ON THE PRIMITIVE DIVISORS OF QUADRATIC POLYNOMIALS

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Abstract. Let x denote a sufficiently large integer. We give a new upper bound for the number of integers $n \leqslant x$ which n^2+1 has a primitive divisor, improving previous result of Harman. This improvement comes from Pascadi's recent work on the largest prime factor of n^2+1.

1. Introduction

Let x, n denote sufficiently large integers, p denote a prime number, P_r denote an integer with at most r prime factors counted with multiplicity, and let f be an irreducible polynomial with degree g. It's conjectured that there are infinitely many n such that $f(n)$ is prime. The simplest case is $g=1$, which is the famous Dirichlet's theorem proved more than 100 years ago. However, for $g \geqslant 2$, this conjecture is still open.

For the second simplest case $g=2$, there are several ways to attack this conjecture. One way is to relax the number of prime factors of $f(n)$, and the best result in this way is due to Iwaniec [7]. Building on the previous work of Richert [11], he showed that for any irreducible polynomial $f(n)=an^2+bn+c$ with $a>0$ and $c \equiv 1(\bmod 2)$, there are infinitely many x such that $f(x)$ is a P_2.

Another possible way is to consider the largest prime factor of $f(n)$. Let $P^{+}(x)$ denote the largest prime factor of x, then we hope to show that the largest prime factor of $f(n)$ is greater than n^g for infinitely many integers n. For general polynomials, the best result is due to Tenenbaum [13], where he showed that for some $0<t<2-\log 4$, the largest prime factor of $f(n)$ is greater than $n \exp((\log n)^t)$ for infinitely many integers n. However, it's rather difficult to prove the same thing holds for $n^{1+\varepsilon}$ even for a small ε.

For the special case $f(n)=n^2+1$, the progress is far more than the general case. In 1967, Hooley [6] first proved the largest prime factor of n^2+1 is greater than $n^{1.10014}$ for infinitely many integers n by using the Weil bound for Kloosterman sums. By applying their new bounds for multilinear forms of Kloosterman sums, Deshouillers and Iwaniec [2] showed in 1982 that the largest prime factor of n^2+1 is greater than $n^{1.202468}$ infinitely often. In 2020, de la Bretèche and Drappeau [1] improved the exponent to 1.2182 by making use of the result of Kim and Sarnak [8]. In 2023, Merikoski [9] proved a new bilinear estimate and used Harman's sieve to get the exponent 1.279. This is the first attempt of using Harman's sieve on this problem. Very recently, Pascadi [10] optimized the exponent to 1.3 by inserting his new arithmetic information.

In this paper we focus on the values of the polynomial n^2+1 with a primitive divisor.

Definition 1.1. Let (A_n) denote a sequence with integer terms. We say an integer $d>1$ is a primitive divisor of A_n if $d \mid A_n$ and $(d, A_m)=1$ for all non-zero terms A_m with $m<n$.

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Proposition 1.2. *For all $n > 1$, the term $n^2 + 1$ has a primitive divisor if and only if $P^+(n^2+1) > 2n$. For all $n > 1$, if $n^2 + 1$ has a primitive divisor then that primitive divisor is a prime and it is unique.*

Contrary to the previous works on the lower bounds for the largest prime factor, a result due to Schinzel [12] showed that for any $\varepsilon > 0$, the largest prime factor of $n^2 + 1$ is less than n^ε infinitely often. In fact, from his result we can easily get the following.

Theorem 1.3. *([3], Theorem 1.2]). The polynomial n^2+1 does not have a primitive divisor for infinitely many terms.*

We are interested in finding good upper and lower bound for the number of terms $n^2 + 1$ with a primitive divisor. We define

$$\rho(x) = \left|\left\{n \leqslant x : n^2 + 1 \text{ has a primitive divisor}\right\}\right|.$$

Then we have the following simple upper bound

$$\rho(x) < x - \frac{Cx}{\log x} \tag{1}$$

for some constant $C > 0$. In [4] the following stronger result is mentioned.

$$\rho(x) < x - \frac{x \log\log x}{\log x}. \tag{2}$$

In [3], Everest and Harman first proved a lower bound with positive density and a better upper bound for $\rho(x)$. More precisely, they got the following bounds:

Theorem 1.4. *([3], Theorem 1.4]). We have*

$$0.5324x < \rho(x) < 0.905x.$$

They also conjectured the asymptotic $\rho(x) \sim (\log 2)x$ in their paper. Recently Harman [5] sharpened the upper and lower bounds for $\rho(x)$.

Theorem 1.5. *([5], Theorem 5.5]). We have*

$$0.5377x < \rho(x) < 0.86x.$$

Note that in [3] they used Deshouillers and Iwaniec's work on the largest prime factor of $n^2 + 1$ to prove the upper bound, while in [5] Harman used Merikoski's work. So it can be easily seen that it is possible to use Pascadi's recent work to improve the upper bound for $\rho(x)$. This is the main work of the present paper.

Theorem 1.6. *We have*

$$\rho(x) < 0.847x.$$

By utilizing Pascadi's recent work, one can also improve some results in [4] pointed out by Harman. We leave them to the readers.

2. Pascadi's sieve decompositions

Let ε denote a sufficient small positive number and P_x denote the largest prime factor of $\prod_{x \leqslant n \leqslant 2x} (n^2+1)$. In this section we briefly introduce Pascadi's work on finding a lower bound for P_x. Let $b(x)$ denote a non-nagative C^∞–smooth function supported on $[x, 2x]$ and its derivatives satisfy $b^{(j)}(x) \ll x^{-j}$ for all $j \geqslant 0$. We define

$$|\mathcal{A}_d| := \sum_{n^2+1 \equiv 0 (\bmod d)} b(n) \quad \text{and} \quad X := \int b(x) dx.$$

Then by the method of Chebyshev–Hooley and the discussion in [9], we only need to find an upper bound for

$$S(x) := \sum_{x < p \leqslant P_x} |\mathcal{A}_p| \log p = X \log x + O(x) \tag{3}$$

with a constant less than 1. By a smooth dyadic partition we have

$$S(x) = \sum_{\substack{x \leqslant P \leqslant P_x \\ P = 2^j x}} S(x, P) + O(x), \tag{4}$$

where

$$S(x, P) = \sum_{P \leqslant p \leqslant 4P} \psi_P(p) \, |\mathcal{A}_p| \log p \tag{5}$$

for some C^∞–smooth functions ψ_P supported on $[P, 4P]$ satisfying $\psi_P^{(l)}(x) \ll P^{-l}$ for all $l \geqslant 0$.

In [10], Pascadi proved the following upper bound for $S(x)$ with $P_x = x^{1.3}$ by using Harman's sieve method together with his new arithmetic information.

Lemma 2.1. *(See [10]). We have*

$$\sum_{\substack{x \leqslant P \leqslant x^{1.3} \\ P = 2^j x}} S(x, P) \leqslant \left(G_0 + G_1 + G_2 + G_3 + G_4 + G_5 - G_6 + G_7\right) X \log x$$
$$< 0.999 X \log x,$$

where

$$G_0 = \int_1^{\frac{7}{6}} 1 d\alpha,$$

$$G_1 = \int_1^{\frac{25}{24}} \int_{\sigma(\alpha)}^{\alpha - 2\sigma(\alpha)} \alpha \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha + \int_1^{\frac{25}{24}} \int_{\xi(\alpha)}^{\frac{\alpha}{2}} \alpha \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha,$$

$$G_2 = \int_{\frac{25}{24}}^{\frac{228}{203}} \int_{\sigma(\alpha)}^{\frac{\alpha}{2}} \alpha \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha,$$

$$G_3 = \int_{\frac{228}{203}}^{\frac{7}{6}} \int_{\sigma_0(\alpha)}^{\frac{\alpha}{2}} \alpha \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha,$$

$$G_4=\int_{\frac{228}{203}}^{\frac{7}{6}}\int_{\sigma_0(\alpha)-\alpha+1}^{\alpha-1}\int_{\sigma_0(\alpha)-\alpha+1}^{\beta_1}\int_{\sigma_0(\alpha)-\alpha+1}^{\beta_2} f_4(\alpha,\beta_1,\beta_2,\beta_3)\,\alpha\frac{\omega\left(\frac{\alpha-\beta_1-\beta_2-\beta_3}{\beta_3}\right)}{\beta_1\beta_2\beta_3^2}d\beta_3 d\beta_2 d\beta_1 d\alpha,$$

$$G_5=4\int_{\frac{7}{6}}^{\frac{139}{114}}\alpha d\alpha,$$

$$G_6=\int_{\frac{7}{6}}^{\frac{139}{114}}\int_{\alpha-1}^{\sigma_0(\alpha)}\alpha\frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2}d\beta d\alpha,$$

$$G_7=4\int_{\frac{139}{114}}^{\frac{5}{4}}\alpha d\alpha+(4-5\theta)\int_{\frac{5}{4}}^{1.3}\frac{\alpha}{1-\theta\alpha}d\alpha,$$

where $\theta=\frac{7}{32}$,

$$\sigma_0(\alpha):=\max\left(\frac{2-(1+\theta)\alpha}{3-2\theta},\frac{(1-\theta)(2-\alpha)}{3-\theta}\right), \tag{6}$$

$$\sigma(\alpha):=\max\left(\frac{4-3\alpha}{3},\sigma_0(\alpha)\right)-\varepsilon, \tag{7}$$

$$\xi(\alpha):=\min\left(\frac{1}{2},\frac{2(1-\theta\alpha)}{4-5\theta}\right)-\alpha+1-2\varepsilon, \tag{8}$$

f_4 *denotes the characteristic function of the set*

$$\{\beta_1+\beta_2,\beta_1+\beta_3,\beta_2+\beta_3,\beta_1+\beta_2+\beta_3\notin[\alpha-1,\sigma_0(\alpha)]\},$$

and $\omega(u)$ *denotes the Buchstab function determined by the following differential-difference equation*

$$\begin{cases}\omega(u)=\frac{1}{u}, & 1\leqslant u\leqslant 2,\\ (u\omega(u))'=\omega(u-1), & u\geqslant 2.\end{cases}$$

3. Proof of the theorem

Let $V(u)$ denote an infinitely differentiable non-negative function such that

$$V(u)\begin{cases}<2, & 1<u<2,\\ =0, & u\leqslant 1 \text{ or } u\geqslant 2,\end{cases}$$

with

$$\frac{d^rV(u)}{du^r}\ll 1\quad\text{and}\quad\int_{\mathbb{R}}V(u)du=1.$$

By the discussion in [3] and [5], we wish to get an upper bound for sum of $\sum_{p|k^2+1}V(k/x)$ of the form

$$\sum_{1\leqslant px^{-\alpha}\leqslant e}\sum_{p|k^2+1}V\left(\frac{k}{x}\right)\leqslant K(\alpha)(1+o(1))\frac{X}{\log x}$$

where $K(\alpha)$ is the sum of sieve theoretical functions related to the sieve decomposition on the problem of the largest prime factor of n^2+1. This requires us to prove that for some τ, we have

$$\int_1^\tau \alpha K(\alpha) d\alpha < 1.$$

By Lemma 2.1 we can take $\tau = 1.3$, and $K(\alpha)$ is defined as the piecewise function in Section 2. Combining this with the bound proved in [3], we have

$$\begin{aligned} \rho(x) &\leqslant (1+o(1))x \int_1^{1.3} K(\alpha) d\alpha \\ &\leqslant \left(G_0' + G_1' + G_2' + G_3' + G_4' + G_5' - G_6' + G_7'\right) x, \end{aligned} \tag{9}$$

where

$$G_0' = \int_1^{\frac{7}{6}} \frac{1}{\alpha} d\alpha,$$

$$G_1' = \int_1^{\frac{25}{24}} \int_{\sigma(\alpha)}^{\alpha-2\sigma(\alpha)} \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha + \int_1^{\frac{25}{24}} \int_{\xi(\alpha)}^{\frac{\alpha}{2}} \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha,$$

$$G_2' = \int_{\frac{25}{24}}^{\frac{228}{203}} \int_{\sigma(\alpha)}^{\frac{\alpha}{2}} \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha,$$

$$G_3' = \int_{\frac{228}{203}}^{\frac{7}{6}} \int_{\sigma_0(\alpha)}^{\frac{\alpha}{2}} \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha,$$

$$G_4' = \int_{\frac{228}{203}}^{\frac{7}{6}} \int_{\sigma_0(\alpha)-\alpha+1}^{\alpha-1} \int_{\sigma_0(\alpha)-\alpha+1}^{\beta_1} \int_{\sigma_0(\alpha)-\alpha+1}^{\beta_2} f_4\left(\alpha, \beta_1, \beta_2, \beta_3\right) \frac{\omega\left(\frac{\alpha-\beta_1-\beta_2-\beta_3}{\beta_3}\right)}{\beta_1\beta_2\beta_3^2} d\beta_3 d\beta_2 d\beta_1 d\alpha,$$

$$G_5' = 4\int_{\frac{7}{6}}^{\frac{139}{114}} 1 d\alpha,$$

$$G_6' = \int_{\frac{7}{6}}^{\frac{139}{114}} \int_{\alpha-1}^{\sigma_0(\alpha)} \frac{\omega\left(\frac{\alpha-\beta}{\beta}\right)}{\beta^2} d\beta d\alpha,$$

$$G_7' = 4\int_{\frac{139}{114}}^{\frac{5}{4}} 1 d\alpha + (4-5\theta) \int_{\frac{5}{4}}^{1.3} \frac{1}{1-\theta\alpha} d\alpha.$$

By numerical integration using Mathematica 14, the value of the right hand side of (8) is less than $0.847x$. Hence the proof of Theorem 1.6 is completed.

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