ON THE EXCEPTIONAL SET IN THE abc CONJECTURE

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Abstract. The abc conjecture states that there are only finitely many triples of coprime positive integers (a, b, c) such that $a + b = c$ and $\mathrm{rad}(abc) < c^{1-\epsilon}$ for any $\epsilon > 0$. Using the optimized methods in a recent work of Browning, Lichtman and Teräväinen, we showed that the number of those triples with $c \leqslant X$ is $O\left(X^{56/85+\varepsilon}\right)$ for any $\varepsilon > 0$, where $\frac{56}{85} \approx 0.658824$. This constitutes an improvement of the previous bound $O\left(X^{33/50}\right)$.

Contents

1. Introduction

Let n denotes a positive integer, p denotes a prime and write

$$\mathrm{rad}(n) = \prod_{p \mid n} p. \tag{1}$$

We say a triple (a, b, c) of coprime positive integers a, b, c is an abc triple of exponent λ if

$$a + b = c \quad \text{and} \quad \mathrm{rad}(abc) < c^{\lambda}.$$

The famous abc conjecture, proposed by Masser and Oesterlé, asserts that there are only finitely many abc triples of exponent λ for any $\lambda < 1$. Now the best result in this direction is due to Stewart and Yu, who showed that there are finitely many abc triples satisfy

$$\mathrm{rad}(abc) < (\log c)^{3-\epsilon}. \tag{2}$$

For more historical progress of the abc conjecture, we refer the readers to [1].

Now, we are focusing on the exceptional set in the abc conjecture. We first define $N_\lambda(X)$ as the number of abc triples of exponent λ in $[1, X]^3$ as $X \to \infty$. A "trivial" bound states that

Theorem 1.1. *("Trivial" bound). Let $\lambda > 0$. Then we have*

$$N_\lambda(X) \ll x^{\frac{2}{3}\lambda+\epsilon}$$

for any $\epsilon > 0$.

For the proof, one can see Lichtman's recent note [2].

In 2024, Browning, Lichtman and Teräväinen [1] developed a system of combinatorial bounds and improved Theorem 1.1. Their result is the first power–saving improvement over the "trivial" bound for λ close to 1 (actually, for $0.99 < \lambda < 1.001$).

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Theorem 1.2. *(Browning–Lichtman–Teräväinen bound). Let $0 < \lambda < 1.001$. Then we have*

$$N_\lambda(X) \ll x^{\frac{33}{50}} = x^{0.66}.$$

In the present paper, we use the method of Browning, Lichtman and Teräväinen [1] to improve their result and show that without further optimization, the best exponent their current method can reach is $\frac{56}{85}$.

Theorem 1.3. *For any $\varepsilon > 0$, there exists a positive constant $\delta = \delta(\varepsilon)$ such that for $0 < \lambda < 1 + \delta$, we have*

$$N_\lambda(X) \ll x^{\frac{56}{85}+\varepsilon}.$$

In this paper, we put $\varepsilon > 0$, $0 < \delta < 10^{-100}$ and $\theta = \frac{56}{85} + \varepsilon$. We also suppose that ϵ is a sufficiently small positive number.

2. Number of solutions to Diophantine equations

We define a counting function $S_{\alpha,\beta,\gamma}(X)$ for $\alpha, \beta, \gamma > 0$ as the same as in [1]: $S_{\alpha,\beta,\gamma}(X)$ denotes the number of $(a, b, c) \in \mathbb{N}^3$ with $\gcd(a, b, c) = 1$ such that

$$a, b, c \in [1, X], \quad a + b = c, \quad \mathrm{rad}(a) \leqslant a^\alpha, \quad \mathrm{rad}(b) \leqslant b^\beta, \quad \mathrm{rad}(c) \leqslant c^\gamma.$$

Then we have

$$N_\lambda(X) \leqslant \max_{\substack{\alpha,\beta,\gamma>0 \\ \alpha+\beta+\gamma\leqslant\lambda}} S_{\alpha,\beta,\gamma}(X). \tag{3}$$

We shall use a standard dyadic decomposition to define a variant of $S_{\alpha,\beta,\gamma}(X)$: Let $S^*_{\alpha,\beta,\gamma}(X)$ denotes the number of $(a, b, c) \in \mathbb{N}^3$ with $\gcd(a, b, c) = 1$ such that

$$c \in \left[\frac{X}{2}, X\right], \quad a + b = c, \quad \mathrm{rad}(a) \sim a^\alpha, \quad \mathrm{rad}(b) \sim b^\beta, \quad \mathrm{rad}(c) \sim c^\gamma.$$

Then, by the pigeonhole principle we have

$$S_{\alpha,\beta,\gamma}(X) \ll (\log X)^4 \max_{\substack{\alpha'\leqslant\alpha \\ \beta'\leqslant\beta \\ \gamma'\leqslant\gamma}} \max_{Y\in[1,X]} S^*_{\alpha',\beta',\gamma'}(Y). \tag{4}$$

Hence, in order to prove Theorem 1.3, we only need to show that

$$S^*_{\alpha,\beta,\gamma}(X) \ll X^\theta (\log X)^{-4}. \tag{5}$$

We need the following important lemma to reduce the problem into bounding the number of solutions to some Diophantine equations.

Lemma 2.1. *([1], Proposition 2.1]). Let $\alpha, \beta, \gamma \in (0, 1]$ be fixed and let $X \geqslant 2$. For any $\epsilon > 0$ there exists an integer $d = d(\epsilon) \geqslant 1$ such that the following holds: There exist $X_1, \ldots, X_d, Y_1, \ldots, Y_d, Z_1, \ldots, Z_d \geqslant 1$ satisfying*

$$X^{\alpha-\epsilon} \leqslant \prod_{1\leqslant j\leqslant d} X_j \leqslant X^{\alpha+\epsilon}, \quad X^{\beta-\epsilon} \leqslant \prod_{1\leqslant j\leqslant d} Y_j \leqslant X^{\beta+\epsilon}, \quad X^{\gamma-\epsilon} \leqslant \prod_{1\leqslant j\leqslant d} Z_j \leqslant X^{\gamma+\epsilon},$$

$$\prod_{1\leqslant j\leqslant d} X_j^j \leqslant X, \quad \prod_{1\leqslant j\leqslant d} Y_j^j \leqslant X, \quad X^{1-\epsilon^2} \leqslant \prod_{1\leqslant j\leqslant d} Z_j^j \leqslant X,$$

and pairwise coprime integers $1 \leqslant c_1, c_2, c_3 \leqslant X^\epsilon$, such that

$$S^*_{\alpha,\beta,\gamma}(X) \ll X^\epsilon B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}),$$

where

$$\begin{aligned} B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \#\Big\{(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbb{N}^{3d} : \ & x_i \sim X_i,\ y_i \sim Y_i,\ z_i \sim Z_i, \\ & c_1 \prod_{j\leqslant d} x_j^j + c_2 \prod_{j\leqslant d} y_j^j = c_3 \prod_{j\leqslant d} z_j^j, \\ & \gcd\left(c_1 \prod_{j\leqslant d} x_j,\ c_2 \prod_{j\leqslant d} y_j,\ c_3 \prod_{j\leqslant d} z_j\right) = 1\Big\} \end{aligned}$$

for $\mathbf{c} \in \mathbb{Z}^3$ *and* $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{R}^d_{>0}$.

Now we give some upper bounds for the integer points $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$. These lemmas are proved in [1] and they will be used to give combinatorial bounds for ν in next section.

Lemma 2.2. *(Fourier bound, [[1], Proposition 3.1]). Let* $d \geqslant 1$, $\epsilon > 0$ *and* $A \geqslant 1$ *be fixed. Let*

$$X_1, \ldots, X_d, Y_1, \ldots, Y_d, Z_1, \ldots, Z_d \geqslant 1.$$

Let $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{Z}^3$ *satisfy* $0 < |c_1|, |c_2|, |c_3| \leqslant \max_{1 \leqslant i \leqslant d}(X_i Y_i Z_i)^A$. *Then we have*

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \max_{1 \leqslant i \leqslant d}(X_i Y_i Z_i)^\epsilon \frac{\prod_{j \leqslant d} (X_j Y_j Z_j (Y_j + Z_j))^{\frac{1}{2}}}{\max_{i=1} \prod_{j \equiv 0 (\bmod i)} Z_j^{\frac{1}{2}}}.$$

Lemma 2.3. *(Geometry bound, [[1], Proposition 3.2]). Let* $d \geqslant 1$ *and* $\epsilon > 0$ *be fixed. Let*

$$X_1, \ldots, X_d, Y_1, \ldots, Y_d, Z_1, \ldots, Z_d \geqslant 1.$$

Let $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{Z}^3$ *have non–zero and pairwise coprime coordinates. Then we have*

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \max_{1 \leqslant i \leqslant d}(X_i Y_i Z_i)^\epsilon \min_{I, I', I'' \subset [d]} \left(\prod_{i \in I} X_i \prod_{i \in I'} Y_i \prod_{i \in I''} Z_i \right) \left(1 + \frac{\prod_{i \notin I} X_i^i \prod_{i \notin I'} Y_i^i \prod_{i \notin I''} Z_i^i}{\max\left(|c_1| \prod_i X_i^i, |c_2| \prod_i Y_i^i, |c_3| \prod_i Z_i^i\right)} \right).$$

Lemma 2.4. *(Determinant bound, [[1], Proposition 3.5]). Let* $d \geqslant 1$ *and let*

$$X_1, \ldots, X_d, Y_1, \ldots, Y_d, Z_1, \ldots, Z_d \geqslant 1.$$

Let $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{Z}^3_{\neq 0}$. *Then we have*

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \max_{1 \leqslant i \leqslant d}(X_i Y_i Z_i)^\epsilon \prod_{i \leqslant d}(X_i Y_i Z_i) \min_{p,q \geqslant 1} \left((X_p Y_q)^{-1} \min\left(X_p^{1/q} Y_q^{1/p} \right) \right).$$

Lemma 2.5. *(Thue bound, [[1], Proposition 3.6]). Let* $d \geqslant 1$ *and let*

$$X_1, \ldots, X_d, Y_1, \ldots, Y_d, Z_1, \ldots, Z_d \geqslant 1.$$

Let $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{Z}^3_{\neq 0}$. *Then we have*

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \max_{1 \leqslant i \leqslant d}(X_i Y_i Z_i)^\epsilon \prod_{i \leqslant d}(X_i Y_i Z_i) \min_{p \geqslant 2} \left(\prod_{\substack{j \leqslant d \\ p | j}} (X_j Y_j)^{-1} \right).$$

3. Upper bounds for ν

In this section we shall use all things proved above to bound $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$ for any pairwise coprime integers $1 \leqslant |c_1|, |c_2|, |c_3| \leqslant X^{\epsilon^2}$, any fixed $d \geqslant 1$ and any choice of $X_i, Y_i, Z_i \geqslant 1$ for $1 \leqslant i \leqslant d$ that satisfies conditions in Lemma 2.1. Moreover, we have

$$\alpha + \beta + \gamma \leqslant \lambda \leqslant 1 + \delta - \epsilon. \tag{6}$$

We define a_i, b_i, c_i by writing

$$X_i = X^{a_i}, \quad Y_i = X^{b_i}, \quad Z_i = X^{c_i} \tag{7}$$

for $i \leqslant d$ and $a_i = b_i = c_i = 0$ for $i > d$. We write $s_i = a_i + b_i + c_i$. By the conditions in Lemma 2.1, we can assume that

$$\sum_{i \leqslant d} i a_i, \ \sum_{i \leqslant d} i b_i \leqslant 1, \quad 1 - \epsilon^2 \leqslant \sum_{i \leqslant d} i c_i \leqslant 1. \tag{8}$$

By (6) and [[1], (1.2)], we can also assume that

$$\sum_{i \leqslant d}(a_i + b_i), \ \sum_{i \leqslant d}(a_i + c_i), \ \sum_{i \leqslant d}(b_i + c_i) \geqslant \theta - \epsilon^2 \tag{9}$$

and

$$\sum_{i \leqslant d} s_i \leqslant 1 + \delta - \epsilon. \tag{10}$$

We define

$$\nu = \frac{\log B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})}{\log X} + 2\epsilon^2. \tag{11}$$

Then we only need to show that

$$\nu \leqslant \theta. \tag{12}$$

Now we shall rewrite Lemmas 2.2–2.5 in terms of an upper bound for ν using parameters a_i, b_i, c_i.

Lemma 3.1. *(Fourier bound). We have*

$$\nu \leqslant \frac{1}{2}\left(1 + \delta + \sum_{i \leqslant d} \max(a_i, b_i) - \max_{m>1}(a_m, b_m)\right).$$

Lemma 3.2. *(Geometry bound). We have*

$$\nu \leqslant \delta + \min_{I, I', I'' \subset [d]} \left(\max\left(1, \sum_{i \in I} i a_i + \sum_{i \in I'} i b_i + \sum_{i \in I''} i c_i\right) - \sum_{i \in I} a_i - \sum_{i \in I'} b_i - \sum_{i \in I''} c_i \right)$$

or

$$\nu \leqslant 4\epsilon^2 + \min_{I, I', I'' \subset [d]} \left(\sum_{i \notin I} a_i + \sum_{i \notin I'} b_i + \sum_{i \notin I''} c_i + \max\left(0, \sum_{i \in I} i a_i + \sum_{i \in I'} i b_i + \sum_{i \in I''} i c_i - 1\right)\right).$$

Lemma 3.3. *(Determinant bound). We have*

$$\nu < \min_{p,q \geqslant 1} \left(1 + \delta - a_p - b_q + \min\left(\frac{a_p}{q}, \frac{b_q}{p}\right)\right).$$

Lemma 3.4. *(Thue bound). We have*

$$\nu < 1 + \delta - \max_{p \geqslant 2} \left(\sum_{p \mid i} (a_i + b_i) \right).$$

We first show that we can assume that

$$2\theta - 1 - \delta \leqslant \sum_{i \leqslant d} a_i,\ \sum_{i \leqslant d} b_i,\ \sum_{i \leqslant d} c_i \leqslant 1 - \theta + \delta - \frac{1}{2}\epsilon. \tag{13}$$

If $\sum_{i \leqslant d} c_i > 1 - \theta + \delta - \frac{1}{2}\epsilon$, then we have

$$\sum_{i \leqslant d} (a_i + b_i) \leqslant \theta - \frac{1}{2}\epsilon. \tag{14}$$

By [[1], (1.2)], Theorem 1.3 is proved. If $\sum_{i \leqslant d} c_i < 2\theta - 1 - \delta$ and all of the three sums are $\leqslant 1 - \theta + \delta - \frac{1}{2}\epsilon$, then we have

$$\sum_{i \leqslant d} (b_i + c_i) \leqslant (2\theta - 1 - \delta) + \left(1 - \theta + \delta - \frac{1}{2}\epsilon\right) = \theta - \frac{1}{2}\epsilon. \tag{15}$$

Again, Theorem 1.3 is proved by [[1], (1.2)].

4. Proof of Theorem 1.1

From now on, we ignore the presence of δ and ϵ in many places, since all the contributions of them can be bounded by ε. We define the parameters $\delta_a, \delta_b, \delta_c, \delta_{ab}, \delta_{ac}, \delta_{bc}, \delta_s$ by

$$\delta_a = \frac{1}{3} - \sum_{i \leqslant d} a_i, \quad \delta_b = \frac{1}{3} - \sum_{i \leqslant d} b_i, \quad \delta_c = \frac{1}{3} - \sum_{i \leqslant d} c_i, \tag{16}$$

$$\delta_{ab} = \delta_a + \delta_b, \quad \delta_{ac} = \delta_a + \delta_c, \quad \delta_{bc} = \delta_b + \delta_c, \quad \delta_s = \delta_a + \delta_b + \delta_c. \tag{17}$$

By (9), (16) and (17) we know that

$$\delta_{ab}, \delta_{ac}, \delta_{bc} \leqslant \frac{2}{3} - \theta. \tag{18}$$

By (13) and (16) we have

$$\theta - \frac{2}{3} \leqslant \delta_a, \delta_b, \delta_c \leqslant \frac{4}{3} - 2\theta. \tag{19}$$

By (10) and (16) we know that

$$1 - \delta_s \leqslant 1 + \delta \tag{20}$$

and

$$2\delta_s = \delta_{ab} + \delta_{ac} + \delta_{bc} \leqslant 2 - 3\theta. \tag{21}$$

Then, by (20) and (21) we have

$$-\delta < \delta_s \leqslant 1 - \frac{3}{2}\theta. \tag{22}$$

Note that these inequalities

$$\sum_{i\geqslant 2}(i-1)a_i \leqslant \frac{2}{3} + \delta_a, \quad \sum_{i\geqslant 3}(i-2)a_i \leqslant \frac{1}{3} + a_1 + 2\delta_a, \quad \sum_{i\geqslant 4}(i-3)a_i \leqslant 2a_1 + a_2 + 3\delta_a \tag{23}$$

follow by (8) and subtracting. Similar inequalities hold for b_i and c_i. By Lemma 3.4, we know that

$$\nu \leqslant 1 + \delta - \max_{p\geqslant 2} \sum_{p|i}(a_i + b_i)$$

and similar results hold for $a_i + c_i$ and $b_i + c_i$. Thus we can assume that

$$a_i + b_i, a_i + c_i, b_i + c_i < 1 - \theta \tag{24}$$

for every $i \geqslant 2$. Moreover, we can assume that

$$a_2 + b_2 + a_4 + b_4, a_2 + c_2 + a_4 + c_4, b_2 + c_2 + b_4 + c_4, < 1 - \theta. \tag{25}$$

Now, (24) and (25) imply that

$$s_2 + s_4, s_3, s_5 \leqslant \frac{3}{2}(1-\theta) = \frac{3}{2} - \frac{3}{2}\theta. \tag{26}$$

By (16) and (23) we also know that

$$\sum_{i\geqslant 1} s_i = 1 - \delta_s, \quad \sum_{i\geqslant 2}(i-1)s_i \leqslant 2 + \delta_s, \quad \sum_{i\geqslant 3}(i-2)s_i \leqslant 1 + s_1 + 2\delta_s, \quad \sum_{i\geqslant 4}(i-3)s_i \leqslant 2s_1 + s_2 + 3\delta_s. \tag{27}$$

If $s_1 + s_2 > 1 - \theta$, then by Lemma 3.2 and (26), we have

$$\begin{aligned}\nu &\leqslant \max(1, s_1 + 2s_2) - s_1 - s_2 + \delta \\ &= \max(1 - s_1 - s_2, s_2) + \delta \\ &< \max\left(\theta, \frac{3}{2} - \frac{3}{2}\theta\right) = \theta \end{aligned} \tag{28}$$

since $\theta > 0.6$. Now we can assume that $s_1 + s_2 \leqslant 1 - \theta$.

For any $i \geqslant 3$, let τ_i be an element in $\{a_i, b_i, c_i, a_i + b_i, a_i + c_i, b_i + c_i, s_i\}$. By Lemma 3.2 we know that

$$\begin{aligned}\nu &\leqslant \max(1, s_1 + 2s_2 + i\tau_i) - s_1 - s_2 - \tau_i + \delta \\ &= \max(1 - s_1 - s_2 - \tau_i, s_2 + (i-1)\tau_i) + \delta \end{aligned} \tag{29}$$

and

$$\begin{aligned}\nu &\leqslant \max(1, s_1 + 3\tau_3) - s_1 - \tau_3 + \delta \\ &= \max(1 - s_1 - \tau_3, 2\tau_3) + \delta. \end{aligned} \tag{30}$$

Combining (29) and (30), we know that $\nu \leqslant \theta$ if

$$\tau_3 \in \left(1 - \theta - s_1 - s_2,\ \frac{1}{2}\theta - \frac{1}{2}s_2\right) \cup \left(1 - \theta - s_1,\ \frac{1}{2}\theta\right). \tag{31}$$

By (23) we know that

$$\sum_{i\geqslant 4} a_i \leqslant \sum_{i\geqslant 4}(i-3)a_i \leqslant 2a_1 + a_2 + 3\delta_a \tag{32}$$

and

$$\sum_{i\geqslant 5} a_i \leqslant \frac{1}{2}\left(\sum_{i\geqslant 4}(i-3)a_i - a_4\right) \leqslant \frac{1}{2}\left(2a_1 + a_2 - a_4 + 3\delta_a\right). \tag{33}$$

By (16), these imply that

$$a_3 = \frac{1}{3} - \delta_a - a_1 - a_2 - \sum_{i\geqslant 4} a_i \geqslant \frac{1}{3} - 3a_1 - 2a_2 - 4\delta_a \tag{34}$$

and

$$a_3 = \frac{1}{3} - \delta_a - a_1 - a_2 - a_4 - \sum_{i\geqslant 5} a_i \geqslant \frac{1}{3} - 2a_1 - \frac{3}{2}a_2 - \frac{1}{2}a_4 - \frac{5}{2}\delta_a. \tag{35}$$

Note that (34) and (35) also hold for b_3 and c_3. Adding up these corresponding lower bounds, we have

$$s_3 \geqslant 1 - 3s_1 - 2s_2 - 4\delta_s \tag{36}$$

and

$$s_3 \geqslant 1 - 2s_1 - \frac{3}{2}s_2 - \frac{1}{2}s_4 - \frac{5}{2}\delta_s. \tag{37}$$

Now, we split the argument according to whether $s_2 \geqslant k$ or $s_2 < k$, where

$$k = \frac{49}{12} - \frac{23}{4}\theta \approx 0.2951. \tag{38}$$

Without loss of generality, we shall assume that $a_3 \geqslant b_3 \geqslant c_3$ in all that follows.

4.1. **Case 1: $s_2 \geqslant k$.** By the assumption $s_1 + s_2 \leqslant 1 - \theta$ we know that

$$s_1 \leqslant 1 - \theta - s_2 \leqslant 1 - \theta - k. \tag{39}$$

By (26) we know that

$$s_4 \leqslant \frac{3}{2} - \frac{3}{2}\theta - s_2 \leqslant \frac{3}{2} - \frac{3}{2}\theta - k. \tag{40}$$

4.1.1. *Subcase 1.1: $b_3 \leqslant 1 - \theta - s_1 - s_2$.* Because $c_3 \leqslant b_3$, we have

$$\begin{aligned} b_3 + c_3 \leqslant 2b_3 &\leqslant 2(1 - \theta - s_1 - s_2) \\ &= 2 - 2\theta - 2s_1 - 2s_2 \\ &\leqslant 2 - 2\theta - 2s_2. \end{aligned} \tag{41}$$

Note that we have

$$2 - 2\theta - 2s_2 \leqslant \frac{1}{2}\theta - \frac{1}{2}s_2 \tag{42}$$

since $s_2 \geqslant \frac{4}{3} - \frac{5}{3}\theta$. We also have

$$\frac{4}{3} - \frac{5}{3}\theta \leqslant k = \frac{49}{12} - \frac{23}{4}\theta. \tag{43}$$

Since $s_2 \geqslant k$ in this case, we have

$$b_3 + c_3 \leqslant \frac{1}{2}\theta - \frac{1}{2}s_2. \tag{44}$$

If $b_3 + c_3$ is in the interval (31), we get $\nu \leqslant \theta$. Otherwise we must have

$$b_3 + c_3 \leqslant 1 - \theta - s_1 - s_2. \tag{45}$$

(We will repeat similar discussions for many times in the following.) Now, by (35), (45) and (39) we can lower bound a_3 by

$$\begin{aligned} a_3 = s_3 - (b_3 + c_3) &\geqslant \left(1 - 2s_1 - \frac{3}{2}s_2 - \frac{1}{2}s_4 - \frac{5}{2}\delta_a\right) - (1 - \theta - s_1 - s_2) \\ &= \theta - \frac{5}{2}\delta_a - s_1 - \frac{1}{2}\left(s_2 + s_4\right) \\ &\geqslant \theta - \frac{5}{2}\delta_a - (1 - \theta - k) - \frac{1}{2}\left(\frac{3}{2} - \frac{3}{2}\theta\right) \end{aligned}$$

$$= \frac{11}{4}\theta - \frac{5}{2}\delta_a - \frac{7}{4} + k. \tag{46}$$

Now, (46) and (16) ensure that

$$\frac{11}{4}\theta - \frac{5}{2}\delta_a - \frac{7}{4} + k \leqslant a_3 \leqslant \frac{1}{3} - \delta_a. \tag{47}$$

By (47) and (19), we know that

$$\begin{aligned} \frac{11}{4}\theta + k - \frac{7}{4} - \frac{1}{3} &\leqslant \frac{3}{2}\delta_a \\ &\leqslant \frac{3}{2}\left(\frac{4}{3} - 2\theta\right) \\ \frac{11}{4}\theta + k - \frac{25}{12} &\leqslant 2 - 3\theta \\ k &\leqslant \frac{49}{12} - \frac{23}{4}\theta. \end{aligned} \tag{48}$$

Now (48) contradicts our assumption since the contributions of δ are omitted.

4.1.2. *Subcase 1.2: $b_3 > 1 - \theta - s_1 - s_2$.* By (31) and a similar discussion as in (44)–(45), we can assume that

$$b_3 \geqslant \frac{1}{2}\theta - \frac{1}{2}s_2. \tag{49}$$

By (23), we have

$$\sum_{i\geqslant 4}(i-2)b_i = \sum_{i\geqslant 3}(i-2)b_i - b_3 \leqslant \frac{1}{3} + b_1 - b_3 + 2\delta_b. \tag{50}$$

We also have

$$b_1 \leqslant s_1 \leqslant 1 - \theta - s_2. \tag{51}$$

Thus, by (50), (51) and (49) we have

$$\begin{aligned} \sum_{i\geqslant 4} b_i &\leqslant \frac{1}{2}\sum_{i\geqslant 4}(i-2)b_i \\ &\leqslant \frac{1}{2}\left(\frac{1}{3} + b_1 - b_3 + 2\delta_b\right) \\ &\leqslant \frac{1}{2}\left(\frac{1}{3} + (1-\theta-s_2) - \left(\frac{1}{2}\theta - \frac{1}{2}s_2\right) + 2\left(\frac{4}{3} - 2\theta\right)\right) \\ &= 2 - \frac{11}{4}\theta - \frac{1}{4}s_2. \end{aligned}$$

We want to show that

$$2 - \frac{11}{4}\theta - \frac{1}{4}s_2 < \frac{1}{2}\theta - \frac{1}{2}s_2. \tag{52}$$

Note that this is equivalent to

$$s_2 < 13\theta - 8. \tag{53}$$

Now, by the assumption above we know that $s_2 < 1 - \theta$, and we have

$$1 - \theta < 13\theta - 8 \tag{54}$$

since $\theta > \frac{9}{14} \approx 0.6428$. Combining (52)–(54) we know that

$$\sum_{i\geqslant 4} a_i,\ \sum_{i\geqslant 4} b_i \leqslant \frac{1}{2}\theta - \frac{1}{2}s_2 \tag{55}$$

and by (31) and a similar discussion as in (44)–(45) we can assume

$$a_4,\ b_4,\ a_5,\ b_5,\ a_6,\ b_6 \leqslant 1 - \theta - s_1 - s_2. \tag{56}$$

Now, by Lemma 3.1 we have

$$\nu < \frac{1}{2}\left(1+\delta+\sum_{i\leqslant d}\max(a_i,b_i)-\max(a_2,b_2)\right). \tag{57}$$

Using (16), this implies that

$$\begin{aligned}2\nu-1-\delta < \sum_{i\neq 2}\max(a_i,b_i) &\leqslant \sum_{i\neq 2,\ i\leqslant 6}\max(a_i,b_i)+\sum_{i\geqslant 7}(a_i+b_i)\\ &= \sum_{i\neq 2,\ i\leqslant 6}\max(a_i,b_i)+\frac{2}{3}-\delta_{ab}-\sum_{i\leqslant 6}(a_i+b_i)\\ &= \frac{2}{3}-\delta_{ab}-\sum_{i\neq 2,\ i\leqslant 6}\min(a_i,b_i)-(a_2+b_2).\end{aligned} \tag{58}$$

We then give a lower bound for a_2+b_2. By (23), we have

$$4\sum_{i\geqslant 7}a_i \leqslant \sum_{i\geqslant 7}(i-3)a_i = \left(\sum_{i\geqslant 4}(i-3)a_i\right)-a_4-2a_5-3a_6 = (2a_1+a_2+3\delta_a)-a_4-2a_5-3a_6, \tag{59}$$

whence

$$\frac{1}{3}-\delta_a = \sum_{i\leqslant 6}a_i+\sum_{i\geqslant 7}a_i \leqslant \sum_{i\leqslant 6}a_i+\frac{1}{4}(2a_1+a_2+3\delta_a-a_4-2a_5-3a_6) = \frac{1}{4}\sum_{i\leqslant 6}(7-i)a_i+\frac{3}{4}\delta_a. \tag{60}$$

Then we have

$$a_2 \geqslant \frac{4}{15}-\frac{1}{5}\sum_{i\neq 2,\ i\leqslant 6}(7-i)a_i-\frac{7}{5}\delta_a \tag{61}$$

and

$$b_2 \geqslant \frac{4}{15}-\frac{1}{5}\sum_{i\neq 2,\ i\leqslant 6}(7-i)b_i-\frac{7}{5}\delta_b. \tag{62}$$

Since $\min(a_3,b_3)=b_3$, we now have

$$\begin{aligned}2\nu-1-\delta &< \frac{2}{3}-\delta_{ab}-\sum_{i\neq 2,\ i\leqslant 6}\min(a_i,b_i)-\left(\frac{8}{15}-\frac{1}{5}\sum_{i\neq 2,\ i\leqslant 6}(7-i)(a_i+b_i)-\frac{7}{5}\delta_{ab}\right)\\ &\leqslant \frac{2}{15}+\frac{2}{5}\delta_{ab}+\frac{1}{5}\left(6\max(a_1,b_1)+\min(a_1,b_1)+4a_3-b_3\right.\\ &\qquad \left.+3\max(a_4,b_4)+2\max(a_5,b_5)+\max(a_6,b_6)\right).\end{aligned} \tag{63}$$

Using (24) and (49), we have

$$\begin{aligned}4a_3-b_3 &\leqslant 4(1-\theta-b_3)-b_3\\ &< 4\left(1-\theta-\left(\frac{1}{2}\theta-\frac{1}{2}s_2\right)\right)-\left(\frac{1}{2}\theta-\frac{1}{2}s_2\right)\\ &= \frac{5}{2}s_2+4-\frac{13}{2}\theta.\end{aligned} \tag{64}$$

Finally, by (63)–(64) we have

$$\begin{aligned}2\nu-1-\delta &< \frac{2}{15}+\frac{2}{5}\delta_{ab}+\frac{1}{5}\left(6\max(a_1,b_1)+\min(a_1,b_1)+4a_3-b_3\right.\\ &\qquad \left.+3\max(a_4,b_4)+2\max(a_5,b_5)+\max(a_6,b_6)\right)\\ &< \frac{2}{15}+\frac{2}{5}\delta_{ab}+\frac{1}{5}\left(6s_1+\left(\frac{5}{2}s_2+4-\frac{13}{2}\theta\right)+6(1-\theta-s_1-s_2)\right)\\ &< \frac{2}{15}+\frac{2}{5}\delta_{ab}+\frac{1}{5}\left(6s_1+\frac{5}{2}s_2+4-\frac{13}{2}\theta+6-6\theta-6s_1-6s_2\right)\end{aligned}$$

$$= \frac{32}{15} - \frac{5}{2}\theta - \frac{7}{10}s_2 + \frac{2}{5}\delta_{ab}$$
$$\leqslant \frac{32}{15} - \frac{5}{2}\theta - \frac{7}{10}\left(\frac{49}{12} - \frac{23}{4}\theta\right) + \frac{2}{5}\left(\frac{2}{3} - \theta\right) < 0.283, \tag{65}$$
$$\nu < \frac{1}{2}(1 + 0.283) < 0.65. \tag{66}$$

4.2. **Case 2:** $s_2 < k$**.** By (24) we have
$$2b_3 \leqslant a_3 + b_3 < 1 - \theta, \tag{67}$$
so that
$$b_3 \leqslant \frac{1}{2} - \frac{1}{2}\theta. \tag{68}$$
We want to show that
$$\frac{1}{2} - \frac{1}{2}\theta \leqslant \frac{1}{2}\theta - \frac{1}{2}s_2. \tag{69}$$
Note that (69) holds if
$$s_2 \leqslant 2\theta - 1. \tag{70}$$
Because we have
$$s_2 < k = \frac{49}{12} - \frac{23}{4}\theta \leqslant 2\theta - 1 \tag{71}$$
when $\theta \geqslant \frac{61}{93} \approx 0.6559$, we deduce that
$$c_3 \leqslant b_3 < \frac{1}{2}\theta - \frac{1}{2}s_2. \tag{72}$$
By (31) and a similar discussion as in (44)–(45), we can assume that
$$c_3 \leqslant b_3 \leqslant 1 - \theta - s_1 - s_2. \tag{73}$$
Then (36) gives that
$$\begin{aligned} a_3 = s_3 - (b_3 + c_3) &\geqslant (1 - 3s_1 - 2s_2 - 4\delta_s) - 2(1 - \theta - s_1 - s_2) \\ &= 2\theta - 1 - s_1 - 4\delta_s. \end{aligned} \tag{74}$$

We first prove the bound (12) in two cases:
$$a_3 \geqslant 2\theta - 1 \qquad \text{and} \qquad b_3 + c_3 < \frac{1}{2}\theta - \frac{1}{2}s_2.$$

4.2.1. *Subcase 2.1:* $a_3 \geqslant 2\theta - 1$. In this case we have
$$b_3,\ c_3 \leqslant 1 - \theta - a_3 \leqslant 2 - 3\theta. \tag{75}$$
Let
$$M = \max_{i \geqslant 4} \max(b_i, c_i). \tag{76}$$
If $M > 3 - \frac{9}{2}\theta$, by Lemma 3.3 we know that
$$\nu = 1 + \delta - a_3 - M + \min\left(\frac{M}{3}, \frac{a_3}{4}\right) \leqslant 1 + \delta - a_3 - \frac{2}{3}M \leqslant 1 - (2\theta - 1) - (2 - 3\theta) = \theta. \tag{77}$$
Thus we can assume that $\max(b_i, c_i) \leqslant 3 - \frac{9}{2}\theta$ for $i \geqslant 4$. Then we have
$$b_i + c_i \leqslant 2\max(b_i, c_i) \leqslant 6 - 9\theta \tag{78}$$
for $i \geqslant 4$. Moreover, by (8) and (16) we know that
$$\sum_{i \leqslant d} (i - 1)(b_i + c_i) \leqslant \frac{4}{3} + \delta_{bc}. \tag{79}$$
Using the second form of Lemma 3.2, we have
$$\nu \leqslant \epsilon + a_3 + b_3 + \min(b_4, c_4) + \sum_{i \geqslant 5} (b_i + c_i)$$

$$+\max\left(0,\ \sum_i is_i - 3(a_3+b_3) - 4\min(b_4,c_4) - \sum_{i\geqslant 5} i(b_i+c_i) - 1\right). \tag{80}$$

Now, define

$$\nu_1 = a_3 + b_3 + \min(b_4,c_4) + \sum_{i\geqslant 5}(b_i+c_i) \tag{81}$$

and

$$\nu_2 = \sum_i is_i - 2(a_3+b_3) - 3\min(b_4,c_4) - \sum_{i\geqslant 5}(i-1)(b_i+c_i) - 1. \tag{82}$$

Then by (80) we have

$$\nu \leqslant \max(\nu_1,\nu_2)+\epsilon. \tag{83}$$

By (18), (23), (67) and (78), we know that

$$\begin{aligned}
\nu_1 &= a_3+b_3+\min(b_4,c_4)+b_5+c_5+\sum_{i\geqslant 6}(b_i+c_i)\\
&\leqslant a_3+b_3+\min(b_4,c_4)+b_5+c_5+\frac{1}{5}\sum_{i\geqslant 6}(i-1)(b_i+c_i)\\
&\leqslant a_3+b_3+\frac{b_4+c_4}{2}+b_5+c_5+\frac{1}{5}\left(\frac{4}{3}+\delta_{bc}-(b_2+c_2)-2(b_3+c_3)-3(b_4+c_4)-4(b_5+c_5)\right)\\
&\leqslant a_3+b_3+\frac{b_4+c_4}{2}+b_5+c_5+\frac{1}{5}\left(\frac{4}{3}+\delta_{bc}-3(b_4+c_4)-4(b_5+c_5)\right)\\
&\leqslant a_3+b_3+\frac{1}{5}\left(\frac{4}{3}+\delta_{bc}\right)-\frac{b_4+c_4}{10}+\frac{b_5+c_5}{5}\\
&\leqslant a_3+b_3+\frac{1}{5}\left(\frac{4}{3}+\delta_{bc}+(b_5+c_5)\right)\\
&\leqslant (1-\theta)+\frac{1}{5}\left(\frac{4}{3}+\left(\frac{2}{3}-\theta\right)+(6-9\theta)\right)\\
&= \frac{13}{5}-3\theta < \theta-\epsilon
\end{aligned} \tag{84}$$

when $\theta > \frac{13}{20} = 0.65$.

Note that we have

$$\sum_i is_i - 1 \leqslant \sum_i is_i - \sum_i ia_i = \sum_i i(b_i+c_i) \tag{85}$$

by (8). Then by (16), (19), (75), (78), (85) and assumptions, for ν_2 we have

$$\begin{aligned}
\nu_2 &= \left(\sum_i is_i - 1\right) - 2(a_3+b_3) - 3\min(b_4,c_4) - \sum_{i\geqslant 5}(i-1)(b_i+c_i)\\
&\leqslant \left(\sum_i i(b_i+c_i) - \sum_{i\geqslant 5}(i-1)(b_i+c_i)\right) - 2(a_3+b_3) - 3\min(b_4,c_4)\\
&= \left(\sum_i (b_i+c_i) + \sum_{i\leqslant 4}(i-1)(b_i+c_i)\right) - 2(a_3+b_3) - 3\min(b_4,c_4)\\
&= \sum_i (b_i+c_i) + (b_2+c_2) + 2(b_3+c_3) + 3(b_4+c_4) - 2(a_3+b_3) - 3\min(b_4,c_4)\\
&= \sum_i (b_i+c_i) + (b_2+c_2) - 2(a_3-c_3) + 3\max(b_4,c_4)\\
&\leqslant \frac{2}{3} - \delta_{bc} + s_2 - 2a_3 + 2c_3 + 3\max(b_4,c_4)
\end{aligned}$$

$$\leqslant \frac{2}{3} - 2\left(\theta - \frac{2}{3}\right) + \left(\frac{49}{12} - \frac{23}{4}\theta\right) - 2(2\theta - 1) + 2(2 - 3\theta) + 3\left(3 - \frac{9}{2}\theta\right)$$
$$= \frac{253}{12} - \frac{125}{4}\theta < \theta - \epsilon \tag{86}$$
when $\theta > \frac{253}{387} \approx 0.6537$. Now by (83), (84) and (86), we get the desired result.

4.2.2. *Subcase 2.2:* $b_3 + c_3 < \frac{1}{2}\theta - \frac{1}{2}s_2$. Now by (31) and a similar discussion as in (44)–(45), we can assume that
$$b_3 + c_3 \leqslant 1 - \theta - s_1 - s_2. \tag{87}$$
By (26), (37) and (87) we know that
$$a_3 = s_3 - (b_3 + c_3) \geqslant \left(1 - 2s_1 - \frac{3}{2}s_2 - \frac{1}{2}s_4 - \frac{5}{2}\delta_s\right) - (1 - \theta - s_1 - s_2)$$
$$= \theta - \frac{5}{2}\delta_s - s_1 - \frac{1}{2}(s_2 + s_4)$$
$$\geqslant \theta - \frac{5}{2}\left(1 - \frac{3}{2}\theta\right) - \frac{1}{2}\left(\frac{3}{2} - \frac{3}{2}\theta\right) - s_1$$
$$= \left(\frac{11}{2}\theta - \frac{13}{4}\right) - s_1. \tag{88}$$
Note that
$$\frac{11}{2}\theta - \frac{13}{4} > 1 - \theta \tag{89}$$
since $\theta > \frac{17}{26} \approx 0.6538$, we have
$$a_3 > 1 - \theta - s_1. \tag{90}$$
By (31) and a similar discussion as in (44)–(45), we can assume that
$$a_3 > \frac{1}{2}\theta. \tag{91}$$
Note that
$$\frac{1}{2}\theta > 2\theta - 1 \tag{92}$$
since $\theta < \frac{2}{3}$, we have
$$a_3 > 2\theta - 1. \tag{93}$$
Now by the discussions in *Subcase 2.1*, we get the desired result.

Now, we will prove **Case 2** by showing that (12) holds for any $(s_1, s_2) \in [0, 1]^2$ (with the assumption $s_2 < k$). We shall consider the following 4 subcases:
$$\begin{cases} (2.3) & 4s_1 + 3s_2 > 4 - 5\theta, \\ (2.4) & 4s_1 + s_2 < 37\theta - 24, \\ (2.5) & 6 - 9\theta \leqslant s_2 \leqslant \frac{7}{3}\theta - \frac{4}{3}, \\ (2.6) & 2s_1 - s_2 > 2 - 3\theta. \end{cases} \tag{94}$$
Note that every point in $[0, 1]^2$ is covered by one of the above 4 subcases when $\theta \geqslant \frac{23}{35} \approx 0.6571$. If $\theta < \frac{23}{35}$, there are two triangles that are not covered by any of the cases.

4.2.3. *Subcase 2.3:* $4s_1 + 3s_2 > 4 - 5\theta$. By (73) we know that
$$b_3 + c_3 \leqslant 2 - 2\theta - 2s_1 - 2s_2. \tag{95}$$
By the assumption we know that
$$-2s_1 - 2s_2 < \frac{5}{2}\theta - 2 - \frac{1}{2}s_2 \tag{96}$$
Now, by (95) and (96) we have
$$b_3 + c_3 < 2 - 2\theta + \frac{5}{2}\theta - 2 - \frac{1}{2}s_2 = \frac{1}{2}\theta - \frac{1}{2}s_2. \tag{97}$$

Hence *Subcase 2.2* completes the proof.

4.2.4. *Subcase 2.4:* $4s_1 + s_2 < 37\theta - 24$. By (24) and (74) we know that

$$\begin{aligned} b_3,\ c_3 &\leqslant 1 - \theta - a_3 \\ &\leqslant 1 - \theta - (2\theta - 1 - s_1 - 4\delta_s) \\ &= 2 - 3\theta + s_1 + 4\delta_s \\ &= 2 - 3\theta + s_1 + 4\left(1 - \frac{3}{2}\theta\right) \\ &= 6 - 9\theta + s_1, \qquad (98) \\ b_3 + c_3 &\leqslant 12 - 18\theta + 2s_1. \qquad (99) \end{aligned}$$

By the assumption we know that

$$2s_1 < \frac{37}{2}\theta - 12 - \frac{1}{2}s_2. \qquad (100)$$

Now, by (99) and (100) we have

$$b_3 + c_3 \leqslant 12 - 18\theta + 2s_1 < 12 - 18\theta + \frac{37}{2}\theta - 12 - \frac{1}{2}s_2 = \frac{1}{2}\theta - \frac{1}{2}s_2. \qquad (101)$$

Hence *Subcase 2.2* completes the proof.

4.2.5. *Subcase 2.5:* $6 - 9\theta \leqslant s_2 \leqslant \frac{7}{3}\theta - \frac{4}{3}$. By (22) and (74) we know that

$$a_3 \geqslant 2\theta - 1 - s_1 - 4\delta_s \geqslant 2\theta - 1 - s_1 - 4\left(1 - \frac{3}{2}\theta\right) \geqslant 8\theta - 5 - s_1. \qquad (102)$$

If $6 - 9\theta \leqslant s_2$, we have

$$a_3 \geqslant 8\theta - 5 - s_1 \geqslant 8\theta - 5 - s_1 + (6 - 9\theta - s_2) = 1 - \theta - s_1 - s_2. \qquad (103)$$

By (31) and a similar discussion as in (44)–(45) we can assume

$$a_3 \geqslant \frac{1}{2}\theta - \frac{1}{2}s_2. \qquad (104)$$

Now, (24) yields

$$\begin{aligned} b_3,\ c_3 &\leqslant 1 - \theta - a_3 \\ &\leqslant 1 - \theta - \left(\frac{1}{2}\theta - \frac{1}{2}s_2\right) \\ &= 1 - \frac{3}{2}\theta + \frac{1}{2}s_2, \qquad (105) \\ b_3 + c_3 &\leqslant 2 - 3\theta + s_2. \qquad (106) \end{aligned}$$

If $s_2 \leqslant \frac{7}{3}\theta - \frac{4}{3}$, we have

$$s_2 \leqslant \frac{7}{2}\theta - 2 - \frac{1}{2}s_2. \qquad (107)$$

Now, by (106) and (107) we have

$$b_3 + c_3 \leqslant 2 - 3\theta + s_2 \leqslant 2 - 3\theta + \frac{7}{2}\theta - 2 - \frac{1}{2}s_2 = \frac{1}{2}\theta - \frac{1}{2}s_2. \qquad (108)$$

Hence *Subcase 2.2* completes the proof.

4.2.6. *Subcase 2.6: $2s_1 - s_2 > 2 - 3\theta$.* In this case the two intervals in (31) overlap, hence we have $\nu \leqslant \theta$ if

$$\tau_3 \in \left(1 - \theta - s_1 - s_2,\ \frac{1}{2}\theta\right). \tag{109}$$

In *Subcases 2.3 and 2.4* we prove the cases $4s_1 + 3s_2 > 4 - 5\theta$ and $4s_1 + s_2 < 37\theta - 24$, so we can assume that

$$4s_1 + 3s_2 \leqslant 4 - 5\theta \tag{110}$$

and

$$4s_1 + s_2 \geqslant 37\theta - 24. \tag{111}$$

By (110) we have

$$s_1 \leqslant \frac{4 - 5\theta}{4} = 1 - \frac{5}{4}\theta \tag{112}$$

and

$$s_2 \leqslant \frac{1}{3}(4 - 5\theta - 4s_1). \tag{113}$$

Now, (111) and (113) give that

$$\begin{aligned} s_2 &\leqslant \frac{1}{3}(4 - 5\theta - 4s_1) \\ &\leqslant \frac{1}{3}(4 - 5\theta - (37\theta - 24 - s_2)) \\ &= \frac{1}{3}(28 - 42\theta + s_2), &(114) \\ s_2 &\leqslant 14 - 21\theta. &(115) \end{aligned}$$

Note that

$$14 - 21\theta < \frac{7}{3}\theta - \frac{4}{3} \tag{116}$$

when $\theta > \frac{23}{35} \approx 0.6571$, we have

$$s_2 \leqslant \frac{7}{3}\theta - \frac{4}{3}. \tag{117}$$

If $6 - 9\theta \leqslant s_2 \leqslant \frac{7}{3}\theta - \frac{4}{3}$, by *Subcase 2.5* we have the desired result. Otherwise we have

$$s_2 \leqslant 6 - 9\theta. \tag{118}$$

By the result proved in *Subcase 2.1*, we can also assume that $a_3 < 2\theta - 1$. Since $2\theta - 1 < \frac{1}{2}\theta$ when $\theta < \frac{2}{3}$, by (31) and a similar discussion as in (44)–(45), we have

$$a_3 < 1 - \theta - s_1 - s_2. \tag{119}$$

We shall consider the following two cases.

Subcase 2.6.1: $b_3 + c_3 < 1 - \theta - s_1 - s_2$. In this case we have, by the assumption and (119),

$$s_3 < 2 - 2\theta - 2s_1 - 2s_2. \tag{120}$$

Now, by (36) and (120) we have

$$1 - 3s_1 - 2s_2 - 4\delta_s < 2 - 2\theta - 2s_1 - 2s_2 \tag{121}$$

and thus

$$2\theta - 1 - 4\delta_s < s_1. \tag{122}$$

By (112) and (122), we have

$$2\theta - 1 + 4\delta < 1 - \frac{5}{4}\theta, \tag{123}$$

which holds true only when $\theta < \frac{8}{13} \approx 0.6154$. This contradicts with our value of θ.

Subcase 2.6.2: $b_3 + c_3 \geqslant 1 - \theta - s_1 - s_2$. By the assumption and (109), after a similar discussion as in (44)–(45) we have

$$b_3 + c_3 \geqslant \frac{1}{2}\theta. \tag{124}$$

Since $b_3 > c_3$, we have $b_3 > \frac{1}{4}\theta$. Now $a_3 > b_3$ yields

$$\frac{1}{4}\theta < b_3 < a_3 < 1 - \theta - s_1 - s_2, \tag{125}$$

$$s_1 + s_2 < 1 - \frac{5}{4}\theta. \tag{126}$$

By Lemma 3.1, we know that

$$\nu \leqslant \frac{1}{2}\left(1 + \delta + \sum_{i\neq 3} \max(a_i, b_i)\right). \tag{127}$$

Note that

$$\sum_i (\max(a_i, b_i) + \min(a_i, b_i)) = \sum_i (a_i + b_i) = \frac{2}{3} - \delta_{ab}, \tag{128}$$

we have

$$\begin{aligned}
2\nu - 1 - \delta &\leqslant \sum_{i\neq 3} \max(a_i, b_i) \\
&\leqslant \max(a_1, b_1) + \max(a_2, b_2) + \sum_{i\geqslant 4} (\max(a_i, b_i) + \min(a_i, b_i)) \\
&= \max(a_1, b_1) + \max(a_2, b_2) + \left(\frac{2}{3} - \delta_{ab} - \sum_{i\leqslant 3} (\max(a_i, b_i) + \min(a_i, b_i))\right) \\
&= \frac{2}{3} - \delta_{ab} - \min(a_1, b_1) - \min(a_2, b_2) - \min(a_3, b_3) - \max(a_3, b_3).
\end{aligned} \tag{129}$$

By (34) we know that

$$3a_1 + 2a_2 + a_3 \geqslant \frac{1}{3} - 4\delta_a \tag{130}$$

and

$$3b_1 + 2b_2 + b_3 \geqslant \frac{1}{3} - 4\delta_b. \tag{131}$$

Thus,

$$3\min(a_1, b_1) \geqslant \frac{1}{3} - 2\max(a_2, b_2) - \max(a_3, b_3) - 4\max(\delta_a, \delta_b). \tag{132}$$

Now, by (19), (125) and (132) we have

$$\begin{aligned}
2\nu - 1 - \delta &\leqslant \frac{2}{3} - \delta_{ab} - \min(a_1, b_1) - \min(a_2, b_2) - \min(a_3, b_3) - \max(a_3, b_3) \\
&\leqslant \frac{2}{3} - \delta_{ab} - \frac{1}{3}\left(\frac{1}{3} - 2\max(a_2, b_2) - \max(a_3, b_3) - 4\max(\delta_a, \delta_b)\right) \\
&\quad - \min(a_2, b_2) - \min(a_3, b_3) - \max(a_3, b_3) \\
&\leqslant \frac{5}{9} + \frac{2}{3}\max(a_2, b_2) + \left(\frac{4}{3}\max(\delta_a, \delta_b) - \delta_{ab}\right) - \left(\min(a_3, b_3) + \frac{2}{3}\max(a_3, b_3)\right) \\
&\leqslant \frac{5}{9} + \frac{2}{3}\max(a_2, b_2) + \frac{1}{3}\max(\delta_a, \delta_b) - \min(\delta_a, \delta_b) - \frac{5}{3}\min(a_3, b_3) \\
&\leqslant \frac{5}{9} + \frac{2}{3}\max(a_2, b_2) + \frac{1}{3}\left(\frac{4}{3} - 2\theta\right) - \left(\theta - \frac{2}{3}\right) - \frac{5}{3}\left(\frac{1}{4}\theta\right) \\
&= \frac{5}{3} - \frac{25}{12}\theta + \frac{2}{3}\max(a_2, b_2),
\end{aligned} \tag{133}$$

$$\nu \leqslant \frac{4}{3} - \frac{25}{24}\theta + \frac{1}{3}\max(a_2, b_2) + \frac{1}{2}\delta. \tag{134}$$

By (134), we know that (12) holds if we have

$$\max(a_2, b_2) < \frac{49}{8}\theta - 4. \tag{135}$$

Now we assume that

$$\max(a_2, b_2) \geqslant \frac{49}{8}\theta - 4. \tag{136}$$

By similar arguments as above, we also have

$$\max(a_2, c_2) \geqslant \frac{49}{8}\theta - 4 \tag{137}$$

and

$$\max(b_2, c_2) \geqslant \frac{49}{8}\theta - 4, \tag{138}$$

which mean that at least two of a_2, b_2, c_2 are $\geqslant \frac{49}{8}\theta - 4$, but then we have

$$s_2 \geqslant 2\left(\frac{49}{8}\theta - 4\right) = \frac{49}{4}\theta - 8, \tag{139}$$

which is larger than $6 - 9\theta$ when $\theta > \frac{56}{85}$ and thus contradicts with (118). That is why we stop at this point.

Finally, combining all above cases, Theorem 1.3 is proved.

References

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