ON CHEN'S THEOREM, GOLDBACH'S CONJECTURE AND ALMOST PRIME TWINS

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ABSTRACT. Let N denote a sufficiently large even integer, we define $D_{1,2}(N)$ as the same as those in previous articles about Chen's theorem. In this paper, we show that $D_{1,2}(N) \geqslant 1.733 \frac{C(N)N}{(\log N)^2}$, improving previous record of Wu about 93%. We also get similar results on twin prime problem and additive representations of integers. An important step in the proof is the application of the theorems of Lichtman.

CONTENTS

1. INTRODUCTION

Let N denote a sufficiently large even integer, p denote a prime number, and let P_2 denote an integer with at most two prime factors counted with multiplicity. We define

$$D_{1,2}(N) := |\{p : p \leqslant N, N - p = P_2\}|. \tag{1}$$

In 1973 Chen [5] established his remarkable Chen's theorem:

$$D_{1,2}(N) \geqslant 0.67 \frac{C(N)N}{(\log N)^2}, \tag{2}$$

where

$$C(N) := \prod_{\substack{p \mid N \\ p>2}} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right). \tag{3}$$

Chen's constant 0.67 was improved successively to

$$0.689,\ 0.7544,\ 0.81,\ 0.8285,\ 0.836,\ 0.867,\ 0.899$$

by Halberstam and Richert [11] [10], Chen [7] [6], Cai and Lu [4], Wu [20], Cai [2] and Wu [21] respectively. Chen [8] announced a better constant 0.9, but this work has not been published.

In this paper, we obtain the following sharper result.

Theorem 1.1.

$$D_{1,2}(N) \geqslant 1.733 \frac{C(N)N}{(\log N)^2}.$$

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One important significance of our Theorem 1.1 is to make us truly achieve and exceed the constant 0.9 claimed by Chen [8]. Our constant 1.733 gives a 92.7% refinement of Wu's prior record 0.899. This is the greatest refinement on the problem since Chen [5] from 1973.

Furthermore, for two relatively prime square-free positive integers a, b, let M denote a sufficiently large integer that is relatively prime to both a and b, $a, b < M^{\varepsilon}$ and let M be even if a and b are both odd. Let $R_{a,b}(M)$ denote the number of primes p such that ap and $M - ap$ are both square-free, $b \mid (M - ap)$, and $\frac{M-ap}{b} = P_2$. In 1976, Ross [[18], Chapter 3] established that

$$R_{a,b}(M) \geqslant 0.608 \frac{C(abM)M}{ab(\log M)^2}, \tag{4}$$

where

$$C(abM) := \prod_{\substack{p \mid abM \\ p>2}} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right), \tag{5}$$

and the constant 0.608 was improved successively to 0.68 and 0.8671 by Li [13] and Li [14] respectively. By using the same sieve process and methods in [14], we have the following sharper.

Theorem 1.2.

$$R_{a,b}(M) \geqslant 1.733 \frac{C(abM)M}{ab(\log M)^2}.$$

Let x denote a sufficiently large integer and define

$$\pi_{1,2}(x) := \left|\{p : p \leqslant x, p+2 = P_2\}\right|. \tag{6}$$

In 1973 Chen [5] showed simultaneously that

$$\pi_{1,2}(x) \geqslant 0.335 \frac{C_2 x}{(\log x)^2}, \tag{7}$$

where

$$C_2 := 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right), \tag{8}$$

and the constant 0.608 was improved successively to

$$0.3445,\ 0.3772,\ 0.405,\ 0.71,\ 1.015,\ 1.05,\ 1.0974,\ 1.104,\ 1.123,\ 1.13$$

by Halberstam [10], Chen [7] [6], Fouvry and Grupp [9], Liu [17], Wu [19], Cai [1], Wu [20], Cai [2] and Cai [3] respectively.

In this paper, we get the following sharper.

Theorem 1.3.

$$\pi_{1,2}(x) \geqslant 1.238 \frac{C_2 x}{(\log x)^2}.$$

2. Lichtman's Distribution Theorems

In this section we put $A, B > 0$, $\theta = \frac{7}{32}$ from Kim–Sarnak [12], and we define the functions $\boldsymbol{\vartheta}_\alpha(t_1)$ and $\boldsymbol{\vartheta}_\alpha(t_1, t_2, t_3)$ with $\alpha = 0$ or 1 as the same as those in [16]. We consider the analogous set of well–factorable vectors $\mathbf{D}_r^{\text{well}}$:

$$\mathbf{D}_r^{\text{well}}(D) = \left\{(D_1, \ldots, D_r) : D_1 \cdots D_{m-1} D_m^2 < D \text{ for all } m \leqslant r\right\}.$$

Lemma 2.1. *Let $(D_1, \ldots, D_r) \in \mathbf{D}_r^{well}(D)$ and write $D = N^{\boldsymbol{\vartheta}}, D_i = N^{t_i}$ for $i \leqslant r$. If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1) - \varepsilon$, then*

$$\sum_{\substack{b=p_1\cdots p_r \\ D_i < p_i \leqslant D_i^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c \mid P(p_r) \\ (q,N)=1}} \widetilde{\lambda}^{\pm}(q) \left(\pi(N; q, N) - \frac{\pi(N)}{\varphi(q)}\right) \ll \frac{N}{(\log N)^A}. \tag{i}$$

Moreover if $t_1 \leqslant \frac{1-\theta}{4}$ and $r \geqslant 3$, then (i) holds if $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1, t_2, t_3) - \varepsilon$.

If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1) - \varepsilon$ *and* $r = 2$*, then*

$$\sum_{\substack{b=p_1p_2 \\ D_1<p_1\leqslant D_1^{1+\varepsilon^9} \\ D_2<p_2\leqslant D_2^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c\mid P(N^u) \\ (q,N)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(N;q,N) - \frac{\pi(N)}{\varphi(q)}\right) \ll \frac{N}{(\log N)^A}. \tag{ii}$$

Moreover if $t_1 \leqslant \frac{1-\theta}{4}$*, then (ii) holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1, t_2, u) - \varepsilon$.

If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1) - \varepsilon$ *and* $r = 1$*, then*

$$\sum_{\substack{b=p_1 \\ D_1<p_1\leqslant D_1^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c\mid P(N^u) \\ (q,N)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(N;q,N) - \frac{\pi(N)}{\varphi(q)}\right) \ll \frac{N}{(\log N)^A}. \tag{iii}$$

Moreover if $t_1 \leqslant \frac{1-\theta}{4}$*, then (iii) holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1, u, u) - \varepsilon$.

If $r = 0$ *and* $u = \frac{1}{500}$*, this simplifies as*

$$\sum_{\substack{q\leqslant N^{\frac{19101}{32000}} \\ q\mid P\left(N^{1/500}\right) \\ (q,N)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(N;q,N) - \frac{\pi(N)}{\varphi(q)}\right) \ll \frac{N}{(\log N)^A}. \tag{iv}$$

Lemma 2.2. *Let* $(D_1, \ldots, D_r) \in \mathbf{D}_r^{well}(D)$ *and write* $D = x^{\boldsymbol{\vartheta}}, D_i = x^{t_i}$ *for* $i \leqslant r$*. If* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1) - \varepsilon$*, then*

$$\sum_{\substack{b=p_1\cdots p_r \\ D_i<p_i\leqslant D_i^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c\mid P(p_r) \\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(x;q,-2) - \frac{\pi(x)}{\varphi(q)}\right) \ll \frac{x}{(\log x)^A}. \tag{v}$$

Moreover if $t_1 \leqslant \frac{1-\theta}{4-3\theta}$ *and* $r \geqslant 3$*, then (v) holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1, t_2, t_3) - \varepsilon$.

If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1) - \varepsilon$ *and* $r = 2$*, then*

$$\sum_{\substack{b=p_1p_2 \\ D_1<p_1\leqslant D_1^{1+\varepsilon^9} \\ D_2<p_2\leqslant D_2^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c\mid P(x^u) \\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(x;q,-2) - \frac{\pi(x)}{\varphi(q)}\right) \ll \frac{x}{(\log x)^A}. \tag{vi}$$

Moreover if $t_1 \leqslant \frac{1-\theta}{4-3\theta}$*, then (vi) holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1, t_2, u) - \varepsilon$.

If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1) - \varepsilon$ *and* $r = 1$*, then*

$$\sum_{\substack{b=p_1 \\ D_1<p_1\leqslant D_1^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c\mid P(x^u) \\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(x;q,-2) - \frac{\pi(x)}{\varphi(q)}\right) \ll \frac{x}{(\log x)^A}. \tag{vii}$$

Moreover if $t_1 \leqslant \frac{1-\theta}{4-3\theta}$*, then (vii) holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1, u, u) - \varepsilon$.

If $r = 0$ *and* $u = \frac{1}{500}$*, this simplifies as*

$$\sum_{\substack{q\leqslant x^{\frac{16483}{26750}} \\ q\mid P\left(x^{1/500}\right) \\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(x;q,-2) - \frac{\pi(x)}{\varphi(q)}\right) \ll \frac{x}{(\log x)^A}. \tag{viii}$$

Lemma 2.3. *Let $(D_1,\dots,D_r)\in \mathbf{D}_r^{well}(D)$ and write $D=N^{\boldsymbol{\vartheta}}, D_i=N^{t_i}$ for $i\leqslant r$. Let $\varepsilon>0$ and real numbers $\varepsilon_1,\dots,\varepsilon_k\geqslant\varepsilon$ such that $\sum_{i\leqslant k}\varepsilon_i=1$, and let $\Delta=1+(\log N)^{-B}$. If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1)-\varepsilon$, then*

$$\sum_{\substack{b=p_1'\cdots p_r'\\ D_i<p_i'\leqslant D_i^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P(p_r')\\ (q,N)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv N(\bmod q)\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,N)=1\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{N}{(\log N)^A}. \tag{ix}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4}$ and $r\geqslant 3$, then (ix) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-\varepsilon$.

If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1)-\varepsilon$ and $r=2$, then

$$\sum_{\substack{b=p_1'p_2'\\ D_1<p_1'\leqslant D_1^{1+\varepsilon^9}\\ D_2<p_2'\leqslant D_2^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P(N^u)\\ (q,N)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv N(\bmod q)\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,N)=1\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{N}{(\log N)^A}. \tag{x}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4}$, then (x) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1,t_2,u)-\varepsilon$.

If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1)-\varepsilon$ and $r=1$, then

$$\sum_{\substack{b=p_1'\\ D_1<p_1'\leqslant D_1^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P(N^u)\\ (q,N)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv N(\bmod q)\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,N)=1\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{N}{(\log N)^A}. \tag{xi}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4}$, then (xi) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1,u,u)-\varepsilon$.

If $r=0$ and $u=\frac{1}{500}$, this simplifies as

$$\sum_{\substack{q\leqslant N^{\frac{19101}{32000}}\\ q\mid P(N^{1/500})\\ (q,N)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv N(\bmod q)\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,N)=1\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{N}{(\log N)^A}. \tag{xii}$$

Lemma 2.4. *Let $(D_1,\dots,D_r)\in \mathbf{D}_r^{well}(D)$ and write $D=x^{\boldsymbol{\vartheta}}, D_i=x^{t_i}$ for $i\leqslant r$. Let $\varepsilon>0$ and real numbers $\varepsilon_1,\dots,\varepsilon_k\geqslant\varepsilon$ such that $\sum_{i\leqslant k}\varepsilon_i=1$, and let $\Delta=1+(\log x)^{-B}$. If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1)-\varepsilon$, then*

$$\sum_{\substack{b=p_1'\cdots p_r'\\ D_i<p_i'\leqslant D_i^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P(p_r')\\ (q,2)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv 2(\bmod q)\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,2)=1\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{x}{(\log x)^A}. \tag{xiii}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4-3\theta}$ and $r\geqslant 3$, then (xiii) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-\varepsilon$.

If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1)-\varepsilon$ and $r=2$, then

$$\sum_{\substack{b=p_1'p_2'\\ D_1<p_1'\leqslant D_1^{1+\varepsilon^9}\\ D_2<p_2'\leqslant D_2^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P(x^u)\\ (q,2)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv 2(\bmod q)\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,2)=1\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{x}{(\log x)^A}. \tag{xiv}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4-3\theta}$, then (xiv) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1,t_2,u)-\varepsilon$.

If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1) - \varepsilon$ *and* $r = 1$, *then*

$$\sum_{\substack{b=p_1' \\ D_1<p_1'\leqslant D_1^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c|P(x^u) \\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q) \left(\sum_{\substack{p_1\cdots p_k\equiv 2(\bmod q) \\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \ \forall i\leqslant k}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{(p_1\cdots p_k,2)=1 \\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \ \forall i\leqslant k}} 1 \right) \ll \frac{x}{(\log x)^A}. \tag{xv}$$

Moreover if $t_1 \leqslant \frac{1-\theta}{4-3\theta}$, *then (xv) holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1,u,u) - \varepsilon$.

If $r = 0$ *and* $u = \frac{1}{500}$, *this simplifies as*

$$\sum_{\substack{q\leqslant x^{\frac{16483}{26750}} \\ q|P\left(x^{1/500}\right) \\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q) \left(\sum_{\substack{p_1\cdots p_k\equiv 2(\bmod q) \\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \ \forall i\leqslant k}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{(p_1\cdots p_k,2)=1 \\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \ \forall i\leqslant k}} 1 \right) \ll \frac{x}{(\log x)^A}. \tag{xvi}$$

3. Weighted Sieve Method

Let $\mathcal{A}$ and $\mathcal{B}$ denote finite sets of positive integers, $\mathcal{P}$ denote an infinite set of primes and $z \geqslant 2$. Put

$$\mathcal{A} = \{N - p : p \leqslant N\}, \quad \mathcal{B} = \{p + 2 : p \leqslant x\},$$

$$\mathcal{P} = \{p : (p,2) = 1\}, \quad \mathcal{P}(q) = \{p : p \in \mathcal{P}, (p,q) = 1\},$$

$$P(z) = \prod_{\substack{p\in\mathcal{P} \\ p<z}} p, \quad \mathcal{A}_d = \{a : a \in \mathcal{A}, a \equiv 0(\bmod d)\}, \quad S(\mathcal{A};\mathcal{P},z) = \sum_{\substack{a\in\mathcal{A} \\ (a,P(z))=1}} 1.$$

Lemma 3.1. *([21], Lemma 2.2]). We have*

$$\begin{aligned}
4D_{1,2}(N) \geqslant\ & 3S\left(\mathcal{A};\mathcal{P}(N),N^{\frac{1}{13.27}}\right) + S\left(\mathcal{A};\mathcal{P}(N),N^{\frac{1}{8.24}}\right) \\
& - 2\sum_{\substack{N^{\frac{1}{13.27}}\leqslant p<N^{\frac{25}{128}} \\ (p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right) \\
& - 2\sum_{\substack{N^{\frac{25}{128}}\leqslant p<N^{\frac{1}{4}} \\ (p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right) \\
& - 2\sum_{\substack{N^{\frac{1}{4}}\leqslant p<N^{\frac{57}{224}} \\ (p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right) \\
& - \sum_{\substack{N^{\frac{57}{224}}\leqslant p<N^{\frac{1}{3}} \\ (p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right) \\
& - \sum_{\substack{N^{\frac{57}{224}}\leqslant p<N^{\frac{1}{2}-\frac{3}{13.27}} \\ (p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right) \\
& + \sum_{\substack{N^{\frac{1}{13.27}}\leqslant p_2<p_1<N^{\frac{1}{8.24}} \\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(N),N^{\frac{1}{13.27}}\right) \\
& + \sum_{\substack{N^{\frac{1}{13.27}}\leqslant p_2<N^{\frac{1}{8.24}}\leqslant p_1<N^{\frac{25}{128}} \\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(N),N^{\frac{1}{13.27}}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{N^{\frac{1}{13.27}} \leqslant p_2 < N^{\frac{1}{8.24}} < N^{\frac{25}{128}} \leqslant p_1 < N^{\frac{57}{224}} \\ (p_1p_2, N)=1}} S\left(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^{\frac{1}{13.27}}\right) \\
&+ \sum_{\substack{N^{\frac{1}{13.27}} \leqslant p_2 < N^{\frac{1}{8.24}} < N^{\frac{57}{224}} \leqslant p_1 < N^{\frac{1}{2}-\frac{3}{13.27}} \\ (p_1p_2, N)=1}} S\left(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^{\frac{1}{13.27}}\right) \\
&- 2 \sum_{\substack{N^{\frac{1}{2}-\frac{3}{13.27}} \leqslant p_1 < p_2 < (\frac{N}{p_1})^{\frac{1}{2}} \\ (p_1p_2, N)=1}} S\left(\mathcal{A}_{p_1p_2}; \mathcal{P}(Np_1), p_2\right) \\
&- \sum_{\substack{N^{\frac{1}{13.27}} \leqslant p_1 < N^{\frac{1}{3}} \leqslant p_2 < (\frac{N}{p_1})^{\frac{1}{2}} \\ (p_1p_2, N)=1}} S\left(\mathcal{A}_{p_1p_2}; \mathcal{P}(Np_1), p_2\right) \\
&- \sum_{\substack{N^{\frac{1}{8.24}} \leqslant p_1 < N^{\frac{1}{2}-\frac{3}{13.27}} \leqslant p_2 < (\frac{N}{p_1})^{\frac{1}{2}} \\ (p_1p_2, N)=1}} S\left(\mathcal{A}_{p_1p_2}; \mathcal{P}(Np_1), \left(\frac{N}{p_1p_2}\right)^{\frac{1}{2}}\right) \\
&- \sum_{\substack{N^{\frac{1}{13.27}} \leqslant p_1 < p_2 < p_3 < p_4 < N^{\frac{1}{8.24}} \\ (p_1p_2p_3p_4, N)=1}} S\left(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(N), p_2\right) \\
&- \sum_{\substack{N^{\frac{1}{13.27}} \leqslant p_1 < p_2 < p_3 < N^{\frac{1}{8.24}} \leqslant p_4 < N^{\frac{1}{2}-\frac{2}{13.27}} p_3^{-1} \\ (p_1p_2p_3p_4, N)=1}} S\left(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(N), p_2\right) \\
&+ O\left(N^{\frac{12.27}{13.27}}\right) \\
=\ & 3S_1 + S_2 - 2S_3 - 2S_4 - 2S_5 - S_6 - S_7 + S_8 + S_9 \\
&+ S_{10} + S_{11} - 2S_{12} - S_{13} - S_{14} - S_{15} - S_{16} + O\left(N^{\frac{12.27}{13.27}}\right).
\end{aligned}
$$

Lemma 3.2. *([3], Lemma 3.2]). We have*

$$
\begin{aligned}
4\pi_{1,2}(x) \geqslant\ & 3S\left(\mathcal{B}; \mathcal{P}, x^{\frac{1}{12}}\right) + S\left(\mathcal{B}; \mathcal{P}, x^{\frac{1}{7.2}}\right) \\
&+ \sum_{x^{\frac{1}{12}} \leqslant p_2 < p_1 < x^{\frac{1}{7.2}}} S\left(\mathcal{B}_{p_1p_2}; \mathcal{P}, x^{\frac{1}{12}}\right) \\
&+ \sum_{x^{\frac{1}{12}} \leqslant p_2 < x^{\frac{1}{7.2}} \leqslant p_1 < x^{\frac{25}{107}}} S\left(\mathcal{B}_{p_1p_2}; \mathcal{P}, x^{\frac{1}{12}}\right) \\
&+ \sum_{x^{\frac{1}{12}} \leqslant p_2 < x^{\frac{1}{7.2}} < x^{\frac{25}{107}} \leqslant p_1 < \min(x^{\frac{2}{7}}, x^{\frac{17}{42}} p_2^{-1})} S\left(\mathcal{B}_{p_1p_2}; \mathcal{P}, x^{\frac{1}{12}}\right) \\
&- 2 \sum_{x^{\frac{1}{12}} \leqslant p < x^{\frac{25}{107}}} S\left(\mathcal{B}_{p}; \mathcal{P}, x^{\frac{1}{12}}\right) - 2 \sum_{x^{\frac{25}{107}} \leqslant p < x^{\frac{2}{7}-\varepsilon}} S\left(\mathcal{B}_{p}; \mathcal{P}, x^{\frac{1}{12}}\right) \\
&- \sum_{x^{\frac{2}{7}-\varepsilon} \leqslant p < x^{\frac{2}{7}}} S\left(\mathcal{B}_{p}; \mathcal{P}, x^{\frac{1}{12}}\right) - \sum_{x^{\frac{2}{7}-\varepsilon} \leqslant p < x^{\frac{29}{100}}} S\left(\mathcal{B}_{p}; \mathcal{P}, x^{\frac{1}{12}}\right) \\
&- \sum_{x^{\frac{29}{100}} \leqslant p < x^{\frac{1}{3}-\varepsilon}} S\left(\mathcal{B}_{p}; \mathcal{P}, x^{\frac{1}{12}}\right) - \sum_{x^{\frac{1}{3}-\varepsilon} \leqslant p < x^{\frac{1}{3}}} S\left(\mathcal{B}_{p}; \mathcal{P}, x^{\frac{1}{12}}\right) \\
&- \sum_{x^{\frac{1}{12}} \leqslant p_1 < x^{\frac{1}{3}} \leqslant p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1p_2}; \mathcal{P}(p_1), p_2\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{x^{\frac{1}{7.2}} \leqslant p_1 < x^{\frac{2}{7}} \leqslant p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1p_2}; \mathcal{P}(p_1), \left(\frac{x}{p_1p_2}\right)^{\frac{1}{2}}\right) \\
&- 2 \sum_{x^{\frac{2}{7}} \leqslant p_1 < p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1p_2}; \mathcal{P}(p_1), p_2\right) \\
&- \sum_{x^{\frac{1}{12}} \leqslant p_1 < p_2 < p_3 < p_4 < x^{\frac{1}{7.2}}} S\left(\mathcal{B}_{p_1p_2p_3p_4}; \mathcal{P}(p_1), p_2\right) \\
&- \sum_{x^{\frac{1}{12}} \leqslant p_1 < p_2 < p_3 < x^{\frac{5}{42}} < x^{\frac{1}{7.2}} < p_4 < x^{\frac{2}{7}}} S\left(\mathcal{B}_{p_1p_2p_3p_4}; \mathcal{P}(p_1), p_2\right) \\
&- \sum_{x^{\frac{1}{12}} \leqslant p_1 < p_2 < x^{\frac{5}{42}} \leqslant p_3 < x^{\frac{1}{7.2}} \leqslant p_4 < x^{\frac{17}{42}} p_3^{-1}} S\left(\mathcal{B}_{p_1p_2p_3p_4}; \mathcal{P}(p_1), p_2\right) \\
&- \sum_{x^{\frac{1}{12}} \leqslant p_1 < x^{\frac{5}{42}} \leqslant p_2 < p_3 < x^{\frac{1}{7.2}} \leqslant p_4 < x^{\frac{17}{42}} p_3^{-1}} S\left(\mathcal{B}_{p_1p_2p_3p_4}; \mathcal{P}(p_1), p_2\right) \\
&- \sum_{x^{\frac{5}{42}} \leqslant p_1 < p_2 < p_3 < x^{\frac{1}{7.2}} \leqslant p_4 < x^{\frac{17}{42}} p_3^{-1}} S\left(\mathcal{B}_{p_1p_2p_3p_4}; \mathcal{P}(p_1), p_2\right) \\
&+ O\left(x^{\frac{11}{12}}\right) \\
=& \; 3S_1' + S_2' + S_3' + S_4' + S_5' - 2S_6' - 2S_7' - S_8' - S_9' - S_{10}' - S_{11}' \\
&- S_{12}' - S_{13}' - 2S_{14}' - S_{15}' - S_{16}' - S_{17}' - S_{18}' - S_{19}' + O\left(x^{\frac{11}{12}}\right).
\end{aligned}
$$

4. Proof of Theorem 1.1

In this section, sets $\mathcal{A}$ and $\mathcal{P}$ are defined respectively. Let γ denote the Euler's constant, $F(s)$ and $f(s)$ are determined by the following differential-difference equation

$$
\begin{cases}
F(s) = \frac{2e^{\gamma}}{s}, \quad f(s) = 0, & 0 < s \leqslant 2, \\
(sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), & s \geqslant 2,
\end{cases}
$$

and $\omega(u)$ denote the Buchstab function determined by the following differential-difference equation

$$
\begin{cases}
\omega(u) = \frac{1}{u}, & 1 \leqslant u \leqslant 2, \\
(u\omega(u))' = \omega(u-1), & u \geqslant 2.
\end{cases}
$$

We first consider S_1 and S_2. By Buchstab's identity, we have

$$
\begin{aligned}
S_1 = S\left(\mathcal{A}; \mathcal{P}(N), N^{\frac{1}{13.27}}\right) = S\left(\mathcal{A}; \mathcal{P}(N), N^{\frac{1}{500}}\right) &- \sum_{\substack{N^{\frac{1}{500}} \leqslant p < N^{\frac{1}{13.27}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}(N), N^{\frac{1}{500}}\right) \\
&+ \sum_{\substack{N^{\frac{1}{500}} \leqslant p_2 < p_1 < N^{\frac{1}{13.27}} \\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^{\frac{1}{500}}\right) \\
&- \sum_{\substack{N^{\frac{1}{500}} \leqslant p_3 < p_2 < p_1 < N^{\frac{1}{13.27}} \\ (p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(N), p_3\right) \qquad (9)
\end{aligned}
$$

and

$$
S_2 = S\left(\mathcal{A}; \mathcal{P}(N), N^{\frac{1}{8.24}}\right) = S\left(\mathcal{A}; \mathcal{P}(N), N^{\frac{1}{500}}\right) - \sum_{\substack{N^{\frac{1}{500}} \leqslant p < N^{\frac{1}{8.24}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}(N), N^{\frac{1}{500}}\right)
$$

$$
\begin{aligned}
&+\sum_{\substack{N^{\frac{1}{500}} \leqslant p_2<p_1<N^{\frac{1}{8.24}} \\ (p_1 p_2, N)=1}} S\left(\mathcal{A}_{p_1 p_2} ; \mathcal{P}(N), N^{\frac{1}{500}}\right) \\
&-\sum_{\substack{N^{\frac{1}{500}} \leqslant p_3<p_2<p_1<N^{\frac{1}{8.24}} \\ (p_1 p_2 p_3, N)=1}} S\left(\mathcal{A}_{p_1 p_2 p_3} ; \mathcal{P}(N), p_3\right). \qquad (10)
\end{aligned}
$$

By Lemma 2.1, Iwaniec's linear sieve method and arguments in [15] and [16] we have

$$
\begin{aligned}
S_1 \geqslant (1+o(1)) \frac{2}{e^\gamma} & \left(500 f\left(500 \boldsymbol{\vartheta}_{\frac{1}{500}}\right)-500 \int_{\frac{1}{500}}^{\frac{1}{13.27}} \frac{F(500(\boldsymbol{\vartheta}_1(t, \frac{1}{500}, \frac{1}{500})-t))}{t} d t\right. \\
& +500 \int_{\frac{1}{500}}^{\frac{1}{13.27}} \int_{\frac{1}{500}}^{t_1} \frac{f(500(\boldsymbol{\vartheta}_1(t_1, t_2, \frac{1}{500})-t_1-t_2))}{t_1 t_2} d t_2 d t_1 \\
& \left.-\int_{\frac{1}{500}}^{\frac{1}{13.27}} \int_{\frac{1}{500}}^{t_1} \int_{\frac{1}{500}}^{t_2} \frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1, t_2, t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1 t_2 t_3^2} d t_3 d t_2 d t_1\right) \frac{C(N) N}{(\log N)^2} \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
S_2 \geqslant (1+o(1)) \frac{2}{e^\gamma} & \left(500 f\left(500 \boldsymbol{\vartheta}_{\frac{1}{500}}\right)-500 \int_{\frac{1}{500}}^{\frac{1}{8.24}} \frac{F(500(\boldsymbol{\vartheta}_1(t, \frac{1}{500}, \frac{1}{500})-t))}{t} d t\right. \\
& +500 \int_{\frac{1}{500}}^{\frac{1}{8.24}} \int_{\frac{1}{500}}^{t_1} \frac{f(500(\boldsymbol{\vartheta}_1(t_1, t_2, \frac{1}{500})-t_1-t_2))}{t_1 t_2} d t_2 d t_1 \\
& \left.-\int_{\frac{1}{500}}^{\frac{1}{8.24}} \int_{\frac{1}{500}}^{t_1} \int_{\frac{1}{500}}^{t_2} \frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1, t_2, t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1 t_2 t_3^2} d t_3 d t_2 d t_1\right) \frac{C(N) N}{(\log N)^2}, \qquad (12)
\end{aligned}
$$

where $\boldsymbol{\vartheta}_{\frac{1}{500}}=\frac{19101}{32000}$. By numerical calculations we get that

$$
S_1 \geqslant 14.901125 \frac{C(N) N}{(\log N)^2} \qquad (13)
$$

and

$$
S_2 \geqslant 9.228483 \frac{C(N) N}{(\log N)^2}. \qquad (14)
$$

For S_3, we can either use Buchstab's identity and Lichtman's method to estimate S_3 with a better distribution level as in [16] or use Chen's double sieve technique as in [21]. The first option leads to

$$
\begin{aligned}
\sum_{\substack{p \\ (p, N)=1}} S\left(\mathcal{A}_p ; \mathcal{P}(N), N^{\frac{1}{13.27}}\right)= & \sum_{\substack{p \\ (p, N)=1}} S\left(\mathcal{A}_p ; \mathcal{P}(N), N^{\frac{1}{k}}\right) \\
& -\sum_{\substack{p_1 \\ N^{\frac{1}{k}} \leqslant p_2<N^{\frac{1}{13.27}} \\ (p_1 p_2, N)=1}} S\left(\mathcal{A}_{p_1 p_2} ; \mathcal{P}(N), N^{\frac{1}{k}}\right) \\
& +\sum_{\substack{p_1 \\ N^{\frac{1}{k}} \leqslant p_3<p_2<N^{\frac{1}{13.27}} \\ (p_1 p_2 p_3, N)=1}} S\left(\mathcal{A}_{p_1 p_2 p_3} ; \mathcal{P}(N), p_3\right) \qquad (15)
\end{aligned}
$$

for some $k \geqslant 13.27$, while the second option creates a small saving on S_3 itself. We can also use Chen's double sieve on the first two sums on the right–hand side of (15) after applying Buchstab's identity. We don't know which of these options gives a smaller value, hence we take a minimum. By Lemma 2.1, Iwaniec's linear sieve method and arguments in [15] and [16] we have

$$\begin{aligned}
S_3 \leqslant{} & (1+o(1))\frac{2}{e^\gamma}\left(\int_{\frac{1}{13.27}}^{\frac{25}{128}} \min\left(13.27\frac{F(13.27(\boldsymbol{\vartheta}_1(t_1,\frac{1}{13.27},\frac{1}{13.27})-t_1))}{t_1}\right.\right.\\
& -\frac{26.54e^\gamma H(13.27(\frac{1}{2}-t_1))}{(13.27(\frac{1}{2}-t_1))t_1}, \min_{13.27\leqslant k\leqslant 500}\left(k\frac{F(k(\boldsymbol{\vartheta}_1(t_1,\frac{1}{k},\frac{1}{k})-t_1))}{t_1}\right.\\
& -\frac{2ke^\gamma H(k(\frac{1}{2}-t_1))}{(k(\frac{1}{2}-t_1))t_1} - k\int_{\frac{1}{k}}^{\frac{1}{13.27}} \frac{f(k(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1t_2}dt_2\\
& -2ke^\gamma\int_{\frac{1}{k}}^{\frac{1}{13.27}} \frac{h(k(\frac{1}{2}-t_1-t_2))}{(k(\frac{1}{2}-t_1-t_2))t_1t_2}dt_2\\
& \left.\left.\left.+\int_{\frac{1}{k}}^{\frac{1}{13.27}}\int_{\frac{1}{k}}^{t_2} \frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2\right)\right)dt_1\right)\frac{C(N)N}{(\log N)^2}\\
\leqslant{} & 14.192163\frac{C(N)N}{(\log N)^2},
\end{aligned} \tag{16}$$

where we choose $k = 14.4$ and $H(s) = H_{1/2}(s)$ and $h(s) = h_{1/2}(s)$ are defined as the same in [21]. We have used the following lower bounds of $H(s)$ and $h(s)$ for $2.0 \leqslant s \leqslant 4.9$. These values can be found at Tables 1 and 2 of [21]. We remark that we have $H_{\boldsymbol{\vartheta}}(s) \geqslant H_{1/2}(s)$ and $h_{\boldsymbol{\vartheta}}(s) \geqslant h_{1/2}(s)$ for $\boldsymbol{\vartheta} > \frac{1}{2}$.

$$H(s) \geqslant \begin{cases}
0.0223939, & 2.0 < s \leqslant 2.2,\\
0.0217196, & 2.2 < s \leqslant 2.3,\\
0.0202876, & 2.3 < s \leqslant 2.4,\\
0.0181433, & 2.4 < s \leqslant 2.5,\\
0.0158644, & 2.5 < s \leqslant 2.6,\\
0.0129923, & 2.6 < s \leqslant 2.7,\\
0.0100686, & 2.7 < s \leqslant 2.8,\\
0.0078162, & 2.8 < s \leqslant 2.9,\\
0.0072943, & 2.9 < s \leqslant 3.0,\\
0.0061642, & 3.0 < s \leqslant 3.1,\\
0.0052233, & 3.1 < s \leqslant 3.2,\\
0.0044073, & 3.2 < s \leqslant 3.3,\\
0.0036995, & 3.3 < s \leqslant 3.4,\\
0.0030860, & 3.4 < s \leqslant 3.5,
\end{cases}
\begin{cases}
0.0025551, & 3.5 < s \leqslant 3.6,\\
0.0020972, & 3.6 < s \leqslant 3.7,\\
0.0017038, & 3.7 < s \leqslant 3.8,\\
0.0013680, & 3.8 < s \leqslant 3.9,\\
0.0010835, & 3.9 < s \leqslant 4.0,\\
0.0008451, & 4.0 < s \leqslant 4.1,\\
0.0006482, & 4.1 < s \leqslant 4.2,\\
0.0004882, & 4.2 < s \leqslant 4.3,\\
0.0003602, & 4.3 < s \leqslant 4.4,\\
0.0002592, & 4.4 < s \leqslant 4.5,\\
0.0001803, & 4.5 < s \leqslant 4.6,\\
0.0001187, & 4.6 < s \leqslant 4.7,\\
0.0000702, & 4.7 < s \leqslant 4.8,\\
0.0000313, & 4.8 < s \leqslant 4.9,
\end{cases} \tag{17}$$

$$h(s) \geqslant \begin{cases} 0.0232385, & s = 2.0, \\ 0.0211041, & 2.0 < s \leqslant 2.1, \\ 0.0191556, & 2.1 < s \leqslant 2.2, \\ 0.0173631, & 2.2 < s \leqslant 2.3, \\ 0.0157035, & 2.3 < s \leqslant 2.4, \\ 0.0141585, & 2.4 < s \leqslant 2.5, \\ 0.0127132, & 2.5 < s \leqslant 2.6, \\ 0.0113556, & 2.6 < s \leqslant 2.7, \\ 0.0100756, & 2.7 < s \leqslant 2.8, \\ 0.0088648, & 2.8 < s \leqslant 2.9, \\ 0.0077612, & 2.9 < s \leqslant 3.0, \\ 0.0066236, & 3.0 < s \leqslant 3.1, \\ 0.0055818, & 3.1 < s \leqslant 3.2, \\ 0.0046164, & 3.2 < s \leqslant 3.3, \\ 0.0037529, & 3.3 < s \leqslant 3.4, \end{cases} \begin{cases} 0.0030123, & 3.4 < s \leqslant 3.5, \\ 0.0023901, & 3.5 < s \leqslant 3.6, \\ 0.0018997, & 3.6 < s \leqslant 3.7, \\ 0.0015336, & 3.7 < s \leqslant 3.8, \\ 0.0012593, & 3.8 < s \leqslant 3.9, \\ 0.0010120, & 3.9 < s \leqslant 4.0, \\ 0.0008099, & 4.0 < s \leqslant 4.1, \\ 0.0006440, & 4.1 < s \leqslant 4.2, \\ 0.0005084, & 4.2 < s \leqslant 4.3, \\ 0.0003980, & 4.3 < s \leqslant 4.4, \\ 0.0003085, & 4.4 < s \leqslant 4.5, \\ 0.0002365, & 4.5 < s \leqslant 4.6, \\ 0.0001791, & 4.6 < s \leqslant 4.7, \\ 0.0001396, & 4.7 < s \leqslant 4.8, \\ 0.0000981, & 4.8 < s \leqslant 4.9. \end{cases} \tag{18}$$

Similarly, for S_4 and S_5 we have

$$\begin{aligned} S_4 \leqslant{} & (1+o(1))\frac{2}{e^\gamma}\left(\int_{\frac{25}{128}}^{\frac{1}{4}} \min\left(13.27\frac{F(13.27(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1} - \frac{26.54e^\gamma H(13.27(\frac{1}{2}-t_1))}{(13.27(\frac{1}{2}-t_1))t_1},\right.\right. \\ & \min_{13.27\leqslant k\leqslant 500}\left(k\frac{F(k(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1} - \frac{2ke^\gamma H(k(\frac{1}{2}-t_1))}{(k(\frac{1}{2}-t_1))t_1}\right. \\ & \left.\left.\left. - \int_{\frac{1}{k}}^{\frac{1}{13.27}} \frac{f\left(\frac{(\boldsymbol{\vartheta}_1(t_1)-t_1-t_2)}{t_2}\right)}{t_1t_2^2}dt_2\right)\right) dt_1\right) \frac{C(N)N}{(\log N)^2} \\ \leqslant{} & 3.721794\frac{C(N)N}{(\log N)^2}, \end{aligned} \tag{19}$$

$$\begin{aligned} S_5 \leqslant{} & (1+o(1))\frac{2}{e^\gamma}\left(\int_{\frac{1}{4}}^{\frac{57}{224}} \min\left(13.27\frac{F(13.27(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1},\right.\right. \\ & \left.\left.\min_{13.27\leqslant k\leqslant 500}\left(k\frac{F(k(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1} - \int_{\frac{1}{k}}^{\frac{1}{13.27}} \frac{f\left(\frac{(\boldsymbol{\vartheta}_1(t_1)-t_1-t_2)}{t_2}\right)}{t_1t_2^2}dt_2\right)\right) dt_1\right) \frac{C(N)N}{(\log N)^2} \\ \leqslant{} & 0.282907\frac{C(N)N}{(\log N)^2}. \end{aligned} \tag{20}$$

We shall use Chen's double sieve to gain a small saving on S_6. By the discussion in [21], we know that [[21], Proposition 4.4] can be used to handle the following sum:

$$\sum_{\substack{N^{\frac{1}{2}-\frac{2.9}{13.27}}\leqslant p<N^{\frac{1}{3}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}(N), N^{\frac{1}{13.27}}\right). \tag{21}$$

By the same process as in [21] we get that

$$\begin{aligned} S_6 \leqslant{} & (1+o(1))\frac{2}{e^\gamma}\left(13.27\int_{\frac{57}{224}}^{\frac{1}{3}} \frac{F(13.27(\frac{1}{2}-t))}{t}dt\right)\frac{C(N)N}{(\log N)^2} - G_1 \\ \leqslant{} & (5.265577-0.031029)\frac{C(N)N}{(\log N)^2} \end{aligned}$$

$$\leqslant 5.234548\frac{C(N)N}{(\log N)^2}, \tag{22}$$

where

$$G_1 = 8\left(\log\left(\frac{\frac{9.2}{13.27}}{1-\frac{4.6}{13.27}}\right)\Psi_2(2.3) + \sum_{4\leqslant i\leqslant 5}\log\left(\frac{(2+0.1i)\left(1-\frac{3.8+0.2i}{13.27}\right)}{(1.9+0.1i)\left(1-\frac{4+0.2i}{13.27}\right)}\right)\Psi_2(2+0.1i)\right.$$
$$\left.+\sum_{6\leqslant i\leqslant 9}\log\left(\frac{(2+0.1i)\left(1-\frac{3.8+0.2i}{13.27}\right)}{(1.9+0.1i)\left(1-\frac{4+0.2i}{13.27}\right)}\right)\Psi_1(2+0.1i)\right)\frac{C(N)N}{(\log N)^2}, \tag{23}$$

where $\Psi_1(s)$ and $\Psi_2(s)$ are defined as the same in [[20], Lemmas 5.1–5.2] and we have used the following lower bounds of them. These values can be found at Table 1 of [20].

$$\Psi_2(s) \geqslant \begin{cases} 0.015247971, & s=2.3,\\ 0.013898757, & s=2.4,\\ 0.011776059, & s=2.5,\end{cases} \quad \Psi_1(s) \geqslant \begin{cases} 0.009405211, & s=2.6,\\ 0.006558950, & s=2.7,\\ 0.003536751, & s=2.8,\\ 0.001056651, & s=2.9.\end{cases} \tag{24}$$

Similarly, for S_7 we have

$$S_7 \leqslant (1+o(1))\frac{2}{e^\gamma}\left(13.27\int_{\frac{57}{224}}^{\frac{1}{2}-\frac{3}{13.27}}\frac{F(13.27(\frac{1}{2}-t))}{t}dt\right)\frac{C(N)N}{(\log N)^2}$$
$$\leqslant 1.256371\frac{C(N)N}{(\log N)^2}. \tag{25}$$

For S_8 we can take a maximum of the lower bounds obtained by those two methods we used on the estimation of S_3.

$$S_8 \geqslant (1+o(1))\frac{2}{e^\gamma}\left(\int_{\frac{1}{13.27}}^{\frac{1}{8.24}}\int_{\frac{1}{13.27}}^{t_1}\max\left(13.27\frac{f(13.27(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{13.27})-t_1-t_2))}{t_1t_2}\right.\right.$$
$$+\frac{26.54e^\gamma h(13.27(\frac{1}{2}-t_1-t_2))}{(13.27(\frac{1}{2}-t_1-t_2))t_1t_2}, \max_{13.27\leqslant k\leqslant 500}\left(k\frac{f(13.27(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1t_2}\right.$$
$$\left.\left.\left.+\frac{2ke^\gamma h(k(\frac{1}{2}-t_1-t_2))}{(k(\frac{1}{2}-t_1-t_2))t_1t_2} - \int_{\frac{1}{k}}^{\frac{1}{13.27}}\frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3\right)\right)dt_2dt_1\right)\frac{C(N)N}{(\log N)^2}$$
$$\geqslant 1.691493\frac{C(N)N}{(\log N)^2}. \tag{26}$$

Similarly, for S_9–S_{11} we have

$$S_9 \geqslant (1+o(1))\frac{2}{e^\gamma}\left(\int_{\frac{1}{8.24}}^{\frac{25}{128}}\int_{\frac{1}{13.27}}^{\frac{1}{8.24}}\max\left(13.27\frac{f(13.27(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{13.27})-t_1-t_2))}{t_1t_2}\right.\right.$$
$$+\frac{26.54e^\gamma h(13.27(\frac{1}{2}-t_1-t_2))}{(13.27(\frac{1}{2}-t_1-t_2))t_1t_2}, \max_{13.27\leqslant k\leqslant 500}\left(k\frac{f(13.27(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1t_2}\right.$$
$$\left.\left.\left.+\frac{2ke^\gamma h(k(\frac{1}{2}-t_1-t_2))}{(k(\frac{1}{2}-t_1-t_2))t_1t_2} - \int_{\frac{1}{k}}^{\frac{1}{13.27}}\frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3\right)\right)dt_2dt_1\right)\frac{C(N)N}{(\log N)^2}$$
$$\geqslant 3.367923\frac{C(N)N}{(\log N)^2}, \tag{27}$$

$$S_{10}+S_{11} \geqslant (1+o(1))\frac{2}{e^\gamma}\left(13.27\int_{\frac{25}{128}}^{\frac{57}{224}}\int_{\frac{1}{13.27}}^{\frac{1}{8.24}}\frac{f(13.27(\boldsymbol{\vartheta}_1(t_1)-t_1-t_2))}{t_1t_2}dt_2dt_1\right.$$

$$+\,13.27\int_{\frac{57}{224}}^{\frac{1}{2}-\frac{3}{13.27}}\int_{\frac{1}{13.27}}^{\frac{1}{8.24}}\frac{f(13.27(\frac{1}{2}-t_1-t_2))}{t_1t_2}dt_2dt_1\Bigg)\frac{C(N)N}{(\log N)^2}+G_2$$
$$\geqslant (1.462958+0.041633)\frac{C(N)N}{(\log N)^2}$$
$$\geqslant 1.504591\frac{C(N)N}{(\log N)^2}, \tag{28}$$

where

$$G_2=4\Bigg(13.27\int_{\frac{25}{128}}^{\frac{1}{2}-\frac{2}{8.24}}\int_{\frac{1}{13.27}}^{\frac{1}{8.24}}\frac{h(13.27(\frac{1}{2}-t_1-t_2))}{(13.27(\frac{1}{2}-t_1-t_2))t_1t_2}dt_2dt_1$$
$$+13.27\int_{\frac{1}{2}-\frac{2}{8.24}}^{\frac{1}{2}-\frac{3}{13.27}}\int_{\frac{1}{13.27}}^{\frac{1.5}{13.27}}\frac{h(13.27(\frac{1}{2}-t_1-t_2))}{(13.27(\frac{1}{2}-t_1-t_2))t_1t_2}dt_2dt_1\Bigg)\frac{C(N)N}{(\log N)^2}. \tag{29}$$

For the remaining terms, we can use Chen's switching principle together with Lichtman's distribution level to estimate them. Namely, for S_{12} we have

$$S_{12}=\sum_{\substack{N^{\frac{1}{2}-\frac{3}{13.27}}\leqslant p_1<p_2<(\frac{N}{p_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}}S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(Np_1),p_2\right)=S\left(\mathcal{A}';\mathcal{P}(N),N^{\frac{1}{2}}\right), \tag{30}$$

where the set $\mathcal{A}'$ is defined as

$$\mathcal{A}'=\left\{N-p_1p_2m : N^{\frac{1}{2}-\frac{3}{13.27}}\leqslant p_1<p_2<(N/p_1)^{\frac{1}{2}},\ p'\mid m\Rightarrow p'>p_2 \text{ or } p'=p_1\right\}.$$

We note that each m above must be a prime number since $\frac{1}{2}-\frac{3}{13.27}>\frac{1}{4}$. By Buchstab's identity, we have

$$\begin{aligned}S_{12}=S\left(\mathcal{A}';\mathcal{P}(N),N^{\frac{1}{2}}\right)&\leqslant S\left(\mathcal{A}';\mathcal{P}(N),N^{\frac{25}{128}}\right)\\&=S\left(\mathcal{A}';\mathcal{P}(N),N^{\frac{1}{500}}\right)-\sum_{\substack{N^{\frac{1}{500}}\leqslant p'<N^{\frac{25}{128}}\\(p',N)=1}}S\left(\mathcal{A}'_{p'};\mathcal{P}(N),N^{\frac{1}{500}}\right)\\&\quad+\sum_{\substack{N^{\frac{1}{500}}\leqslant p_2'<p_1'<N^{\frac{25}{128}}\\(p_1'p_2',N)=1}}S\left(\mathcal{A}'_{p_1'p_2'};\mathcal{P}(N),N^{\frac{1}{500}}\right)\\&\quad-\sum_{\substack{N^{\frac{1}{500}}\leqslant p_3'<p_2'<p_1'<N^{\frac{25}{128}}\\(p_1'p_2'p_3',N)=1}}S\left(\mathcal{A}'_{p_1'p_2'p_3'};\mathcal{P}(N),p_3'\right).\end{aligned} \tag{31}$$

Then by Lemma 2.3, Iwaniec's linear sieve method and arguments in [15] and [16] we have

$$S_{12}\leqslant(1+o(1))\frac{2C(N)\left|\mathcal{A}'\right|}{e^{\gamma}\log N}\Bigg(500F\left(500\boldsymbol{\vartheta}_{\frac{1}{500}}\right)-500\int_{\frac{1}{500}}^{\frac{25}{128}}\frac{f(500(\boldsymbol{\vartheta}_1(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt$$
$$+\,500\int_{\frac{1}{500}}^{\frac{25}{128}}\int_{\frac{1}{500}}^{t_1}\frac{F(500(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1$$
$$-\int_{\frac{1}{500}}^{\frac{25}{128}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2}\frac{f\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\Bigg)$$
$$\leqslant(1+o(1))\frac{2G_3}{e^{\gamma}}\left(\int_2^{\frac{1927}{727}}\frac{\log(t-1)}{t}dt\right)\frac{C(N)N}{(\log N)^2}$$
$$\leqslant 0.498525\frac{C(N)N}{(\log N)^2}, \tag{32}$$

where

$$G_3 = 500F\left(500\boldsymbol{\vartheta}_{\frac{1}{500}}\right) - 500\int_{\frac{1}{500}}^{\frac{25}{128}} \frac{f(500(\boldsymbol{\vartheta}_1(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt$$
$$+ 500\int_{\frac{1}{500}}^{\frac{25}{128}}\int_{\frac{1}{500}}^{t_1} \frac{F(500(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1$$
$$- \int_{\frac{1}{500}}^{\frac{25}{128}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2} \frac{f\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1. \tag{33}$$

Similarly, for S_{13}–S_{16} we have

$$S_{13} \leqslant (1+o(1))\frac{2G_3}{e^\gamma}\left(\int_2^{12.27}\frac{\log\left(2-\frac{3}{t+1}\right)}{t}dt\right)\frac{C(N)N}{(\log N)^2}$$
$$\leqslant 4.514343\frac{C(N)N}{(\log N)^2}, \tag{34}$$

$$S_{14} \leqslant (1+o(1))\frac{2G_3}{e^\gamma}\left(\int_{\frac{1927}{727}}^{7.24}\frac{\log\left(\frac{1927}{727}-\frac{\frac{2654}{727}}{t+1}\right)}{t}dt\right)\frac{C(N)N}{(\log N)^2}$$
$$\leqslant 4.576860\frac{C(N)N}{(\log N)^2}, \tag{35}$$

$$S_{15} \leqslant (1+o(1))\frac{2G_3}{e^\gamma}\left(\int_{\frac{1}{13.27}}^{\frac{1}{8.24}}\int_{t_1}^{\frac{1}{8.24}}\int_{t_2}^{\frac{1}{8.24}}\int_{t_3}^{\frac{1}{8.24}}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C(N)N}{(\log N)^2}$$
$$\leqslant 0.090595\frac{C(N)N}{(\log N)^2}, \tag{36}$$

$$S_{16} \leqslant (1+o(1))\frac{2G_3}{e^\gamma}\left(\int_{\frac{1}{13.27}}^{\frac{1}{8.24}}\int_{t_1}^{\frac{1}{8.24}}\int_{t_2}^{\frac{1}{8.24}}\int_{\frac{1}{8.24}}^{\frac{1}{2}-\frac{2}{13.27}-t_3}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C(N)N}{(\log N)^2}$$
$$\leqslant 0.499530\frac{C(N)N}{(\log N)^2}. \tag{37}$$

Finally, by Lemma 3.1 and (9)–(37) we get

$$3S_1+S_2+S_8+S_9+S_{10}+S_{11} \geqslant 60.495865\frac{C(N)N}{(\log N)^2},$$

$$2S_3+2S_4+2S_5+S_6+S_7+2S_{12}+S_{13}+S_{14}+S_{15}+S_{16} \leqslant 53.563025\frac{C(N)N}{(\log N)^2},$$

$$4D_{1,2}(N) \geqslant (3S_1+S_2+S_8+S_9+S_{10}+S_{11})$$
$$-(2S_3+2S_4+2S_5+S_6+S_7+2S_{12}+S_{13}+S_{14}+S_{15}+S_{16})$$
$$\geqslant 6.932\frac{C(N)N}{(\log N)^2},$$

$$D_{1,2}(N) \geqslant 1.733\frac{C(N)N}{(\log N)^2}.$$

Theorem 1.1 is proved. Since the detail of the proof of Theorem 1.2 is similar to those of Theorem 1.1 and Theorem 1.1 in [14] so we omit it in this paper.

5. Proof of Theorem 1.3

In this section, sets $\mathcal{B}$ and $\mathcal{P}$ are defined respectively. For S_1' and S_2', by Buchstab's identity, we have

$$\begin{aligned} S_1' = S\left(\mathcal{B};\mathcal{P},x^{\frac{1}{12}}\right) = S\left(\mathcal{B};\mathcal{P},x^{\frac{1}{500}}\right) &- \sum_{x^{\frac{1}{500}}\leqslant p<x^{\frac{1}{12}}} S\left(\mathcal{B}_p;\mathcal{P},x^{\frac{1}{500}}\right) \\ &+ \sum_{x^{\frac{1}{500}}\leqslant p_2<p_1<x^{\frac{1}{12}}} S\left(\mathcal{B}_{p_1p_2};\mathcal{P},x^{\frac{1}{500}}\right) \\ &- \sum_{x^{\frac{1}{500}}\leqslant p_3<p_2<p_1<x^{\frac{1}{12}}} S\left(\mathcal{B}_{p_1p_2p_3};\mathcal{P},p_3\right) \end{aligned} \tag{38}$$

and

$$\begin{aligned} S_2' = S\left(\mathcal{B};\mathcal{P},x^{\frac{1}{7.2}}\right) = S\left(\mathcal{B};\mathcal{P},x^{\frac{1}{500}}\right) &- \sum_{x^{\frac{1}{500}}\leqslant p<x^{\frac{1}{7.2}}} S\left(\mathcal{B}_p;\mathcal{P},x^{\frac{1}{500}}\right) \\ &+ \sum_{x^{\frac{1}{500}}\leqslant p_2<p_1<x^{\frac{1}{7.2}}} S\left(\mathcal{B}_{p_1p_2};\mathcal{P},x^{\frac{1}{500}}\right) \\ &- \sum_{x^{\frac{1}{500}}\leqslant p_3<p_2<p_1<x^{\frac{1}{7.2}}} S\left(\mathcal{B}_{p_1p_2p_3};\mathcal{P},p_3\right). \end{aligned} \tag{39}$$

By Lemma 2.2, Iwaniec's linear sieve method and arguments in [15] and [16] we have

$$\begin{aligned} S_1' \geqslant{}& (1+o(1))\frac{1}{e^\gamma}\Bigg(500f\left(500\boldsymbol{\vartheta}'_{\frac{1}{500}}\right) - 500\int_{\frac{1}{500}}^{\frac{1}{12}} \frac{F(500(\boldsymbol{\vartheta}_0(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt \\ &+ 500\int_{\frac{1}{500}}^{\frac{1}{12}}\int_{\frac{1}{500}}^{t_1} \frac{f(500(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1 \\ &- \int_{\frac{1}{500}}^{\frac{1}{12}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2} \frac{F\left(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\Bigg)\frac{C_2x}{(\log x)^2} \\ \geqslant{}& 6.737438\frac{C_2x}{(\log x)^2} \end{aligned} \tag{40}$$

and

$$\begin{aligned} S_2' \geqslant{}& (1+o(1))\frac{1}{e^\gamma}\Bigg(500f\left(500\boldsymbol{\vartheta}'_{\frac{1}{500}}\right) - 500\int_{\frac{1}{500}}^{\frac{1}{7.2}} \frac{F(500(\boldsymbol{\vartheta}_0(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt \\ &+ 500\int_{\frac{1}{500}}^{\frac{1}{7.2}}\int_{\frac{1}{500}}^{t_1} \frac{f(500(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1 \\ &- \int_{\frac{1}{500}}^{\frac{1}{7.2}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2} \frac{F\left(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\Bigg)\frac{C_2x}{(\log x)^2} \\ \geqslant{}& 4.008831\frac{C_2x}{(\log x)^2}, \end{aligned} \tag{41}$$

where $\boldsymbol{\vartheta}'_{\frac{1}{500}} = \frac{16483}{26750}$. For S_3'–S_7', by Lemma 2.2, Iwaniec's linear sieve method and above discussion, we have

$$\begin{aligned} S_3' \geqslant{}& (1+o(1))\frac{1}{e^\gamma}\Bigg(\int_{\frac{1}{12}}^{\frac{1}{7.2}}\int_{\frac{1}{12}}^{t_1} \max\Bigg(12\frac{f(12(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{12})-t_1-t_2))}{t_1t_2}, \\ &\max_{12\leqslant k\leqslant 500}\Bigg(k\frac{f(k(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1t_2} \end{aligned}$$

$$
\begin{aligned}
&\quad - \int_{\frac{1}{k}}^{\frac{1}{12}} \frac{F\left(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 \Bigg)\Bigg) dt_2 dt_1 \Bigg) \frac{C_2 x}{(\log x)^2} \\
&\geqslant 0.874702 \frac{C_2 x}{(\log x)^2}, \qquad (42)\\
S_4' &\geqslant (1+o(1)) \frac{1}{e^\gamma} \Bigg(\int_{\frac{1}{7.2}}^{\frac{25}{107}} \int_{\frac{1}{12}}^{\frac{1}{7.2}} \max \Bigg(12 \frac{f(12(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{12})-t_1-t_2))}{t_1 t_2}, \\
&\max_{12 \leqslant k \leqslant 500} \Bigg(k \frac{f(k(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1 t_2} \\
&\quad - \int_{\frac{1}{k}}^{\frac{1}{12}} \frac{F\left(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 \Bigg)\Bigg) dt_2 dt_1 \Bigg) \frac{C_2 x}{(\log x)^2} \\
&\geqslant 1.704764 \frac{C_2 x}{(\log x)^2}, \qquad (43)\\
S_5' &\geqslant (1+o(1)) \frac{1}{e^\gamma} \left(12 \int_{\frac{1}{12}}^{\frac{1}{7.2}} \int_{\frac{25}{107}}^{\min\left(\frac{2}{7},\frac{17}{42}-t_1\right)} \frac{f(12(\boldsymbol{\vartheta}_0(t_2)-t_1-t_2))}{t_1 t_2} dt_2 dt_1 \right) \frac{C_2 x}{(\log x)^2} \\
&\geqslant 0.448166 \frac{C_2 x}{(\log x)^2}, \qquad (44)\\
S_6' &\leqslant (1+o(1)) \frac{1}{e^\gamma} \Bigg(\int_{\frac{1}{12}}^{\frac{25}{107}} \min \Bigg(12 \frac{F(12(\boldsymbol{\vartheta}_0(t_1,\frac{1}{12},\frac{1}{12})-t_1))}{t_1}, \\
&\min_{12 \leqslant k \leqslant 500} \Bigg(k \frac{F(k(\boldsymbol{\vartheta}_0(t_1,\frac{1}{k},\frac{1}{k})-t_1))}{t_1} - k \int_{\frac{1}{k}}^{\frac{1}{12}} \frac{f(k(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1 t_2} dt_2 \\
&\quad + \int_{\frac{1}{k}}^{\frac{1}{12}} \int_{\frac{1}{k}}^{t_2} \frac{F\left(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 \Bigg)\Bigg) dt_1 \Bigg) \frac{C_2 x}{(\log x)^2} \\
&\leqslant 6.953322 \frac{C_2 x}{(\log x)^2}, \qquad (45)\\
S_7' &\leqslant (1+o(1)) \frac{1}{e^\gamma} \Bigg(\int_{\frac{25}{107}}^{\frac{2}{7}} \min \Bigg(12 \frac{F(12(\boldsymbol{\vartheta}_0(t_1)-t_1))}{t_1}, \\
&\min_{12 \leqslant k \leqslant 500} \Bigg(k \frac{F(k(\boldsymbol{\vartheta}_0(t_1)-t_1))}{t_1} - \int_{\frac{1}{k}}^{\frac{1}{12}} \frac{f\left(\frac{(\boldsymbol{\vartheta}_0(t_1)-t_1-t_2)}{t_2}\right)}{t_1 t_2^2} dt_2 \Bigg)\Bigg) dt_1 \Bigg) \frac{C_2 x}{(\log x)^2} \\
&\leqslant 1.390939 \frac{C_2 x}{(\log x)^2}. \qquad (46)
\end{aligned}
$$

For S_{12}'–S_{19}', by Chen's switching principle, Lemma 2.4 and above arguments on estimating S_{12}–S_{16} we have

$$
\begin{aligned}
S_{12}' &\leqslant (1+o(1)) \frac{G_4}{e^\gamma} \left(\int_2^{11} \frac{\log\left(2-\frac{3}{t+1}\right)}{t} dt \right) \frac{C_2 x}{(\log x)^2} \\
&\leqslant 1.981662 \frac{C_2 x}{(\log x)^2}, \qquad (47)\\
S_{13}' &\leqslant (1+o(1)) \frac{G_4}{e^\gamma} \left(\int_{2.5}^{6.2} \frac{\log\left(2.5-\frac{3.5}{t+1}\right)}{t} dt \right) \frac{C_2 x}{(\log x)^2}
\end{aligned}
$$

$$\leqslant 1.717054\frac{C_2x}{(\log x)^2}, \tag{48}$$

$$S'_{14} \leqslant (1+o(1))\frac{G_4}{e^\gamma}\left(\int_2^{2.5}\frac{\log(t-1)}{t}dt\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 0.153821\frac{C_2x}{(\log x)^2}, \tag{49}$$

$$S'_{15} \leqslant (1+o(1))\frac{G_4}{e^\gamma}\left(\int_{\frac{1}{12}}^{\frac{1}{7.2}}\int_{t_1}^{\frac{1}{7.2}}\int_{t_2}^{\frac{1}{7.2}}\int_{t_3}^{\frac{1}{7.2}}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 0.051713\frac{C_2x}{(\log x)^2}, \tag{50}$$

$$S'_{16} \leqslant (1+o(1))\frac{G_4}{e^\gamma}\left(\int_{\frac{1}{12}}^{\frac{5}{42}}\int_{t_1}^{\frac{5}{42}}\int_{t_2}^{\frac{5}{42}}\int_{\frac{1}{7.2}}^{\frac{2}{7}}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 0.101840\frac{C_2x}{(\log x)^2}, \tag{51}$$

$$S'_{17} \leqslant (1+o(1))\frac{G_4}{e^\gamma}\left(\int_{\frac{1}{12}}^{\frac{5}{42}}\int_{t_1}^{\frac{5}{42}}\int_{\frac{5}{42}}^{\frac{1}{7.2}}\int_{\frac{1}{7.2}}^{\frac{17}{42}-t_3}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 0.118478\frac{C_2x}{(\log x)^2}, \tag{52}$$

$$S'_{18} \leqslant (1+o(1))\frac{G_4}{e^\gamma}\left(\int_{\frac{1}{12}}^{\frac{5}{42}}\int_{\frac{5}{42}}^{\frac{1}{7.2}}\int_{t_2}^{\frac{1}{7.2}}\int_{\frac{1}{7.2}}^{\frac{17}{42}-t_3}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 0.042337\frac{C_2x}{(\log x)^2}, \tag{53}$$

$$S'_{19} \leqslant (1+o(1))\frac{G_4}{e^\gamma}\left(\int_{\frac{5}{42}}^{\frac{1}{7.2}}\int_{t_1}^{\frac{1}{7.2}}\int_{t_2}^{\frac{1}{7.2}}\int_{\frac{1}{7.2}}^{\frac{17}{42}-t_3}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 0.005901\frac{C_2x}{(\log x)^2}. \tag{54}$$

where

$$G_4 = 500F\left(500\boldsymbol{\vartheta}'_{\frac{1}{500}}\right) - 500\int_{\frac{1}{500}}^{\frac{21}{107}}\frac{f(500(\boldsymbol{\vartheta}_0(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt$$
$$+500\int_{\frac{1}{500}}^{\frac{21}{107}}\int_{\frac{1}{500}}^{t_1}\frac{F(500(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1$$
$$-\int_{\frac{1}{500}}^{\frac{21}{107}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2}\frac{f\left(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1. \tag{55}$$

For the remaining terms, by the arguments in [3] and [21], we have

$$S'_8 \ll \frac{\varepsilon C_2x}{(\log x)^2}, \tag{56}$$

$$S'_9 \leqslant (1+o(1))\frac{12}{e^\gamma}\left(\int_{(\frac{11}{20}-\frac{29}{100})12}^{(\frac{4}{7}-\frac{2}{7})12}\frac{F(t)}{2\times 12-t}dt\right) \leqslant 0.111039\frac{C_2x}{(\log x)^2}, \tag{57}$$

$$S'_{10} \leqslant (1+o(1))\frac{12}{e^{\gamma}}\left(\int_{(\frac{11}{20}-\frac{1}{3})12}^{(\frac{11}{20}-\frac{29}{100})12} \frac{F(t)}{\frac{11}{20}\times 12 - t}dt\right) \leqslant 1.169696\frac{C_2 x}{(\log x)^2}, \tag{58}$$

$$S'_{11} \ll \frac{\varepsilon C_2 x}{(\log x)^2}. \tag{59}$$

Finally, by Lemma 3.2 and (38)–(59) we get

$$3S'_1 + S'_2 + S'_3 + S'_4 + S'_5 \geqslant 27.248777\frac{C_2 x}{(\log x)^2},$$

$$\begin{aligned}
&2S'_6 + 2S'_7 + S'_8 + S'_9 + S'_{10} + S'_{11} + S'_{12} + S'_{13}\\
&+ 2S'_{14} + S'_{15} + S'_{16} + S'_{17} + S'_{18} + S'_{19} \leqslant 22.295884\frac{C_2 x}{(\log x)^2},\\
&4\pi_{1,2}(x) \geqslant (3S'_1 + S'_2 + S'_3 + S'_4 + S'_5)\\
&\qquad - (2S'_6 + 2S'_7 + S'_8 + S'_9 + S'_{10} + S'_{11} + S'_{12} + S'_{13}\\
&\qquad + 2S'_{14} + S'_{15} + S'_{16} + S'_{17} + S'_{18} + S'_{19})\\
&\qquad \geqslant 4.952\frac{C_2 x}{(\log x)^2},
\end{aligned}$$

$$\pi_{1,2}(x) \geqslant 1.238\frac{C_2 x}{(\log x)^2}.$$

Theorem 1.3 is proved.

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