ON A CONJECTURE INVOLVING TWIN PRACTICAL NUMBERS

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Abstract. In this paper, we prove that for every prime number $7 \leqslant p < 100$, there are infinitely many practical numbers q such that both q^p and $q^p + 2$ are practical numbers. This refine a result involving twin practical numbers of Wang and Sun. We also state a conjecture that the above theorem holds for all positive integers.

1. Introduction

A positive integer m is called a practical number if each $n = 1, \ldots, m$ can be written as the sum of some distinct divisors of n. Clearly all practical numbers are even except 1, and all even perfect numbers and powers of 2 are practical. Like the famous twin prime conjecture, Margenstern [1] established that there are infinitely many twin practical numbers m and $m+2$, and another detailed proof was given by Melfi [2] in 1996.

In 2022, Wang and Sun [3] generalized Margenstern's result and showed that there are infinitely many practical numbers q such that q^4 and q^4+2 are also practical numbers. They showed their result by modifying Melfi's cyclotomic method.

In this paper, we further state a conjecture and showed some new results.

Conjecture 1.1. *For any positive integer C, there are infinitely many practical numbers q such that q^C and $q^C + 2$ are also practical numbers.*

Theorem 1.2. $2^{135\times 7^k+1} + 2$, $2^{165\times 3^k+1} + 2$, $2^{175\times 3^k+1} + 2$ *and* $2^{189\times 5^k+1} + 2$ *are practical for every integer* $k \geqslant 0$.

Theorem 1.3. *Let p denotes a prime number, $7 \leqslant p < 100$. Then there are infinitely many practical numbers q such that q^p and $q^p + 2$ are also practical numbers.*

Clearly the result of Melfi is equivalent to the case $C = 1$ of Conjecture 1.1, and Wang and Sun solved the case $C = 2, 4$.

2. Proof of Theorems 1.2 and 1.3

In order to prove Theorem 1.2, we need the following structure theorem:

Lemma 2.1. *[[2], Lemma 1]. Let m be any practical number. Then mn is practical for every $n = 1, \ldots, \sigma(m) + 1$. In particular, mn is practical for every $1 \leqslant n \leqslant 2m$.*

We first prove $2^{135\times 7^k+1} + 2$ are practical for every $k \geqslant 0$. write $m_k = 2^{135\times 7^k+1} + 2$ for $k = 0, 1, 2, \ldots$, then we only need to show that m_k is practical for every $k = 0, 1, 2, \ldots$. Let

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$x=2^{7^k}$, then we have $m_k=2(x^{135}+1)$ and $m_{k+1}=2(x^{945}+1)$. Let $\Phi_m(x)$ denotes the mth cyclotomic polynomial, then we have

$$\frac{x^{1890}+1}{x^{945}+1}=\frac{x^{270}+1}{x^{135}+1}\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)\Phi_{378}(x)\Phi_{630}(x)\Phi_{1890}(x). \tag{1}$$

By similar arguments as in [3], we get that

$$\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)<8x^{126}<4(x^{135}+1), \tag{2}$$

$$\Phi_{378}(x)\Phi_{630}(x)<2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x), \tag{3}$$

$$\Phi_{1890}(x)<2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)\Phi_{378}(x)\Phi_{630}(x). \tag{4}$$

Then by Lemma 2.1 and similar arguments as in [3], we know that m_k is practical for every $k\geqslant 0$. For other cases, note that

$$\frac{x^{990}+1}{x^{495}+1}=\frac{x^{330}+1}{x^{165}+1}\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x)\Phi_{990}(x) \tag{5}$$

$$\frac{x^{1050}+1}{x^{525}+1}=\frac{x^{350}+1}{x^{175}+1}\Phi_{6}(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x)\Phi_{1050}(x) \tag{6}$$

$$\frac{x^{1890}+1}{x^{945}+1}=\frac{x^{378}+1}{x^{189}+1}\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)\Phi_{630}(x)\Phi_{1890}(x) \tag{7}$$

and the proof is very similar to the above one. Now Theorem 1.2 is proved.

By the following identities (where m is any positive integer)

$$\begin{aligned}
165*3^{5+6m}+1&\equiv 0(\mathrm{mod}7),\\
35*3^{3+10m}+1&\equiv 0(\mathrm{mod}11),\\
135*7^{3+12m}+1&\equiv 0(\mathrm{mod}13),\\
35*3^{8+16m}+1&\equiv 0(\mathrm{mod}17),\\
35*3^{17+18m}+1&\equiv 0(\mathrm{mod}19),\\
135*7^{20+22m}+1&\equiv 0(\mathrm{mod}23),\\
35*3^{24+28m}+1&\equiv 0(\mathrm{mod}29),\\
35*3^{27+30m}+1&\equiv 0(\mathrm{mod}31),\\
189*5^{32+36m}+1&\equiv 0(\mathrm{mod}37),\\
135*7^{7+40m}+1&\equiv 0(\mathrm{mod}41),\\
35*3^{3+42m}+1&\equiv 0(\mathrm{mod}43),\\
35*3^{11+46m}+1&\equiv 0(\mathrm{mod}47),\\
35*3^{1+52m}+1&\equiv 0(\mathrm{mod}53),\\
165*3^{21+58m}+1&\equiv 0(\mathrm{mod}59),\\
135*7^{50+60m}+1&\equiv 0(\mathrm{mod}61),\\
135*7^{33+66m}+1&\equiv 0(\mathrm{mod}67),\\
35*3^{11+70m}+1&\equiv 0(\mathrm{mod}71),\\
189*5^{57+72m}+1&\equiv 0(\mathrm{mod}73),
\end{aligned}$$

$$35 * 3^{2+78m} + 1 \equiv 0 (\mathrm{mod} 79),$$
$$35 * 3^{24+82m} + 1 \equiv 0 (\mathrm{mod} 83),$$
$$35 * 3^{69+88m} + 1 \equiv 0 (\mathrm{mod} 89),$$
$$35 * 3^{40+96m} + 1 \equiv 0 (\mathrm{mod} 97),$$

we complete the proof of Theorem 1.3. By combining those identities, we can also show that Conjecture 1.1 is true for some other positive integers. However, we are not able to give a positive answer for $C = 3, 5$ now. We hope someone can accomplish this work.

References

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