ON ALMOST PRIMES IN PIATETSKI–SHAPIRO SEQUENCES

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ABSTRACT. The author proves that for $0.9985 < \gamma < 1$, there exist infinitely many primes p such that $[p^{1/\gamma}]$ has at most 5 prime factors counted with multiplicity. This gives an improvement upon the previous results of Banks–Guo–Shparlinski and Xue–Li–Zhang.

CONTENTS

1. INTRODUCTION

Let P_r denotes an integer with at most r prime factors counted with multiplicity, p denotes a prime and $\gamma \in (0,1)$. The Piatetski–Shapiro sequences are sequences of the form

$$\mathscr{N}_\gamma = \left\{ [n^{1/\gamma}] : n \in \mathbb{N}^+ \right\}. \tag{1}$$

In 1953, Piatetski–Shapiro [3] established that there are infinitely many primes in $\mathscr{N}_\gamma$ if $\gamma \in \left(\frac{11}{12}, 1\right)$. This range of γ has been improved by many authors, and the best record now is due to Li , where he showed that the same result holds for any $\gamma \in \left(\frac{775}{919}, 1\right)$.

In 1987, Balog [1] considered the following subsequence of $\mathscr{N}_\gamma$:

$$\mathscr{P}_\gamma = \left\{ [p^{1/\gamma}] : p \text{ prime} \right\}. \tag{2}$$

He showed that for almost all $\gamma \in (0,1)$, we have

$$\limsup_{x\to\infty} \frac{\sum_{\substack{q \leqslant x \\ q \in \mathscr{P}_\gamma \\ q \text{ prime}}} 1}{x^\gamma \gamma^{-1} (\log x)^{-2}} \geqslant 1, \tag{3}$$

but this result gives no information for any specific choice of γ.

In 2016, Banks, Guo and Shparlinski [2] generalized and enhanced Balog's result. They showed explicitly that for every $\gamma \in (0,1) \setminus \left\{ \frac{1}{z} : z \in \mathbb{Z}^+ \right\}$, there exist infinitely many $q \in \mathscr{P}_\gamma$ such that $q = P_{R(\gamma)}$ for some finite $R(\gamma)$. As a part of their result, they showed that there are infinitely many $q \in \mathscr{P}_\gamma$ with $q = P_8$ for any $\gamma \in (0.9505, 1)$. In 2024, Xue, Li and Zhang [4] improved this to $q = P_7$ for any $\gamma \in (0.989, 1)$. In this paper, we shall further improve this to $q = P_5$ when γ is near 1.

Theorem 1.1. *There are infinitely many $q \in \mathscr{P}_\gamma$ with $q = P_5$ for any $\gamma \in (0.9985, 1)$. Moreover, the following estimates*

$$\sum_{\substack{q \leqslant x \\ q \in \mathscr{P}_\gamma \\ \Omega(q) \leqslant 5}} 1 \gg \frac{x^\gamma}{(\log x)^2}$$

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holds for all sufficiently large x provided that $\gamma \in (0.9985, 1)$.

Throughout this paper, we always suppose that x is a sufficiently large integer, ε and η are sufficiently small positive numbers and $0 < \gamma < 1$. The letter p, with or without subscript, is reserved for prime numbers. Let $\xi = \frac{140\gamma - 99}{270} - \eta$, $u = \frac{1}{\xi} + \varepsilon$ and $\lambda = \frac{1}{9-u-\varepsilon}$. We define the set $\mathcal{A}$ as

$$\mathcal{A} = \{a : a \leqslant x, a \in \mathscr{P}_\gamma\}$$

and we put

$$\mathcal{A}_d = \{a : ad \in \mathcal{A}\}, \quad P(z) = \prod_{p<z} p, \quad S(\mathcal{A}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z))=1}} 1.$$

2. Two combinatorial lemmas

In this section we shall prove two combinatorial lemmas which will be used to handle some error terms after performing Chen's switching principle on some sieve functions.

Lemma 2.1. *For any positive numbers $t_1, t_2, t_3, t_4, t_5, t_6, t_7$ with*

$$\frac{1}{17.41} \leqslant t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < t_7 \quad \textit{and} \quad t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 = 1,$$

There exists some $I \subset \{1, 2, 3, 4, 5, 6, 7\}$ such that

$$0.611797 \leqslant \sum_{i \in I} t_i \leqslant 0.787393.$$

Proof. Consider the following cases:

1. $0.611797 \leqslant t_4 + t_5 + t_6 + t_7 \leqslant 0.787393$. Take $I = \{4, 5, 6, 7\}$.
2. $t_4 + t_5 + t_6 + t_7 > 0.787393$. Now we have

$$t_1 + t_2 + t_3 < 0.212607.$$

Since

$$\frac{3}{17.41} \leqslant 3t_1 < t_1 + t_2 + t_3,$$

we have

$$0.787393 < t_4 + t_5 + t_6 + t_7 < 1 - \frac{3}{17.41} < 0.82769.$$

Now

$$\frac{1}{17.41} < t_4 < \frac{1}{4}(t_4 + t_5 + t_6 + t_7) = 0.2069225,$$

we have

$$0.5804705 = 0.787393 - 0.2069225 < t_5 + t_6 + t_7 < 0.82769 - \frac{1}{17.41} < 0.770252.$$

If $0.611797 \leqslant t_5 + t_6 + t_7 < 0.770252$, take $I = \{5, 6, 7\}$. Otherwise we have

$$0.5804705 < t_5 + t_6 + t_7 < 0.611797.$$

Since

$$0.05743 < \frac{1}{17.41} \leqslant t_1 < \frac{1}{7}(t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7) = \frac{1}{7} < 0.143,$$

we have

$$0.6379005 = 0.05743 + 0.5804705 < t_1 + t_5 + t_6 + t_7 < 0.611797 + 0.143 = 0.754797.$$

Take $I = \{1, 5, 6, 7\}$.

3. $t_4 + t_5 + t_6 + t_7 < 0.611797$. Now we have

$$\frac{4}{17.41} < t_4 + t_5 + t_6 + t_7 < 0.611797$$

and

$$0.388203 = 1 - 0.611797 < t_1 + t_2 + t_3.$$

Hence

$$0.1294 < \frac{0.388203}{3} < \frac{1}{3}(t_1 + t_2 + t_3) < t_3 < t_4 < t_5.$$

Note that

$$t_1 + t_2 + t_3 < \frac{1}{2}(t_1 + t_2 + t_3 + t_4 + t_5 + t_6) < \frac{1}{2} \times \frac{6}{7} = \frac{3}{7} < 0.4286$$

and

$$t_4 + t_5 < \frac{1}{2}(t_4 + t_5 + t_6 + t_7) < \frac{0.611797}{2} < 0.3059,$$

we have

$$0.647 < 0.388203 + 2 \times 0.1294 < t_1 + t_2 + t_3 + t_4 + t_5$$

and

$$t_1 + t_2 + t_3 + t_4 + t_5 < 0.4286 + 0.3059 = 0.7345.$$

Take $I = \{1, 2, 3, 4, 5\}$.

Combining all above cases, Lemma 2.1 is proved. □

Lemma 2.2. *For any positive numbers $t_1, t_2, t_3, t_4, t_5, t_6$ with*

$$\frac{1}{17.41} \leqslant t_1 < t_2 < t_3 < t_4 < t_5 < t_6 \quad \textit{and} \quad t_1 + t_2 + t_3 + t_4 + t_5 + t_6 = 1,$$

There exists some $I \subset \{1, 2, 3, 4, 5, 6\}$ such that

$$0.611797 \leqslant \sum_{i \in I} t_i \leqslant 0.787393.$$

Proof. Consider the following cases:

1. $0.611797 \leqslant t_4 + t_5 + t_6 \leqslant 0.787393$. Take $I = \{4, 5, 6\}$.
2. $t_4 + t_5 + t_6 > 0.787393$. Now we have

$$\frac{3}{17.41} < t_1 + t_2 + t_3 < 1 - 0.787393 = 0.212607$$

and

$$0.787393 < t_4 + t_5 + t_6 < 1 - \frac{3}{17.41} < 0.8277.$$

We also have

$$t_5 + t_6 < 1 - \frac{4}{17.41} < 0.7703.$$

If $0.611797 \leqslant t_5 + t_6 < 0.7703$, take $I = \{5, 6\}$. Otherwise we have

$$t_5 + t_6 < 0.611797.$$

Since

$$0.787393 < t_4 + t_5 + t_6,$$

we have

$$0.175596 < t_4 < t_5.$$

Note that

$$t_4 + t_5 < \frac{2}{3}(t_4 + t_5 + t_6) < \frac{2}{3}0.8277 = 0.5518,$$

we have

$$0.351192 = 2 \times 0.175596 < t_4 + t_5 < 0.5518.$$

Since

$$0.1723 < \frac{3}{17.41} < t_1 + t_2 + t_3 < 0.212607,$$

we have

$$0.523492 = 0.1723 + 0.351192 < t_1 + t_2 + t_3 + t_4 + t_5 < 0.212607 + 0.5518 = 0.764407.$$

If $0.611797 \leqslant t_1 + t_2 + t_3 + t_4 + t_5 < 0.764407$, take $I = \{1, 2, 3, 4, 5\}$. Otherwise we have

$$0.523492 < t_1 + t_2 + t_3 + t_4 + t_5 < 0.611797,$$

which means that

$$0.388203 = 1 - 0.611797 < t_6 < 1 - 0.523492 = 0.476508.$$

If there exists $i \in \{1,2,3,4,5\}$ such that $0.212667 \leqslant t_i \leqslant 0.388203$, then we can take $I = \{1, \cdots, 6, \text{ except } i\}$. Now we consider the following 3 subcases:

2.1. At least two of t_1, t_2, t_3, t_4, t_5 are larger than 0.388203. Now we have

$$1.16 < 3 \times 0.388203 < t_1 + t_2 + t_3 + t_4 + t_5 + t_6 = 1,$$

which is a contradiction.

2.2. One of t_1, t_2, t_3, t_4, t_5 is larger than 0.388203. Now we must have

$$\frac{1}{17.41} \leqslant t_1 < t_2 < t_3 < t_4 < 0.212667 < 0.388203 < t_5.$$

Hence

$$t_1 + t_2 + t_3 + t_4 < 1 - (t_5 + t_6) < 1 - 2 \times 0.388203 = 0.223594.$$

If $0.212667 \leqslant t_1 + t_2 + t_3 + t_4 < 0.223594$, take $I = \{5, 6\}$. Otherwise we have

$$t_1 + t_2 + t_3 + t_4 < 0.212667.$$

But since

$$0.2297 < \frac{4}{17.41} < t_1 + t_2 + t_3 + t_4,$$

we get

$$0.2297 < t_1 + t_2 + t_3 + t_4 < 0.212667,$$

which is a contradiction.

2.3. None of t_1, t_2, t_3, t_4, t_5 is larger than 0.388203. Now we must have

$$\frac{1}{17.41} \leqslant t_1 < t_2 < t_3 < t_4 < t_5 < 0.212667$$

and

$$0.351192 < t_4 + t_5.$$

Hence

$$\frac{3}{17.41} < t_1 + t_2 + t_3 < 1 - (t_4 + t_5 + t_6) < 1 - 0.351192 - 0.388203 = 0.260605.$$

If $0.212667 \leqslant t_1 + t_2 + t_3 < 0.260605$, take $I = \{4, 5, 6\}$. Otherwise we have

$$\frac{3}{17.41} < t_1 + t_2 + t_3 < 0.212667.$$

Now we have

$$\frac{1}{17.41} \leqslant t_1 < \frac{1}{3}(t_1 + t_2 + t_3) < 0.0709.$$

Since

$$0.175596 < t_4 < 0.212607,$$

we have

$$0.233 < \frac{1}{17.41} + 0.175596 < t_1 + t_4 < 0.0709 + 0.212607 < 0.3.$$

Take $I = \{2, 3, 5, 6\}$.

3. $t_4 + t_5 + t_6 < 0.611797$. Now we have

$$0.5 = \frac{1}{2}(t_1 + t_2 + t_3 + t_4 + t_5 + t_6) < t_4 + t_5 + t_6 < 0.611797$$

and

$$\frac{1}{17.41} \leqslant t_1 < \frac{1}{6}(t_1 + t_2 + t_3 + t_4 + t_5 + t_6) = \frac{1}{6}.$$

Hence

$$0.5574 < \frac{1}{17.41} + 0.5 < t_1 + t_4 + t_5 + t_6 < \frac{1}{6} + 0.611797 < 0.7784.$$

If $0.611797 \leqslant t_1 + t_4 + t_5 + t_6 < 0.7784$, take $I = \{1, 4, 5, 6\}$. Otherwise we have

$$0.5574 < t_1 + t_4 + t_5 + t_6 < 0.611797.$$

Since

$$\frac{2}{17.41} \leqslant t_1 + t_2 < \frac{1}{3}(t_1 + t_2 + t_3 + t_4 + t_5 + t_6) = \frac{1}{3},$$

we have

$$0.6148 < \frac{2}{17.41} + 0.5 < t_1 + t_2 + t_4 + t_5 + t_6 < \frac{1}{3} + 0.611797 < 0.94514.$$

If $0.6148 < t_1 + t_2 + t_4 + t_5 + t_6 \leqslant 0.787393$, take $I = \{1, 2, 4, 5, 6\}$. Otherwise we have

$$0.787393 < t_1 + t_2 + t_4 + t_5 + t_6 < 0.94514.$$

Hence

$$\frac{1}{17.41} < t_3 < 1 - 0.787393 = 0.212607.$$

Note that

$$t_1 + t_4 + t_5 + t_6 < 0.611797,$$

we have

$$0.175596 = 0.787393 - 0.611797 < t_2.$$

Hence

$$0.87798 = 5 \times 0.175596 < 5t_2 < t_2 + t_3 + t_4 + t_5 + t_6$$

and

$$\frac{1}{17.41} \leqslant t_1 < 1 - 0.87798 = 0.12202.$$

Since

$$0.175596 < t_2 < t_3 < 0.212607,$$

we have

$$0.22 < \frac{1}{17.41} + 0.175596 < t_1 + t_3 < 0.12202 + 0.212607 < 0.34.$$

Take $I = \{2, 4, 5, 6\}$.

Combining all above cases, Lemma 2.2 is proved. □

3. Proof of Theorem 1.1

Now we follow the discussion in [4]. Consider the following weighted sum

$$W\left(\mathcal{A}, x^{\frac{1}{17.41}}\right) = \sum_{\substack{a \in \mathcal{A} \\ \left(a, P\left(x^{\frac{1}{17.41}}\right)\right)=1}} \left(1 - \lambda \sum_{\substack{x^{\frac{1}{17.41}} \leqslant p < x^{\frac{1}{u}} \\ p \mid a}} \left(1 - \frac{u \log p}{\log x}\right)\right) = \sum_{\substack{a \in \mathcal{A} \\ \left(a, P\left(x^{\frac{1}{17.41}}\right)\right)=1}} \mathscr{W}_a. \tag{4}$$

We have

$$\begin{aligned} W\left(\mathcal{A}, x^{\frac{1}{17.41}}\right) = & \sum_{\substack{a \in \mathcal{A} \\ \left(a, P\left(x^{\frac{1}{17.41}}\right)\right)=1 \\ \Omega(a) \leqslant 5}} \mathscr{W}_a + \sum_{\substack{a \in \mathcal{A} \\ \left(a, P\left(x^{\frac{1}{17.41}}\right)\right)=1 \\ \Omega(a)=6 \\ \mu(a) \neq 0}} \mathscr{W}_a + \sum_{\substack{a \in \mathcal{A} \\ \left(a, P\left(x^{\frac{1}{17.41}}\right)\right)=1 \\ \Omega(a)=7 \\ \mu(a) \neq 0}} \mathscr{W}_a \\ & + \sum_{\substack{a \in \mathcal{A} \\ \left(a, P\left(x^{\frac{1}{17.41}}\right)\right)=1 \\ \Omega(a)=8 \\ \mu(a) \neq 0}} \mathscr{W}_a + \sum_{\substack{a \in \mathcal{A} \\ \left(a, P\left(x^{\frac{1}{17.41}}\right)\right)=1 \\ 6 \leqslant \Omega(a) \leqslant 8 \\ \mu(a)=0}} \mathscr{W}_a + \sum_{\substack{a \in \mathcal{A} \\ \left(a, P\left(x^{\frac{1}{17.41}}\right)\right)=1 \\ \Omega(a) \geqslant 9}} \mathscr{W}_a \\ = & W_1 + W_2 + W_3 + W_4 + W_5 + W_6. \end{aligned} \tag{5}$$

Our aim is to show that $W_1 > 0$.

Trivially, we have $\mathscr{W}_a < 0$ for $\Omega(a) \geqslant 9$. We also have $W_5 \ll x^{1-\frac{1}{17.41}+\varepsilon}$. Thus,

$$\begin{aligned} W_1 &= W\left(\mathcal{A}, x^{\frac{1}{17.41}}\right) - W_2 - W_3 - W_4 - W_5 - W_6 \\ &= W\left(\mathcal{A}, x^{\frac{1}{17.41}}\right) - W_2 - W_3 - W_4 - W_6 + O\left(x^{1-10^{-10}}\right) \end{aligned}$$

$$> W\left(\mathcal{A}, x^{\frac{1}{17.41}}\right) - W_2 - W_3 - W_4 + O\left(x^{1-10^{-10}}\right). \tag{6}$$

By the same arguments as in [4], we have

$$\begin{aligned} W\left(\mathcal{A}, x^{\frac{1}{17.41}}\right) &= S\left(\mathcal{A}, x^{\frac{1}{17.41}}\right) - \lambda \sum_{x^{\frac{1}{17.41}} \leqslant p < x^{\frac{1}{u}}} \left(1 - \frac{u \log p}{\log x}\right) S\left(\mathcal{A}_p, x^{\frac{1}{17.41}}\right) \\ &\geqslant (1+o(1)) \frac{2\mathcal{C}(\omega)\pi\left(x^{\gamma}\right)}{17.41^2 \log x} \left(\frac{\log(17.41\xi - 1)}{\xi} - \lambda \int_u^{17.41} \frac{t-u}{t(t\xi-1)} dt\right) \end{aligned} \tag{7}$$

and

$$\begin{aligned} W_4 = &\sum_{\substack{a\in\mathcal{A} \\ \left(a, P\left(x^{\frac{1}{17.41}}\right)\right)=1 \\ \Omega(a)=8 \\ \mu(a)\neq 0}} \left(1 - \lambda \sum_{\substack{x^{\frac{1}{17.41}} \leqslant p < x^{\frac{1}{u}} \\ p\mid a}} \left(1 - \frac{u\log p}{\log x}\right)\right) \\ \leqslant &\ \lambda \sum_{\substack{a\in\mathcal{A} \\ \left(a, P\left(x^{\frac{1}{17.41}}\right)\right)=1 \\ \Omega(a)=8 \\ \mu(a)\neq 0}} 1 \\ \leqslant &\ (1+o(1)) \frac{2\mathcal{C}(\omega)\pi\left(x^{\gamma}\right)}{17.41^2 \log x} \left(\frac{\lambda\gamma}{\xi} \int_{\frac{1}{17.41}}^{\frac{1}{8}} \int_{t_1}^{\frac{1}{7}(1-t_1)} \int_{t_2}^{\frac{1}{6}(1-t_1-t_2)} \int_{t_3}^{\frac{1}{5}(1-t_1-t_2-t_3)} \int_{t_4}^{\frac{1}{4}(1-t_1-t_2-t_3-t_4)} \right. \\ &\int_{t_5}^{\frac{1}{3}(1-t_1-t_2-t_3-t_4-t_5)} \int_{t_6}^{\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6)} \\ &\left. \frac{1}{t_1t_2t_3t_4t_5t_6t_7(1-t_1-t_2-t_3-t_4-t_5-t_6-t_7)} dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1 \right) \\ \leqslant &\ (1+o(1)) \frac{2\mathcal{C}(\omega)\pi\left(x^{\gamma}\right)}{17.41^2 \log x} \left(\frac{\lambda\gamma}{\xi} 0.00259\right), \end{aligned} \tag{8}$$

where $\mathcal{C}(\omega)$ is defined in [[4], (2.6)].

Note that we have

$$5 - 5\gamma + 4\xi < 0.611797$$

and

$$\frac{1}{4}(\gamma + \xi + 2) > 0.787393$$

when $0.9985 < \gamma < 1$. Then by Chen's switching principle, Iwaniec's linear sieve, Lemmas 2.1–2.2 and similar arguments as in [4], we can obtain

$$\begin{aligned} W_3 = &\sum_{\substack{a\in\mathcal{A} \\ \left(a, P\left(x^{\frac{1}{17.41}}\right)\right)=1 \\ \Omega(a)=7 \\ \mu(a)\neq 0}} \left(1 - \lambda \sum_{\substack{x^{\frac{1}{17.41}} \leqslant p < x^{\frac{1}{u}} \\ p\mid a}} \left(1 - \frac{u\log p}{\log x}\right)\right) \\ \leqslant &\ \lambda \sum_{\substack{a\in\mathcal{A} \\ \left(a, P\left(x^{\frac{1}{17.41}}\right)\right)=1 \\ \Omega(a)=7 \\ \mu(a)\neq 0}} 1 \\ \leqslant &\ (1+o(1)) \frac{2\mathcal{C}(\omega)\pi\left(x^{\gamma}\right)}{17.41^2 \log x} \left(\frac{\lambda\gamma}{\xi} \int_{\frac{1}{17.41}}^{\frac{1}{7}} \int_{t_1}^{\frac{1}{6}(1-t_1)} \int_{t_2}^{\frac{1}{5}(1-t_1-t_2)} \int_{t_3}^{\frac{1}{4}(1-t_1-t_2-t_3)} \right. \end{aligned}$$

$$
\int_{t_4}^{\frac{1}{3}(1-t_1-t_2-t_3-t_4)} \int_{t_5}^{\frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)} \frac{1}{t_1t_2t_3t_4t_5t_6(1-t_1-t_2-t_3-t_4-t_5-t_6)} dt_6dt_5dt_4dt_3dt_2dt_1 \Bigg)
$$

$$
\leqslant (1+o(1))\frac{2\mathcal{C}(\omega)\pi\left(x^{\gamma}\right)}{17.41^2\log x}\left(\frac{\lambda\gamma}{\xi}0.02571\right) \tag{9}
$$

and

$$
\begin{aligned}
W_2 &= \sum_{\substack{a\in\mathcal{A}\\ \left(a,P\left(x^{\frac{1}{17.41}}\right)\right)=1\\ \Omega(a)=6\\ \mu(a)\neq 0}} \left(1-\lambda\sum_{\substack{x^{\frac{1}{17.41}}\leqslant p<x^{\frac{1}{u}}\\ p\mid a}}\left(1-\frac{u\log p}{\log x}\right)\right)\\
&\leqslant \lambda \sum_{\substack{a\in\mathcal{A}\\ \left(a,P\left(x^{\frac{1}{17.41}}\right)\right)=1\\ \Omega(a)=6\\ \mu(a)\neq 0}} 1\\
&\leqslant (1+o(1))\frac{2\mathcal{C}(\omega)\pi\left(x^{\gamma}\right)}{17.41^2\log x}\left(\frac{\lambda\gamma}{\xi}\int_{\frac{1}{17.41}}^{\frac{1}{6}}\int_{t_1}^{\frac{1}{5}(1-t_1)}\int_{t_2}^{\frac{1}{4}(1-t_1-t_2)}\int_{t_3}^{\frac{1}{3}(1-t_1-t_2-t_3)}\int_{t_4}^{\frac{1}{2}(1-t_1-t_2-t_3-t_4)}\right.\\
&\qquad\left.\frac{1}{t_1t_2t_3t_4t_5(1-t_1-t_2-t_3-t_4-t_5)}dt_5dt_4dt_3dt_2dt_1\right)\\
&\leqslant (1+o(1))\frac{2\mathcal{C}(\omega)\pi\left(x^{\gamma}\right)}{17.41^2\log x}\left(\frac{\lambda\gamma}{\xi}0.16688\right).
\end{aligned} \tag{10}
$$

Finally, combining (6)–(10) we get that

$$
\begin{aligned}
W_1 &> W\left(\mathcal{A},x^{\frac{1}{17.41}}\right)-W_2-W_3-W_4+O\left(x^{1-10^{-10}}\right)\\
&\geqslant (1+o(1))\frac{2\mathcal{C}(\omega)\pi\left(x^{\gamma}\right)}{17.41^2\log x}\left(\frac{\log(17.41\xi-1)}{\xi}-\lambda\int_u^{17.41}\frac{t-u}{t(t\xi-1)}dt\right.\\
&\qquad\left.-\frac{\lambda\gamma}{\xi}(0.00259+0.02571+0.16688)\right)+O\left(x^{1-10^{-10}}\right).
\end{aligned}
$$

For $0.9985<\gamma<1$, we know that

$$
\frac{\log(17.41\xi-1)}{\xi}-\lambda\int_u^{17.41}\frac{t-u}{t(t\xi-1)}dt-\frac{\lambda\gamma}{\xi}(0.00259+0.02571+0.16688)>0.004 \tag{11}
$$

and the proof of Theorem 1.1 is completed.

We remark that the same method fails to prove $[p^{1/\gamma}]=P_4$ because of the following two restrictions:

1. We cannot use the estimation of exponential sums [[4], Lemma 5.1] to handle the error term occurred since we can not obtain a result of Lemma 2.2–type with 5 variables. An obvious counterexample is that each one of p_1,p_2,p_3,p_4,p_5 has size around $x^{0.2}$.

2. Even if we can enlarge the corresponding Type–II range in so that a result of Lemma 2.2–type with 5 variables can be obtained, we cannot get a positive lower bound for

$$
\sum_{\substack{a\in\mathcal{A}\\ \left(a,P\left(x^{\frac{1}{17.41}}\right)\right)=1\\ \Omega(a)\leqslant 4}} \mathscr{W}_a
$$

using Richert's logarithmic sieve weight.

It seems that the limit of our method is to prove $[p^{1/\gamma}]=P_5$.

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