ON THE GENERALIZED DIRICHLET DIVISOR PROBLEM

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Abstract. Using more advanced results on the growth exponent for Riemann zeta–function and accurate numerical estimations, we obtain better upper bounds for α_k $(9 \leqslant k \leqslant 20)$ on the generalized Dirichlet divisor problem. This gives a minor improvement upon the recent result of Trudgian and Yang.

Contents

1. Introduction

Let $k \geqslant 2$ denotes an integer and $d_k(n)$ is the divisor function that represents the number of ways n may be written as a product of exactly k factors. The generalized Dirichlet divisor problem consists of the estimation of the function

$$\Delta_k(x) = \sum_{n \leqslant x} d_k(n) - x P_{k-1}(\log x), \tag{1}$$

where P_{k-1} is an explicit polynomial of degree $k-1$. Clearly we have $\Delta_k(x) = o(x)$. We then define α_k as the least exponent for which

$$\Delta_k(x) \ll x^{\alpha_k + \varepsilon}. \tag{2}$$

In 1916, Hardy [2] first proved a lower bound that $\alpha_k \geqslant \frac{1}{2} - \frac{1}{2k}$ for all $k \geqslant 2$. The generalized Dirichlet divisor problem conjecture states that $\alpha_k = \frac{1}{2} - \frac{1}{2k}$ holds for all $k \geqslant 2$, and this conjecture implies the Lindelöf hypothesis. Now, the best upper bounds for α_k $(k \leqslant 8)$ are

$$\alpha_2 \leqslant 0.3144831759741, \qquad \alpha_3 \leqslant \frac{43}{96}, \qquad \alpha_k \leqslant \frac{3k-4}{4k} \text{ for } 4 \leqslant k \leqslant 8$$

by Li and Yang [8], Kolesnik [7] and Heath–Brown [3] (and Ivić [5]) respectively. Ivić also gave upper bounds with $k \geqslant 9$ in his book. For results with large k, one can see works of Heath–Brown [4] and Bellotti and Yang [1]. We also refer the readers to the blueprint of the new project ANTEDB organized by Tao, Trudgian and Yang.

In 1989, Ivić and Ouellet [6] refined the technique used in and gave better bounds for α_k with $k \geqslant 9$. In [5], Ivić connected this problem with the function $m(\sigma)$ defined as follows: For any fixed $\frac{1}{2} < \sigma < 1$ we define $m(\sigma)$ as the supremum of all numbers $m \geqslant 4$ such that

$$\int_1^T |\zeta(\sigma + it)|^m \, dt \ll T^{1+\varepsilon}. \tag{3}$$

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In order to obtain good bounds for α_k, one need to get lower bounds for $m(\sigma)$. Ivić and Ouellet [6] used a large value theorem and growth exponents for Riemann zeta–function to bound $m(\sigma)$. Specially, for $10 \leqslant k \leqslant 20$ they got

$$\begin{aligned} &\alpha_{10} \leqslant 0.675, && \alpha_{11} \leqslant 0.6957, && \alpha_{12} \leqslant 0.7130, && \alpha_{13} \leqslant 0.7306, \\ &\alpha_{14} \leqslant 0.7461, && \alpha_{15} \leqslant 0.75851, && \alpha_{16} \leqslant 0.7691, && \alpha_{17} \leqslant 0.7785, \\ &\alpha_{18} \leqslant 0.7868, && \alpha_{19} \leqslant 0.7942, && \alpha_{20} \leqslant 0.8009. \end{aligned}$$

In 2024, Trudgian and Yang [9] mentioned a series of new bounds for α_k. They combined the method of Ivić and Ouellet [6] with their new growth exponents for Riemann zeta–function to obtain those bounds.

Theorem 1.1. *([9], Theorem 2.9]). We have*

$$\begin{aligned} &\alpha_9 \leqslant 0.64720, && \alpha_{10} \leqslant 0.67173, && \alpha_{11} \leqslant 0.69156, && \alpha_{12} \leqslant 0.70818, \\ &\alpha_{13} \leqslant 0.72350, && \alpha_{14} \leqslant 0.73696, && \alpha_{15} \leqslant 0.74886, && \alpha_{16} \leqslant 0.75952, \\ &\alpha_{17} \leqslant 0.76920, && \alpha_{18} \leqslant 0.77792, && \alpha_{19} \leqslant 0.78581, && \alpha_{20} \leqslant 0.79297, && \alpha_{21} \leqslant 0.79951. \end{aligned}$$

In this paper, we use the essentially same methods to give a very minor improvement on their results.

Theorem 1.2. *We have*

$$\begin{aligned} &\alpha_9 \leqslant 0.638889, && \alpha_{10} \leqslant 0.663329, && \alpha_{11} \leqslant 0.684349, && \alpha_{12} \leqslant 0.701768, \\ &\alpha_{13} \leqslant 0.717611, && \alpha_{14} \leqslant 0.732262, && \alpha_{15} \leqslant 0.745070, && \alpha_{16} \leqslant 0.756380, \\ &\alpha_{17} \leqslant 0.766588, && \alpha_{18} \leqslant 0.775721, && \alpha_{19} \leqslant 0.783939, && \alpha_{20} \leqslant 0.791374. \end{aligned}$$

2. Growth exponents for Riemann zeta–function

In this section we list the new growth exponents for Riemann zeta–function proved by Trudgian and Yang [9], which is the most powerful and important input in the proof of Theorem 1.2 (and also Theorem 1.1).

Lemma 2.1. *([9], Theorem 2.4]). We have*

$$\mu(\sigma) \leqslant \begin{cases} \frac{31}{36} - \frac{3}{7}\sigma, & \frac{1}{2} \leqslant \sigma \leqslant \frac{88225}{153852}, \\ \frac{220633}{620612} - \frac{62831}{155153}\sigma, & \frac{88225}{153852} \leqslant \sigma \leqslant \frac{521}{796}, \\ \frac{1333}{3825} - \frac{1508}{3825}\sigma, & \frac{521}{796} \leqslant \sigma \leqslant \frac{53141}{76066}, \\ \frac{405}{1202} - \frac{227}{601}\sigma, & \frac{53141}{76066} \leqslant \sigma \leqslant \frac{454}{641}, \\ \frac{779}{2590} - \frac{423}{1295}\sigma, & \frac{454}{641} \leqslant \sigma \leqslant \frac{3473692}{4856993}, \\ \frac{1610593}{5622410} - \frac{861996}{2811205}\sigma, & \frac{3473692}{4856993} \leqslant \sigma \leqslant \frac{52209}{69128}, \\ \frac{157319}{560830} - \frac{251324}{841245}\sigma, & \frac{52209}{69128} \leqslant \sigma \leqslant \frac{1389}{1736}, \\ \frac{2841}{10316} - \frac{754}{2579}\sigma, & \frac{1389}{1736} \leqslant \sigma \leqslant \frac{587779}{702192}, \\ \frac{1691}{6554} - \frac{890}{3277}\sigma, & \frac{587779}{702192} \leqslant \sigma \leqslant \frac{7441}{8695}, \\ \frac{29}{130} - \frac{3}{13}\sigma, & \frac{7441}{8695} \leqslant \sigma \leqslant \frac{277}{300}. \end{cases}$$

3. Ivić large value theorem

Now we provide the large value theorem used by Ivić and Ouellet [6].

Lemma 3.1. *(, Lemma 1). Let $t_1, \ldots, t_R$ be real numbers such that $T \leqslant t_r \leqslant 2T$ for $r = 1, \ldots, R$ and $|t_r - t_s| \geqslant (\log T)^4$ for $1 \leqslant r \neq s \leqslant R$. If*

$$T^\varepsilon < V \leqslant \left| \sum_{m \sim M} a_m m^{-\sigma - it_r} \right|$$

where $a_m \ll M^\varepsilon$ for $m \sim M$, $1 \ll M \ll T^C$, then

$$R \ll T^\varepsilon \left(M^{2-2\sigma} V^{-2} + T V^{-f(\sigma)} \right),$$

where

$$f(\sigma)=\begin{cases}\frac{2}{3-4\sigma}, & \frac{1}{2}<\sigma\leqslant\frac{2}{3},\\ \frac{10}{7-8\sigma}, & \frac{2}{3}\leqslant\sigma\leqslant\frac{11}{14},\\ \frac{34}{15-16\sigma}, & \frac{11}{14}\leqslant\sigma\leqslant\frac{13}{15},\\ \frac{98}{31-32\sigma}, & \frac{13}{15}\leqslant\sigma\leqslant\frac{57}{62},\\ \frac{5}{1-\sigma}, & \frac{57}{62}\leqslant\sigma\leqslant 1-\varepsilon.\end{cases}$$

4. Proof of Theorem 1.2

We shall use the method of Ivić and Ouellet [6] to prove Theorem 1.2. It was shown in [, Chapter 8] that to obtain bounds for $m(\sigma)$ it suffices to obtain bounds of the form

$$R\ll T^{1+\varepsilon}V^{-m(\sigma)}, \tag{4}$$

where R is the number of points $t_r(1\leqslant r\leqslant R)$ such that $|t_r|\leqslant T$, $|t_r-t_s|\geqslant(\log T)^4$ for $1\leqslant r\neq s\leqslant R$ and $|\zeta(\sigma+it_r)|\geqslant V>0$ for any given V. Moreover, by [, (8.97)] we know that

$$R\ll T^{\varepsilon}\left(TV^{-2f(\sigma)}+T^{\frac{4-4\sigma}{1+2\sigma}}V^{\frac{-12}{1+2\sigma}}+T^{\frac{4(1-\sigma)(\kappa+\lambda)}{((2-4\lambda)\sigma-1+2\kappa-2\lambda)}}V^{\frac{-4(1+2\kappa+2\lambda)}{((2-4\lambda)\sigma-1+2\kappa-2\lambda)}}\right), \tag{5}$$

where (κ,λ) is an exponent pair. We shall use $(\kappa,\lambda)=\left(\frac{3}{40},\frac{31}{40}\right)$ in the rest of our paper for the sake of convenience.

Now, for every $\frac{1}{2}\leqslant\sigma\leqslant\frac{277}{300}$ we define $c(\sigma)$ is the piecewise function given by Lemma 2.1. Clearly $c(\sigma)$ is an upper bound for $\mu(\sigma)$. By () and the definitions of $f(\sigma)$ and $c(\sigma)$, we can easily calculate the corresponding $m(\sigma)$ for some σ between $\frac{1}{2}$ and $\frac{277}{300}$. Numerical calculation gives that

$$\begin{array}{llll} m(0.638889)>9, & m(0.663329)>10, & m(0.684349)>11, & m(0.701768)>12,\\ m(0.717611)>13, & m(0.732262)>14, & m(0.745070)>15, & m(0.756380)>16,\\ m(0.766588)>17, & m(0.775721)>18, & m(0.783939)>19, & m(0.791374)>20\end{array}$$

and Theorem 1.2 is now proved.

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