this paper, we only provide a detailed proof of $\pi_{1,3,14}(x) \gg \frac{C_3 x \log\log x}{(\log x)^3}$, which is a simple version of Theorem 1.1. The readers can modify our proof to get Theorems 1.1–1.5.

2. Preliminary lemmas

Let $\mathcal{A}$ denote a finite set of positive integers, $\mathcal{P}$ denote an infinite set of primes, q denote a prime number satisfies $q < x^{\varepsilon}$ and put

$$\mathcal{A}=\left\{\frac{p+2}{q}: 7<p \leqslant x, p \equiv-2(\bmod q),(p+6, P(z))=1\right\}, \quad \mathcal{P}=\{p:(p, q)=1\},$$

$$\mathcal{P}(r)=\{p: p \in \mathcal{P},(p, r)=1\}, \quad P(z)=\prod_{\substack{p \in \mathcal{P} \\ p<z}} p,$$

$$\mathcal{A}_d=\{a: a \in \mathcal{A}, a \equiv 0(\bmod d)\}, \quad S(\mathcal{A} ; \mathcal{P}, z)=\sum_{\substack{a \in \mathcal{A} \\ (a, P(z))=1}} 1,$$

$$\mathcal{M}_k=\left\{n: n=p_1 p_2 \cdots p_k, 13<n \leqslant x+6, x^{0.005} \leqslant p_1<\cdots<p_k, n \equiv 4(\bmod q)\right\},$$

$$\mathcal{A}^{(k)}=\left\{\frac{p+2}{q}: 7<p \leqslant x, p \equiv-2(\bmod q), p+6 \in \mathcal{M}_k\right\},$$

$$\mathcal{E}=\left\{q m p_1 p_2 p_3 p_4: q m p_1 p_2 p_3 p_4 \leqslant x+2,\left(\frac{x}{q}\right)^{\frac{1}{13}} \leqslant p_1<p_2<p_3<p_4<\left(\frac{x}{q}\right)^{\frac{1}{8.4}},\left(m, q p_1^{-1} P\left(p_2\right)\right)=1\right\},$$

$$\mathcal{B}=\{n-2: n \in \mathcal{E}\}, \quad \mathscr{W}=\left\{\left\{\frac{p+2}{q}, p+6\right\}: 7<p \leqslant x, p \equiv-2(\bmod q)\right\},$$

$$\mathscr{W}^{(1)}=\{\{n-2, n+4\}: n \in \mathcal{E}\}, \quad \mathscr{W}^{(2)}=\left\{\left\{n-6, \frac{n-4}{q}\right\}: n \in \mathcal{M}_k\right\}.$$

Lemma 2.1. *([2], Lemma 1], deduced from [7], Proposition 1). Let $\mathscr{W}$ be a finite subset of $\mathbb{N}^2$. Suppose that $z_1, z_2 \geqslant 2$ with $\log z_1 \asymp \log z_2$ and write $\mathbf{z}=\{z_1, z_2\}$. For $\mathbf{d}=\{d_1, d_2\}$ and $\mathbf{n}=\{n_1, n_2\}$, we write $\mathbf{d} \mid \mathbf{n}$ to mean that $d_1 \mid n_1$ and $d_2 \mid n_2$. Set*

$$\mathscr{W}_{\mathbf{d}}=\{\mathbf{n} \in \mathscr{W}: \mathbf{d} \mid \mathbf{n}\}, \quad S(\mathscr{W}, \mathbf{z})=\sum_{\substack{\{n_1, n_2\} \in \mathscr{W} \\ p \mid n_1 \Rightarrow p \geqslant z_1 \\ p \mid n_2 \Rightarrow p \geqslant z_2}} 1.$$

Suppose that

$$|\mathscr{W}_{\mathbf{d}}|=h(\mathbf{d}) X+R(\mathbf{d})$$

for some $X>0$ independent of $\mathbf{d}$ and some multiplicative function $h(\mathbf{d}) \in[0,1)$ such that $h(p, 1)+h(1, p)-1<h(p, p) \leqslant h(p, 1)+h(1, p)$ for all primes p, and

$$h(p, 1), h(1, p) \leqslant c p^{-1}, \quad h(p, p) \leqslant c p^{-2}$$

for some constant $c \geqslant 2$.

Let $h_1(d)=h(d, 1)$ and $h_2(d)=h(1, d)$. Suppose that

$$\prod_{w \leqslant p<z}\left(1-h_j(p)\right)^{-1} \leqslant \frac{\log z}{\log w}\left(1+\frac{L}{\log w}\right) \quad(j=1,2)$$

for $z \geqslant w \geqslant 2$ and some positive constant L. Then:

$$S(\mathscr{W}, \mathbf{z}) \leqslant X V\left(z_0, h^*\right) V_1 V_2\left\{\left(F\left(s_1\right) F\left(s_2\right)\right)\left(1+O\left(\left(\log D_1 D_2\right)^{-1 / 6}\right)\right)\right\}$$
$$+O_{\varepsilon}\left(\sum_{d_1 d_2 \leqslant\left(D_1 D_2\right)^{1+\varepsilon}} \tau^4\left(d_1 d_2\right)\left|R\left(\left\{d_1, d_2\right\}\right)\right|\right),$$

$$S(\mathscr{W}, \mathbf{z}) \geqslant X V\left(z_0, h^*\right) V_1 V_2\left\{\left(f\left(s_1\right) F\left(s_2\right)+F\left(s_1\right) f\left(s_2\right)-F\left(s_1\right) F\left(s_2\right)\right)\left(1+O\left(\left(\log D_1 D_2\right)^{-1 / 6}\right)\right)\right\}$$
$$+O_{\varepsilon}\left(\sum_{d_1 d_2 \leqslant\left(D_1 D_2\right)^{1+\varepsilon}} \tau^4\left(d_1 d_2\right)\left|R\left(\left\{d_1, d_2\right\}\right)\right|\right)$$

for any $\varepsilon > 0$, where

$$z_0 = \exp\left(\sqrt[3]{\log z_1 z_2}\right), \quad s_j = \frac{\log D_j}{\log z_j} \quad (j = 1, 2),$$

$$V(z_0, h^*) = \prod_{p < z_0} (1 - h^*(p)), \quad h^*(p) = h(p, 1) + h(1, p) - h(p, p),$$

$$V_j = \prod_{z_0 \leqslant p < z_j} (1 - h_j(p)) \quad (j = 1, 2).$$

and γ denotes the Euler's constant, $f(s)$ and $F(s)$ are determined by the following differential-difference equation

$$\begin{cases} F(s) = \frac{2e^\gamma}{s}, \quad f(s) = 0, & 0 < s \leqslant 2, \\ (sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), & s \geqslant 2. \end{cases}$$

Lemma 2.2. *([1], Lemma 2], deduced from [6]).*

$$\begin{aligned}
F(s) =& \frac{2e^\gamma}{s}, \quad 0 < s \leqslant 3; \\
F(s) =& \frac{2e^\gamma}{s}\left(1 + \int_2^{s-1} \frac{\log(t-1)}{t} dt\right), \quad 3 \leqslant s \leqslant 5; \\
F(s) =& \frac{2e^\gamma}{s}\left(1 + \int_2^{s-1} \frac{\log(t-1)}{t} dt + \int_2^{s-3} \frac{\log(t-1)}{t} dt \int_{t+2}^{s-1} \frac{1}{u} \log \frac{u-1}{t+1} du\right), \quad 5 \leqslant s \leqslant 7; \\
f(s) =& \frac{2e^\gamma \log(s-1)}{s}, \quad 2 \leqslant s \leqslant 4; \\
f(s) =& \frac{2e^\gamma}{s}\left(\log(s-1) + \int_3^{s-1} \frac{dt}{t} \int_2^{t-1} \frac{\log(u-1)}{u} du\right), \quad 4 \leqslant s \leqslant 6; \\
f(s) =& \frac{2e^\gamma}{s}\left(\log(s-1) + \int_3^{s-1} \frac{dt}{t} \int_2^{t-1} \frac{\log(u-1)}{u} du\right. \\
&\left. + \int_2^{s-4} \frac{\log(t-1)}{t} dt \int_{t+2}^{s-2} \frac{1}{u} \log \frac{u-1}{t+1} \log \frac{s}{u+2} du\right), \quad 6 \leqslant s \leqslant 8.
\end{aligned}$$

Lemma 2.3. *([1], Lemma 4], deduced from [8], [13]). Let*

$$x > 1, \quad z = x^{\frac{1}{u}}, \quad Q(z) = \prod_{p < z} p.$$

Then for $u \geqslant 1$, we have

$$\sum_{\substack{n \leqslant x \\ (n, Q(z)) = 1}} 1 = w(u) \frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right),$$

where $w(u)$ is determined by the following differential-difference equation

$$\begin{cases} w(u) = \frac{1}{u}, & 1 \leqslant u \leqslant 2, \\ (uw(u))' = w(u-1), & u \geqslant 2. \end{cases}$$

Moreover, we have

$$\begin{cases} w(u) \leqslant \frac{1}{1.763}, & u \geqslant 2, \\ w(u) < 0.5644, & u \geqslant 3, \\ w(u) < 0.5617, & u \geqslant 4. \end{cases}$$

Conjecture 2.4. *(GEH(θ), deduced from [[12], Conjecture 1.2]). Let $\theta \in (0,1)$ be a constant and put*

$$P(y_1,\ldots,y_k;z_1,\ldots,z_k)=\{m=p_1\cdots p_k: y_1\leqslant p_1\leqslant z_1,\ldots,y_k\leqslant p_k\leqslant z_k\},$$
$$\pi_k(x;q,a)=\sum_{\substack{m\in P(y_1,\ldots,y_k;z_1,\ldots,z_k)\\ m\leqslant x, m\equiv a(\bmod q)}}1,$$
$$\pi_k(x;q)=\sum_{\substack{m\in P(y_1,\ldots,y_k;z_1,\ldots,z_k)\\ m\leqslant x,(m,q)=1}}1.$$

Then for any $A>0$ and $l\geqslant 1$ there exists $B=B(A,l)>0$ such that

$$\sum_{q\leqslant x^\theta \log^{-B}x}\tau^l(q)\max_{(a,q)=1}\left|\pi_k(x;q,a)-\frac{\pi_k(x;q)}{\varphi(q)}\right|\ll\frac{x}{\log^A x},$$

where the implied constant depends only on k,l and A.

Lemma 2.5. *([[2], Lemma 3], deduced from [7], [5]). GEH(θ) is true for $\theta\leqslant\frac{1}{2}$.*

3. Weighted sieve method

Lemma 3.1. *We have*

$$\begin{aligned}
4\pi_{1,3,14}(x)\geqslant &3S\left(\mathcal{A};\mathcal{P},\left(\frac{x}{q}\right)^{\frac{1}{13}}\right)+S\left(\mathcal{A};\mathcal{P},\left(\frac{x}{q}\right)^{\frac{1}{8.4}}\right)\\
&+\sum_{\left(\frac{x}{q}\right)^{\frac{1}{13}}\leqslant p_1<p_2<\left(\frac{x}{q}\right)^{\frac{1}{8.4}}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P},\left(\frac{x}{q}\right)^{\frac{1}{13}}\right)\\
&+\sum_{\left(\frac{x}{q}\right)^{\frac{1}{13}}\leqslant p_1<\left(\frac{x}{q}\right)^{\frac{1}{8.4}}\leqslant p_2<\left(\frac{x}{q}\right)^{0.475-\frac{2}{13}-\varepsilon}p_1^{-1}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P},\left(\frac{x}{q}\right)^{\frac{1}{13}}\right)\\
&-\sum_{\left(\frac{x}{q}\right)^{\frac{1}{13}}\leqslant p<\left(\frac{x}{q}\right)^{\frac{1}{3.145}}}S\left(\mathcal{A}_{p};\mathcal{P},\left(\frac{x}{q}\right)^{\frac{1}{13}}\right)\\
&-\sum_{\left(\frac{x}{q}\right)^{\frac{1}{13}}\leqslant p<\left(\frac{x}{q}\right)^{\frac{1}{3.81}}}S\left(\mathcal{A}_{p};\mathcal{P},\left(\frac{x}{q}\right)^{\frac{1}{13}}\right)\\
&-\sum_{\left(\frac{x}{q}\right)^{\frac{1}{13}}\leqslant p_1<\left(\frac{x}{q}\right)^{\frac{1}{3.145}}\leqslant p_2<\left(\frac{x}{qp_1}\right)^{\frac{1}{2}}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{\left(\frac{x}{q}\right)^{\frac{1}{8.4}}\leqslant p_1<\left(\frac{x}{q}\right)^{\frac{1}{3.81}}\leqslant p_2<\left(\frac{x}{qp_1}\right)^{\frac{1}{2}}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),\left(\frac{x}{qp_1p_2}\right)^{\frac{1}{2}}\right)\\
&-\sum_{\left(\frac{x}{q}\right)^{\frac{1}{13}}\leqslant p_1<p_2<p_3<p_4<\left(\frac{x}{q}\right)^{\frac{1}{8.4}}}S\left(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{\left(\frac{x}{q}\right)^{\frac{1}{13}}\leqslant p_1<p_2<p_3<\left(\frac{x}{q}\right)^{\frac{1}{8.4}}\leqslant p_4<\left(\frac{x}{q}\right)^{0.475-\frac{2}{13}-\varepsilon}p_3^{-1}}S\left(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right)\\
&-2\sum_{\left(\frac{x}{q}\right)^{\frac{1}{3.145}}\leqslant p_1<p_2<\left(\frac{x}{qp_1}\right)^{\frac{1}{2}}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)\\
&-2\sum_{\left(\frac{x}{q}\right)^{\frac{1}{3.81}}\leqslant p_1<p_2<\left(\frac{x}{qp_1}\right)^{\frac{1}{2}}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(p_1),p_2\right)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{k=15}^{199} S\left(\mathcal{A}^{(k)};\mathcal{P},\left(\frac{x}{q}\right)^{\frac{1}{13}}\right)-\sum_{k=15}^{199} S\left(\mathcal{A}^{(k)};\mathcal{P},\left(\frac{x}{q}\right)^{\frac{1}{8.4}}\right)\\
&-\sum_{k=15}^{199} S\left(\mathcal{A}^{(k)};\mathcal{P},\left(\frac{x}{q}\right)^{\frac{1}{3.81}}\right)-\sum_{k=15}^{199} S\left(\mathcal{A}^{(k)};\mathcal{P},\left(\frac{x}{q}\right)^{\frac{1}{3.145}}\right)+O\left(x^{\frac{12}{13}}\right)\\
=&\left(3S_{11}+S_{12}\right)+\left(S_{21}+S_{22}\right)-\left(S_{31}+S_{32}\right)-\left(S_{41}+S_{42}\right)\\
&-\left(S_{51}+S_{52}\right)-2\left(S_{61}+S_{62}\right)-\left(S_{71}+S_{72}+S_{73}+S_{74}\right)+O\left(x^{\frac{12}{13}}\right)\\
=&S_1+S_2-S_3-S_4-S_5-2S_6-S_7+O\left(x^{\frac{12}{13}}\right).
\end{aligned}
$$

Proof. It is similar to that of [[2], Lemma 6] so we omit it here. □

4. Proof of Theorem 1.1

In this section, sets $\mathcal{A}$, $\mathcal{E}$, $\mathcal{B}$, $\mathcal{M}_k$, $\mathcal{A}^{(k)}$, $\mathscr{W}$, $\mathscr{W}^{(1)}$ and $\mathscr{W}^{(2)}$ are defined respectively.

4.1. **Evaluation of S_1, S_2, S_3.** We are going to use Lemma 2.1 to the set $\mathscr{W}$ and obtain upper and lower bounds of S_1, S_2 and S_3. For a prime $p>7$, we note that $d_1 \mid \left(\frac{p+2}{q}\right)$ and $d_2 \mid (p+6)$ imply that $(d_1,d_2)=(d_1,2)=(d_2,6)=1$. Therefore we can take

$$
\begin{aligned}
&|\mathscr{W}_{\mathbf{d}}|=h(\mathbf{d})X+R(\mathbf{d}),\quad X=\frac{\pi(x)}{\varphi(q)},\\
&h(\mathbf{d})=\begin{cases}\frac{1}{\varphi(d_1d_2)}, & (d_1,d_2)=(d_1,2)=(d_2,6)=1,\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}
$$

It is easy to show that

$$
h_1(p)=\begin{cases}0, & p=2,\\ \frac{1}{p-1}, & p\geqslant 3;\end{cases}\quad h_2(p)=\begin{cases}0, & p=2,3,\\ \frac{1}{p-1}, & p\geqslant 5;\end{cases}\quad h^*(p)=\begin{cases}0, & p=2,\\ \frac{1}{2}, & p=3,\\ \frac{2}{p-1}, & p\geqslant 5.\end{cases}
$$

and

$$
\begin{aligned}
V\left(z_0,h^*\right)V_1V_2&=\frac{1}{2}\prod_{3<p\leqslant z_0}\left(1-\frac{2}{p-1}\right)\prod_{z_0<p\leqslant\left(\frac{x}{q}\right)^{\frac{1}{13}}}\left(1-\frac{1}{p-1}\right)\prod_{z_0<p\leqslant x^{0.005}}\left(1-\frac{1}{p-1}\right)\\
&=\left(1+O\left(z_0^{-1}\right)\right)C_3V\left(\left(\frac{x}{q}\right)^{\frac{1}{13}}\right)V\left(x^{0.005}\right)
\end{aligned}
\tag{12}
$$

with

$$
V(z)=\prod_{p<z}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}}{\log z}\left(1+O\left(\frac{1}{\log z}\right)\right).
$$

To deal with the error term, by the Chinese remainder theorem, we have

$$
|R(\mathbf{d})|\leqslant|r(d_1d_2)|,
$$

where

$$
|r(d)|=\max_{(a,d)=1}\left|\sum_{\substack{p\leqslant x\\ p\equiv a(\bmod d)}}1-\frac{\pi(x)}{\varphi(d)}\right|+O(1).
$$

By Bombieri's theorem we have

$$
\sum_{d_1d_2\leqslant x^{\frac{1}{2}-\varepsilon}}\tau^4\left(d_1d_2\right)|R(\mathbf{d})|\ll x(\log x)^{-5}. \tag{13}
$$

Then by Lemma 2.1 we have

$$\begin{aligned}
S_{11} =& S\left(\mathscr{W},\left\{\left(\frac{x}{q}\right)^{\frac{1}{13}}, x^{0.005}\right\}\right) \\
\geqslant& (1+o(1)) \frac{\pi(x)}{\varphi(q)} C_3 V\left(\left(\frac{x}{q}\right)^{\frac{1}{13}}\right) V\left(x^{0.005}\right)\{f(6.175) F(5)+F(6.175) f(5)-F(6.175) F(5)\} \\
& +O\left(x(\log x)^{-5}\right) \\
=& (1+o(1)) \frac{4 C_3 \pi(x)}{\varphi(q)\left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)}\left\{f_0(6.175) F_0(5)+F_0(6.175) f_0(5)-F_0(6.175) F_0(5)\right\} \\
& +O\left(x(\log x)^{-5}\right) \\
\geqslant& 818.10189 \frac{C_3 x \log \log x}{(\log x)^3},
\end{aligned} \tag{14}$$

where

$$f_0(s)=\frac{s}{2 e^\gamma} f(s), \quad F_0(s)=\frac{s}{2 e^\gamma} F(s).$$

Similarly, we have

$$\begin{aligned}
S_{12} =& S\left(\mathscr{W},\left\{\left(\frac{x}{q}\right)^{\frac{1}{8.4}}, x^{0.005}\right\}\right) \\
\geqslant& (1+o(1)) \frac{\pi(x)}{\varphi(q)} C_3 V\left(\left(\frac{x}{q}\right)^{\frac{1}{8.4}}\right) V\left(x^{0.005}\right)\{f(3.99) F(5)+F(3.99) f(5)-F(3.99) F(5)\} \\
& +O\left(x(\log x)^{-5}\right) \\
=& (1+o(1)) \frac{4 C_3 \pi(x)}{\varphi(q)\left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)}\left\{f_0(3.99) F_0(5)+F_0(3.99) f_0(5)-F_0(3.99) F_0(5)\right\} \\
& +O\left(x(\log x)^{-5}\right) \\
\geqslant& 516.86063 \frac{C_3 x \log \log x}{(\log x)^3},
\end{aligned} \tag{15}$$

$$\begin{aligned}
S_{21} \geqslant& (1+o(1)) \frac{0.475 \times 4 C_3 \pi(x)}{\varphi(q)\left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)}\left\{f_0(5) G+F_0(5) g-F_0(5) G\right\}+O\left(x(\log x)^{-5}\right) \\
\geqslant& 73.9301 \frac{C_3 x \log \log x}{(\log x)^3},
\end{aligned} \tag{16}$$

$$\begin{aligned}
S_{22} \geqslant& (1+o(1)) \frac{0.475 \times 4 C_3 \pi(x)}{\varphi(q)\left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)}\left\{f_0(5) H+F_0(5) h-F_0(5) H\right\}+O\left(x(\log x)^{-5}\right) \\
\geqslant& 149.13684 \frac{C_3 x \log \log x}{(\log x)^3},
\end{aligned} \tag{17}$$

$$\begin{aligned}
S_{31} \leqslant& (1+o(1)) \frac{0.475 \times 4 C_3 \pi(x)}{\varphi(q)\left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)}\left\{F_0(5) J\right\}+O\left(x(\log x)^{-5}\right) \\
\leqslant& 1282.38485 \frac{C_3 x \log \log x}{(\log x)^3},
\end{aligned} \tag{18}$$

$$\begin{aligned}
S_{32} \leqslant& (1+o(1)) \frac{0.475 \times 4 C_3 \pi(x)}{\varphi(q)\left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)}\left\{F_0(5) K\right\}+O\left(x(\log x)^{-5}\right) \\
\leqslant& 1048.20211 \frac{C_3 x \log \log x}{(\log x)^3},
\end{aligned} \tag{19}$$

where

$$G=\int_{1/13}^{1/8.4} \frac{d t_1}{t_1} \int_{t_1}^{1/8.4} \frac{d t_2}{t_2\left(0.475-t_1-t_2\right)}$$

$$+\int_{1/13}^{1/8.4}\frac{dt_1}{t_1}\int_{t_1}^{1/8.4}\frac{dt_2}{t_2\left(0.475-t_1-t_2\right)}\int_{2}^{5.175-13(t_1+t_2)}\frac{\log\left(t_3-1\right)}{t_3}dt_3,$$

$$g=\int_{1/13}^{1/8.4}\frac{dt_1}{t_1}\int_{t_1}^{1/8.4}\frac{\log\left(5.175-13\left(t_1+t_2\right)\right)}{t_2\left(0.475-t_1-t_2\right)}dt_2$$

$$+\int_{1/13}^{2.175/26}\frac{dt_1}{t_1}\int_{t_1}^{2.175/13-t_1}\frac{dt_2}{t_2\left(0.475-t_1-t_2\right)}\int_{3}^{5.175-13(t_1+t_2)}\frac{dt_3}{t_3}\int_{2}^{t_3-1}\frac{\log\left(t_4-1\right)}{t_4}dt_4,$$

$$H=\int_{1/13}^{1/8.4}\frac{dt_1}{t_1}\int_{1/8.4}^{0.475-2/13-t_1}\frac{dt_2}{t_2\left(0.475-t_1-t_2\right)}$$

$$+\int_{1/13}^{1/8.4}\frac{dt_1}{t_1}\int_{1/8.4}^{0.475-2/13-t_1}\frac{dt_2}{t_2\left(0.475-t_1-t_2\right)}\int_{2}^{5.175-13(t_1+t_2)}\frac{\log\left(t_3-1\right)}{t_3}dt_3,$$

$$h=\int_{1/13}^{1/8.4}\frac{dt_1}{t_1}\int_{1/8.4}^{0.475-2/13-t_1}\frac{\log\left(5.175-13\left(t_1+t_2\right)\right)}{t_2\left(0.475-t_1-t_2\right)}dt_2,$$

$$J=\int_{1/13}^{1/3.145}\frac{dt}{t(0.475-t)}+\int_{1/13}^{0.475-3/13}\frac{dt_1}{t_1\left(0.475-t_1\right)}\int_{2}^{5.175-13t_1}\frac{\log\left(t_2-1\right)}{t_2}dt_2$$

$$+\int_{1/13}^{0.475-5/13}\frac{dt_1}{t_1\left(0.475-t_1\right)}\int_{2}^{3.175-13t}\frac{\log\left(t_2-1\right)}{t_2}dt_2\int_{t_2+2}^{5.175}\frac{1}{t_3}\log\frac{t_3-1}{t_2+1}dt_3,$$

$$K=\int_{1/13}^{1/3.81}\frac{dt}{t(0.475-t)}+\int_{1/13}^{0.475-3/13}\frac{dt_1}{t_1\left(0.475-t_1\right)}\int_{2}^{5.175-13t_1}\frac{\log\left(t_2-1\right)}{t_2}dt_2$$

$$+\int_{1/13}^{0.475-5/13}\frac{dt_1}{t_1\left(0.475-t_1\right)}\int_{2}^{3.175-13t}\frac{\log\left(t_2-1\right)}{t_2}dt_2\int_{t_2+2}^{5.175}\frac{1}{t_3}\log\frac{t_3-1}{t_2+1}dt_3.$$

4.2. **Evaluation of S_4, S_5, S_6.** By Chen's role-reversal trick we know that

$$\begin{aligned}S_{51}=&\sum_{\substack{p\in\mathcal{B}\\ \left(p+6,P\left(x^{0.005}\right)\right)=1}}1\\ \leqslant&\sum_{\substack{m\in\mathcal{B}\\ \left(m,P\left(x^{\frac{0.475-\varepsilon}{2}}\right)\right)=1\\ \left(m+6,P\left(x^{0.005}\right)\right)=1}}1+O\left(x^{\frac{1}{2}}\right)\\ =&S\left(\mathscr{W}^{(1)},\left\{x^{\frac{0.475-\varepsilon}{2}},x^{0.005}\right\}\right)+O\left(x^{\frac{1}{2}}\right).\end{aligned}\tag{20}$$

we may write

$$\left|\mathscr{W}_{\mathbf{d}}^{(1)}\right|=h(\mathbf{d})|\mathcal{E}|+R^{(1)}(\mathbf{d}),$$

where

$$h(\mathbf{d})=\begin{cases}\frac{1}{\varphi(d_1d_2)}, & (d_1,d_2)=(d_1,2)=(d_2,6)=1,\\ 0, & \text{otherwise}\end{cases}$$

and

$$\begin{aligned}\left|R^{(1)}(\mathbf{d})\right|\leqslant&\max_{(a,d_1d_2)=1}\left|\sum_{\substack{n\in\mathcal{E}\\ n\equiv a(\bmod d_1d_2)}}1-\frac{1}{\varphi\left(d_1d_2\right)}\sum_{\substack{n\in\mathcal{E}\\ (n,d_1d_2)=1}}1\right|+\frac{1}{\varphi\left(d_1d_2\right)}\sum_{\substack{n\in\mathcal{E}\\ (n,d_1d_2)>1}}1\\ =&R_1^{(1)}(\mathbf{d})+R_2^{(1)}(\mathbf{d}).\end{aligned}$$

To deal with the error term, by the arguments used in [12], we have

$$\sum_{d_1 d_2 \leqslant x^{\frac{1}{2}-\varepsilon}} \tau^4 (d_1 d_2) R^{(1)}(\mathbf{d}) \ll x(\log x)^{-5}. \tag{21}$$

By Lemma 2.3 we have

$$\begin{aligned}
|\mathcal{E}| &= \sum_{(\frac{x}{q})^{\frac{1}{13}} \leqslant p_1 < p_2 < p_3 < p_4 < (\frac{x}{q})^{\frac{1}{8.4}}} \sum_{\substack{1 \leqslant m \leqslant \frac{x}{q p_1 p_2 p_3 p_4} \\ (m, q p_1^{-1} P(p_2))=1}} 1 \\
&\leqslant (1+o(1)) \frac{0.5617x}{q \log x} \int_{\frac{1}{13}}^{\frac{1}{8.4}} \frac{dt_1}{t_1} \int_{t_1}^{\frac{1}{8.4}} \frac{1}{t_2} \left(\frac{1}{t_1} - \frac{1}{t_2} \right) \log \frac{1}{8.4 t_2} dt_2. \\
&\leqslant \frac{0.00934x}{q \log x}.
\end{aligned} \tag{22}$$

Then by Lemma 2.1 we have

$$\begin{aligned}
S_{51} \leqslant & S\left(\mathscr{W}^{(1)}, \left\{x^{\frac{0.475-\varepsilon}{2}}, x^{0.005}\right\}\right) \\
\leqslant & (1+o(1)) \frac{4C_3|\mathcal{E}|}{\left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)} \left\{F_0(2)F_0(5)\right\} + O\left(x(\log x)^{-5}\right) \\
\leqslant & 4.41937 \frac{C_3 x \log\log x}{(\log x)^3}.
\end{aligned} \tag{23}$$

Similarly, we have

$$\begin{aligned}
S_{52} \leqslant & (1+o(1)) \frac{4C_3 x L}{q \log x \left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)} \left\{F_0(2)F_0(5)\right\} + O\left(x(\log x)^{-5}\right) \\
\leqslant & 22.91504 \frac{C_3 x \log\log x}{(\log x)^3},
\end{aligned} \tag{24}$$

$$\begin{aligned}
S_{41} \leqslant & (1+o(1)) \frac{4C_3 x \left(\int_{2.145}^{12} \frac{\log\left(2.145 - \frac{3.145}{t+1}\right)}{t} dt \right)}{q \log x \left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)} \left\{F_0(2)F_0(5)\right\} + O\left(x(\log x)^{-5}\right) \\
\leqslant & 371.11243 \frac{C_3 x \log\log x}{(\log x)^3},
\end{aligned} \tag{25}$$

$$\begin{aligned}
S_{42} \leqslant & (1+o(1)) \frac{4C_3 x \left(\int_{2.81}^{7.4} \frac{\log\left(2.81 - \frac{3.81}{t+1}\right)}{t} dt \right)}{q \log x \left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)} \left\{F_0(2)F_0(5)\right\} + O\left(x(\log x)^{-5}\right) \\
\leqslant & 341.31874 \frac{C_3 x \log\log x}{(\log x)^3},
\end{aligned} \tag{26}$$

$$\begin{aligned}
S_{61} \leqslant & (1+o(1)) \frac{4C_3 x \left(\int_{2}^{2.145} \frac{\log(t-1)}{t} dt \right)}{q \log x \left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)} \left\{F_0(2)F_0(5)\right\} + O\left(x(\log x)^{-5}\right) \\
\leqslant & 2.27032 \frac{C_3 x \log\log x}{(\log x)^3},
\end{aligned} \tag{27}$$

$$\begin{aligned}
S_{62} \leqslant & (1+o(1)) \frac{4C_3 x \left(\int_{2}^{2.81} \frac{\log(t-1)}{t} dt \right)}{q \log x \left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)} \left\{F_0(2)F_0(5)\right\} + O\left(x(\log x)^{-5}\right) \\
\leqslant & 49.78864 \frac{C_3 x \log\log x}{(\log x)^3},
\end{aligned} \tag{28}$$

where

$$L = \int_{1/13}^{1/8.4} \frac{dt_1}{t_1} \int_{t_1}^{1/8.4} \frac{dt_2}{t_2^2} \int_{t_2}^{1/8.4} \frac{dt_3}{t_3} \int_{1/8.4}^{0.475-2/13-t_3} \frac{w\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_4} dt_4$$

$$\leqslant 0.5644 \int_{1/13}^{1/8.4} \frac{dt_1}{t_1} \int_{t_1}^{1/8.4} \frac{1}{t_2} \left(\frac{1}{t_1} - \frac{1}{t_2} \right) \log 8.4 \left(0.475 - \frac{2}{13} - t_3 \right) dt_2$$
$$\leqslant 0.04839.$$

4.3. **Evaluation of S_7.** By Chen's role-reversal trick we know that

$$\begin{aligned} S_{71} = & \sum_{\substack{p+6 \in \mathcal{M}_k \\ \left(\frac{p+2}{q}, P\left(\left(\frac{x}{q} \right)^{\frac{1}{13}} \right) \right) = 1}} 1 \\ \leqslant & \sum_{\substack{m \in \mathcal{M}_k \\ \left(m-6, P\left(x^{\frac{0.025}{2}} \right) \right) = 1 \\ \left(\frac{m-4}{q}, P\left(\left(\frac{x}{q} \right)^{\frac{1}{13}} \right) \right) = 1}} 1 + O\left(x^{\frac{1}{2}} \right) \\ = & S\left(\mathscr{W}^{(2)}, \left\{ x^{\frac{0.025}{2}}, \left(\frac{x}{q} \right)^{\frac{1}{13}} \right\} \right) + O\left(x^{\frac{1}{2}} \right). \end{aligned} \tag{29}$$

we may write

$$\left| \mathscr{W}_{\mathbf{d}}^{(2)} \right| = h(\mathbf{d}) |\mathcal{M}_k| + R^{(2)}(\mathbf{d}),$$

where

$$h(\mathbf{d}) = \begin{cases} \frac{1}{\varphi(d_1 d_2)}, & (d_1, d_2) = (d_1, 2) = (d_2, 6) = 1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} \left| R^{(2)}(\mathbf{d}) \right| \leqslant & \max_{(a, d_1 d_2) = 1} \left| \sum_{\substack{n \in \mathcal{M}_k \\ n \equiv a (\bmod d_1 d_2)}} 1 - \frac{1}{\varphi(d_1 d_2)} \sum_{\substack{n \in \mathcal{M}_k \\ (n, d_1 d_2) = 1}} 1 \right| + \frac{1}{\varphi(d_1 d_2)} \sum_{\substack{n \in \mathcal{M}_k \\ (n, d_1 d_2) > 1}} 1 \\ = & R_1^{(2)}(\mathbf{d}) + R_2^{(2)}(\mathbf{d}). \end{aligned}$$

To deal with the error term, by the arguments similar to those for S_{51}, we have

$$\sum_{d_1 d_2 \leqslant x^{\frac{1}{2} - \varepsilon}} \tau^4(d_1 d_2) R^{(2)}(\mathbf{d}) \ll x (\log x)^{-5}.$$

By the prime number theorem and summation by parts we have

$$\begin{aligned} |\mathcal{M}_k| = & \frac{1}{\varphi(q)} \sum_{z \leqslant p_1 \leqslant \cdots \leqslant p_{k-1} \leqslant \left(\frac{x+6}{p_1 \cdots p_{k-2}} \right)^{1/2}} \frac{x}{p_1 \cdots p_{k-1} \log \frac{x}{p_1 \cdots p_{k-1}}} \\ = & \left(1 + O\left(\frac{1}{\log x} \right) \right) c_k \frac{\pi(x)}{\varphi(q)}, \end{aligned} \tag{30}$$

where

$$c_k = \int_{k-1}^{199} \frac{dt_1}{t_1} \int_{k-2}^{t_1 - 1} \frac{dt_2}{t_2} \cdots \int_3^{t_{k-4} - 1} \frac{dt_{k-3}}{t_{k-3}} \int_2^{t_{k-3} - 1} \frac{\log(t_{k-2} - 1) \, dt_{k-2}}{t_{k-2}}.$$

By similar numerical integration used in [2], we have

$$C_0 = \sum_{k=15}^{199} c_k < 0.00408. \tag{31}$$

Then from (29)–(31) we have

$$\begin{aligned} S_{71} \leqslant & (1 + o(1)) \frac{4 C_0 C_3 \pi(x)}{\varphi(q) (\log x^{0.475 - \varepsilon}) (\log x^{0.025})} \{ F_0(2) F_0(6.175) \} + O\left(x (\log x)^{-5} \right) \\ \leqslant & 2.38485 \frac{C_3 x \log \log x}{(\log x)^3}. \end{aligned} \tag{32}$$

Similarly, we have

$$\begin{aligned}
S_{72} \leqslant &(1+o(1)) \frac{4C_0C_3\pi(x)}{\varphi(q)\left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)}\left\{F_0(2)F_0(3.99)\right\} + O\left(x(\log x)^{-5}\right) \\
\leqslant &1.57643 \frac{C_3 x \log\log x}{(\log x)^3}, \qquad (33) \\
S_{73} \leqslant &(1+o(1)) \frac{4C_0C_3\pi(x)}{\varphi(q)\left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)}\left\{F_0(2)F_0(2)\right\} + O\left(x(\log x)^{-5}\right) \\
\leqslant &1.37432 \frac{C_3 x \log\log x}{(\log x)^3}, \qquad (34) \\
S_{74} \leqslant &(1+o(1)) \frac{4C_0C_3\pi(x)}{\varphi(q)\left(\log x^{0.475-\varepsilon}\right)\left(\log x^{0.025}\right)}\left\{F_0(2)F_0(2)\right\} + O\left(x(\log x)^{-5}\right) \\
\leqslant &1.37432 \frac{C_3 x \log\log x}{(\log x)^3}. \qquad (35)
\end{aligned}$$

4.4. **Proof of theorem 1.1.** By (14)–(19), (23)–(28) and (32)–(35) we get

$$S_1 + S_2 \geqslant 3194.23324 \frac{C_3 x \log\log x}{(\log x)^3},$$

$$S_3 + S_4 + S_5 + 2S_6 + S_7 \leqslant 3181.18071 \frac{C_3 x \log\log x}{(\log x)^3},$$

$$4\pi_{1,3,14}(x) \geqslant (S_1 + S_2) - (S_3 + S_4 + S_5 + 2S_6 + S_7) \geqslant 13.05253 \frac{C_3 x \log\log x}{(\log x)^3},$$

$$\pi_{1,3,14}(x) \geqslant 3.26313 \frac{C_3 x \log\log x}{(\log x)^3}.$$

Now the proof of $\pi_{1,3,14}(x) \gg \frac{C_3 x \log\log x}{(\log x)^3}$ is completed. Then we can prove Theorem 1.1 by replacing q by products of small primes $q_1 q_2 \cdots q_{a-1}$ where q_i denote a prime number satisfies

$$a \geqslant 2, \quad q_i < x^{\varepsilon} \text{ for every } 1 \leqslant i \leqslant a-1.$$

5. An upper bound result

Now we finish this paper with a look at the upper bound estimate. Let

$$\mathscr{W}' = \{\{p+2, p+6\} : 7 < p \leqslant x\},$$

then we have

$$\pi_{1,1,1}(x) \leqslant S\left(\mathscr{W}', \left\{x^{\frac{1}{10}}, x^{\frac{1}{10}}\right\}\right) + O\left(x^{\frac{1}{10}}\right). \qquad (36)$$

By Lemma 2.1 and some routine arguments we have

$$\begin{aligned}
&S\left(\mathscr{W}', \left\{x^{\frac{1}{10}}, x^{\frac{1}{10}}\right\}\right) \\
\leqslant &(1+o(1)) C_3 \pi(x) V\left(x^{\frac{1}{10}}\right) V\left(x^{\frac{1}{10}}\right)\{F(2)F(2)\} + O\left(x(\log x)^{-5}\right) \\
\leqslant &100 \frac{C_3 x}{(\log x)^3}. \qquad (37)
\end{aligned}$$

Finally by (36)–(37) we get the following theorem of the upper bound orders of Hardy-Littlewood prime triples.

Theorem 5.1.

$$\pi_{1,1,1}(x) \ll \frac{C_3 x}{(\log x)^3} \quad \textit{and} \quad D_{1,1,1}(N) \ll \frac{N}{(\log N)^3}.$$

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