# LARGEST SQUARE DIVISORS OF SHIFTED PRIMES

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ABSTRACT. The author shows that there are infinitely many primes $p$ such that for any nonzero integer $a$, $p-a$ is divisible by a square $d^2 > p^{\frac{1}{2}+\frac{1}{700}}$. The exponent $\frac{1}{2}+\frac{1}{700}$ improves Merikoski's $\frac{1}{2}+\frac{1}{2000}$. Many powerful devices in Harman's sieve are used for this improvement.

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## 1. INTRODUCTION

The Euler's conjecture, which states that there are infinitely many primes of the form $n^2+1$, is one of Landau's problems on prime numbers. There are several ways to attack this conjecture. One way is to relax the number of prime factors of $f(n)$, and the best result in this way is due to Iwaniec [4]. Building on the previous work of Richert [11], he showed that for any irreducible polynomial $f(n) = an^2+bn+c$ with $a>0$ and $c \equiv 1(\bmod 2)$, there are infinitely many $x$ such that $f(x)$ has at most 2 prime factors.

Another possible way is to consider the square divisors of $p-1$. If we can show that there are infinitely many primes such that $p-1$ is divisible by a large square $d^2 \geqslant p^\theta$ with $\theta = 1$, then the Euler's conjecture is solved. The first result on this direction is due to Baier and Zhao [1] [2], where they proved the above statement holds with $d$ prime and $\theta < \frac{4}{9}$ as an application of their large sieve for sparse sets of moduli. They interpret the problem as an equidistribution problem for primes $p \equiv 1(\bmod\ d^2)$, after which the result follows from their Bombieri–Vinogradov type theorem for sparse sets of moduli [[1], Theorem 3].

In 2009, Matomäki [9] improved the above result to $\theta < \frac{1}{2}$ using Harman's sieve [3] and Type–II information obtained using the large sieve of Baier and Zhao [2]. Note that the exponent $\theta = \frac{1}{2}$ is the limit of what can be obtained under the Generalized Riemann Hypothesis (GRH). In [10], Merikoski first broke this $\frac{1}{2}$–barrier and successfully got $\theta \leqslant \frac{1}{2}+\frac{1}{2000}$ without the restriction that $d$ is a prime. In the article [10], he mentioned that the "extra" exponent $\frac{1}{2000}$ has not been fully optimized, and one should be able to increase this to some value between $\frac{1}{500}$ and $\frac{1}{1000}$. In this paper, we increase this to $\frac{1}{700}$ by a careful decomposition on Harman's sieve.

**Theorem 1.1.** *Let $a \neq 0$ be an integer. There are infinitely many primes $p$ such that $d^2 \mid (p-a)$ for some integer $d$ with*

$$d^2 \geqslant p^{\frac{1}{2}+\frac{1}{700}}.$$

Throughout this paper, we always suppose that $\varepsilon$ is a sufficiently small positive constant and $X$ is sufficiently large. The letter $p$, with or without subscript, is reserved for prime numbers. Let $\varpi = \frac{1}{1400}$, $D = X^{\frac{1}{2}+2\varpi}$, $K = \lceil \frac{1}{\varepsilon} \rceil$ and $P = D^{\frac{1}{K}}$. Let $\sigma$ be a number satifsies the condition $19\sigma + 90\varpi + 71\varepsilon < 1$. Define

$$I_j = \left(2^{j-1}P^{\frac{1}{2}}, 2^j P^{\frac{1}{2}}\right] \text{ for } j = 1, 2, \ldots, K.$$

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We set

$$\mathcal{D}=\{p_1^2p_2^2\cdots p_K^2 : p_j\in I_j \text{ for } j=1,2,\ldots,K\}$$

so that $d^2\in D$ is of size $\asymp D$ and is a square of a squarefree integer.

Fix an integer $a\neq 0$ and a $C^\infty$–smooth function $0\leqslant\psi\leqslant 1$, supported on the interval $[1,2]$ and satisfying $\psi(x)=1$ for $1+\eta\leqslant x\leqslant 2-\eta$ for some sufficiently small positive $\eta$. For $d^2\in D$ and $z<X$, denote

$$S\left(\mathcal{A}^d,z\right)=\sum_{\substack{n\equiv a(\bmod d^2)\\(n,P(z))=1}}\psi\left(\frac{n}{X}\right)\quad\text{and}\quad S\left(\mathcal{B}^d,z\right)=\frac{1}{\varphi(d^2)}\sum_{\substack{(n,d^2)=1\\(n,P(z))=1}}\psi\left(\frac{n}{X}\right).$$

Then Theorem 1.1 holds if there exists $\varepsilon,\eta,c>0$ such that for all but $O\left(D^{\frac{1}{2}}X^{-\eta}\right)$ of the moduli $d^2\in\mathcal{D}$, we have

$$S\left(\mathcal{A}^d,2X^{\frac{1}{2}}\right)\geqslant cS\left(\mathcal{B}^d,2X^{\frac{1}{2}}\right). \tag{1}$$

## 2. Asymptotic formulas

**Lemma 2.1.** *([10], Proposition 7]). Let $U\leqslant X^{\frac{1}{2}+2\varpi+\varepsilon}$ and let $a_u$ be divisor–bounded. Then for all but $O\left(D^{\frac{1}{2}}X^{-\eta}\right)$ of $d^2\in\mathcal{D}$, we have*

$$\sum_{u\sim U}a_uS\left(\mathcal{A}_u^d,X^{\sigma-2\varpi}\right)=(1+o(1))\sum_{u\sim U}a_uS\left(\mathcal{B}_u^d,X^{\sigma-2\varpi}\right).$$

**Lemma 2.2.** *([10], Proposition 6]). Let $U\leqslant X^{\frac{1}{2}-\sigma}$, $V\leqslant X^{\frac{1}{8}+\frac{\sigma}{2}-\frac{5\varpi}{2}-\eta}$ and let $a_u$, $b_v$ be divisor–bounded. Then for all but $O\left(D^{\frac{1}{2}}X^{-\eta}\right)$ of $d^2\in\mathcal{D}$, we have*

$$\sum_{\substack{u\sim U\\v\sim V}}a_ub_vS\left(\mathcal{A}_{uv}^d,X^{\sigma-2\varpi}\right)=(1+o(1))\sum_{\substack{u\sim U\\v\sim V}}a_ub_vS\left(\mathcal{B}_{uv}^d,X^{\sigma-2\varpi}\right).$$

**Lemma 2.3.** *([10], Proposition 4]). Let $UV=X$, $X^{\frac{1}{2}-\sigma}\leqslant U\leqslant X^{\frac{1}{2}-2\varpi-\varepsilon}$ and let $a_u$, $b_v$ be divisor–bounded. Then we have*

$$\sum_{d^2\in\mathcal{D}}\left|\sum_{\substack{uv\equiv a(\bmod d^2)\\u\sim U,v\sim V}}a_ub_v\psi\left(\frac{uv}{X}\right)-\frac{1}{\varphi(d^2)}\sum_{\substack{(uv,d^2)=1\\u\sim U,v\sim V}}a_ub_v\psi\left(\frac{uv}{X}\right)\right|\ll D^{-\frac{1}{2}}X^{1-\eta}.$$

## 3. The final decomposition

Before decomposing, we define asymptotic regions $I$ and $II$ as

$$I(m,n):=\left\{m+n\leqslant\frac{1}{2}+2\varpi,\ \text{or } m\leqslant\frac{1}{2}-\sigma \text{ and } n<\frac{1}{8}+\frac{\sigma}{2}-\frac{5\varpi}{2}\right\},$$

$$II(m,n):=\left\{\frac{1}{2}-\sigma\leqslant m\leqslant\frac{1}{2}-2\varpi \text{ or } \frac{1}{2}-\sigma\leqslant n\leqslant\frac{1}{2}-2\varpi\right.$$
$$\left.\text{or } \frac{1}{2}-\sigma\leqslant m+n\leqslant\frac{1}{2}-2\varpi \text{ or } \frac{1}{2}+2\varpi\leqslant m+n\leqslant\frac{1}{2}+\sigma\right\}.$$

Let $\omega(u)$ denotes the Buchstab function determined by the following differential–difference equation

$$\begin{cases}\omega(u)=\frac{1}{u}, & 1\leqslant u\leqslant 2,\\ (u\omega(u))'=\omega(u-1), & u\geqslant 2.\end{cases}$$

Moreover, we have the upper and lower bounds for $\omega(u)$:

$$\omega(u)\geqslant\omega_0(u)=\begin{cases}\frac{1}{u}, & 1\leqslant u<2,\\ \frac{1+\log(u-1)}{u}, & 2\leqslant u<3,\\ \frac{1+\log(u-1)}{u}+\frac{1}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt, & 3\leqslant u<4,\\ 0.5612, & u\geqslant 4,\end{cases}$$

$$\omega(u) \leqslant \omega_1(u) = \begin{cases} \frac{1}{u}, & 1 \leqslant u < 2, \\ \frac{1+\log(u-1)}{u}, & 2 \leqslant u < 3, \\ \frac{1+\log(u-1)}{u} + \frac{1}{u}\int_2^{u-1} \frac{\log(t-1)}{t} dt, & 3 \leqslant u < 4, \\ 0.5617, & u \geqslant 4. \end{cases}$$

We shall use $\omega_0(u)$ and $\omega_1(u)$ to give numerical bounds for some sieve functions discussed below. We shall also use the simple upper bound $\omega(u) \leqslant \max(\frac{1}{u}, 0.5672)$ (see Lemma 8(iii) of [5]) to estimate high–dimensional integrals. Fix $\sigma = \frac{1}{20.31}$ and let $p_j = X^{\alpha_j}$. By Buchstab's identity, we have

$$\begin{aligned} S\left(\mathcal{A}^d, 2X^{\frac{1}{2}}\right) &= S\left(\mathcal{A}^d, X^{\sigma-2\varpi}\right) - \sum_{\sigma-2\varpi \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^d_{p_1}, X^{\sigma-2\varpi}\right) + \sum_{\substack{\sigma-2\varpi \leqslant \alpha_1 < \frac{1}{2} \\ \sigma-2\varpi \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1-\alpha_1}{2}\right)}} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right) \\ &= S\left(\mathcal{A}^d, X^{\sigma-2\varpi}\right) - \sum_{\sigma-2\varpi \leqslant \alpha_1 < \frac{1}{2}} S\left(\mathcal{A}^d_{p_1}, X^{\sigma-2\varpi}\right) + \sum_{(\alpha_1,\alpha_2)\in II} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right) \\ &\quad + \sum_{(\alpha_1,\alpha_2)\in A} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right) + \sum_{(\alpha_1,\alpha_2)\in B} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right) + \sum_{(\alpha_1,\alpha_2)\in C} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right) \\ &= S_1 - S_2 + S_{II} + S_A + S_B + S_C, \end{aligned} \tag{2}$$

where

$$\begin{aligned} A(\alpha_1, \alpha_2) = &\left\{\sigma - 2\varpi \leqslant \alpha_1 < \frac{1}{2},\ \sigma - 2\varpi \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1-\alpha_1}{2}\right),\ (\alpha_1,\alpha_2) \notin II,\right. \\ &\left.\quad (\alpha_1,\alpha_2,\alpha_2) \text{ can be partitioned into } (m,n) \in I\right\}, \\ B(\alpha_1, \alpha_2) = &\left\{\sigma - 2\varpi \leqslant \alpha_1 < \frac{1}{2},\ \sigma - 2\varpi \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1-\alpha_1}{2}\right),\ (\alpha_1,\alpha_2) \notin II,\right. \\ &\left.\quad (\alpha_1,\alpha_2,\alpha_2) \text{ cannot be partitioned into } (m,n) \in I,\ (\alpha_1,\alpha_2) \in I,\ (1-\alpha_1-\alpha_2,\alpha_2) \in I\right\}, \\ C(\alpha_1, \alpha_2) = &\left\{\sigma - 2\varpi \leqslant \alpha_1 < \frac{1}{2},\ \sigma - 2\varpi \leqslant \alpha_2 < \min\left(\alpha_1, \frac{1-\alpha_1}{2}\right),\ (\alpha_1,\alpha_2) \notin II \cup A \cup B\right\}. \end{aligned}$$

We have asymptotic formulas for $S_1$ and $S_2$ by Lemma 2.1. We can also give an asymptotic formula for $S_{II}$ by Lemma 2.3. For $S_A$, we can apply Buchstab's identity to get

$$\begin{aligned} S_A &= \sum_{(\alpha_1,\alpha_2)\in A} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right) \\ &= \sum_{(\alpha_1,\alpha_2)\in A} S\left(\mathcal{A}^d_{p_1p_2}, X^{\sigma-2\varpi}\right) - \sum_{\substack{(\alpha_1,\alpha_2)\in A \\ \sigma-2\varpi \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right) \\ (\alpha_1,\alpha_2,\alpha_3) \text{ can be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3}, p_3\right) \\ &\quad - \sum_{\substack{(\alpha_1,\alpha_2)\in A \\ \sigma-2\varpi \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right) \\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3}, X^{\sigma-2\varpi}\right) \\ &\quad + \sum_{\substack{(\alpha_1,\alpha_2)\in A \\ \sigma-2\varpi \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right) \\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II \\ \sigma-2\varpi \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1-\alpha_1-\alpha_2-\alpha_3}{2}\right) \\ (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \text{ can be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3p_4}, p_4\right) \\ &\quad + \sum_{\substack{(\alpha_1,\alpha_2)\in A \\ \sigma-2\varpi \leqslant \alpha_3 < \min\left(\alpha_2, \frac{1-\alpha_1-\alpha_2}{2}\right) \\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II \\ \sigma-2\varpi \leqslant \alpha_4 < \min\left(\alpha_3, \frac{1-\alpha_1-\alpha_2-\alpha_3}{2}\right) \\ (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \text{ cannot be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3p_4}, p_4\right) \end{aligned}$$

$$= S_{A1} - S_{A2} - S_{A3} + S_{A4} + S_{A5}. \tag{3}$$

By Lemmas 2.1–2.2 we have asymptotic formulas for $S_{A1}$ and $S_{A3}$. For $S_{A2}$ and $S_{A4}$ we can also give asymptotic formulas by Lemma 2.3. We discard part of $S_{A5}$ if we can neither give an asymptotic formula nor decompose it further. For the remaining part, we can perform Buchstab's identity twice more to reach a six–dimensional sum if we can group $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4)$ into $(m, n) \in I$, and we can use reversed Buchstab's identity to make some almost–primes visible. Working as in [7] and [8], the total loss from $S_A$ can be bounded by

$$\begin{aligned}
&\Bigg(\int_{\sigma-2\varpi}^{\frac{1}{2}} \int_{\sigma-2\varpi}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4) \in S_{A51}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1 t_2 t_3 t_4^2} dt_4 dt_3 dt_2 dt_1\Bigg) \\
&- \Bigg(\int_{\sigma-2\varpi}^{\frac{1}{2}} \int_{\sigma-2\varpi}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \int_{t_4}^{\frac{1-t_1-t_2-t_3-t_4}{2}} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5) \in S_{A52}] \frac{\omega_0\left(\frac{1-t_1-t_2-t_3-t_4-t_5}{t_5}\right)}{t_1 t_2 t_3 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\Bigg) \\
&+ \Bigg(\int_{\sigma-2\varpi}^{\frac{1}{2}} \int_{\sigma-2\varpi}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \\
&\qquad \int_{\sigma-2\varpi}^{\min\left(t_4, \frac{1-t_1-t_2-t_3-t_4}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_5, \frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6) \in S_{A53}] \frac{\omega_1\left(\frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{t_6}\right)}{t_1 t_2 t_3 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1\Bigg) \\
&+ \Bigg(\int_{\sigma-2\varpi}^{\frac{1}{2}} \int_{\sigma-2\varpi}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_3, \frac{1-t_1-t_2-t_3}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_4, \frac{1-t_1-t_2-t_3-t_4}{2}\right)} \\
&\qquad \int_{\sigma-2\varpi}^{\min\left(t_5, \frac{1-t_1-t_2-t_3-t_4-t_5}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_6, \frac{1-t_1-t_2-t_3-t_4-t_5-t_6}{2}\right)} \\
&\qquad \int_{\sigma-2\varpi}^{\min\left(t_7, \frac{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7}{2}\right)} \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \in S_{A54}] \times \\
&\qquad \frac{\max\left(\frac{t_8}{1-t_1-t_2-t_3-t_4-t_5-t_6-t_7-t_8}, 0.5672\right)}{t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8^2} dt_8 dt_7 dt_6 dt_5 dt_4 dt_3 dt_2 dt_1\Bigg) \\
&< 0.002515,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
S_{A51}(t_1,t_2,t_3,t_4) := \{ & (t_1,t_2) \in S_A, \\
& \sigma - 2\varpi \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
& (t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n) \in II, \\
& \sigma - 2\varpi \leqslant t_4 < \min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right), \\
& (t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in II, \\
& (t_1,t_2,t_3,t_4,t_4) \text{ cannot be partitioned into } (m,n) \in I,
\end{aligned}$$

$$\sigma-2\varpi \leqslant t_1<\frac{1}{2},\ \sigma-2\varpi \leqslant t_2<\min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\},$$

$$\begin{aligned}
S_{A52}(t_1,t_2,t_3,t_4,t_5):= \ \{&(t_1,t_2)\in S_A,\\
&\sigma-2\varpi \leqslant t_3<\min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in II,\\
&\sigma-2\varpi \leqslant t_4<\min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in II,\\
&(t_1,t_2,t_3,t_4,t_4) \text{ cannot be partitioned into } (m,n)\in I,\\
&t_4<t_5<\frac{1}{2}(1-t_1-t_2-t_3-t_4),\\
&(t_1,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (m,n)\in II,\\
&\sigma-2\varpi \leqslant t_1<\frac{1}{2},\ \sigma-2\varpi \leqslant t_2<\min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\},
\end{aligned}$$

$$\begin{aligned}
S_{A53}(t_1,t_2,t_3,t_4,t_5,t_6):= \ \{&(t_1,t_2)\in S_A,\\
&\sigma-2\varpi \leqslant t_3<\min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in II,\\
&\sigma-2\varpi \leqslant t_4<\min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in II,\\
&(t_1,t_2,t_3,t_4,t_4) \text{ can be partitioned into } (m,n)\in I,\\
&\sigma-2\varpi \leqslant t_5<\min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
&(t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n)\in II,\\
&\sigma-2\varpi \leqslant t_6<\min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right),\\
&(t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n)\in II,\\
&(t_1,t_2,t_3,t_4,t_5,t_6,t_6) \text{ cannot be partitioned into } (m,n)\in I,\\
&\sigma-2\varpi \leqslant t_1<\frac{1}{2},\ \sigma-2\varpi \leqslant t_2<\min\left(t_1, \frac{1}{2}(1-t_1)\right)\Bigg\},
\end{aligned}$$

$$\begin{aligned}
S_{A54}(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8):= \ \{&(t_1,t_2)\in S_A,\\
&\sigma-2\varpi \leqslant t_3<\min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n)\in II,\\
&\sigma-2\varpi \leqslant t_4<\min\left(t_3, \frac{1}{2}(1-t_1-t_2-t_3)\right),\\
&(t_1,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n)\in II,\\
&(t_1,t_2,t_3,t_4,t_4) \text{ can be partitioned into } (m,n)\in I,\\
&\sigma-2\varpi \leqslant t_5<\min\left(t_4, \frac{1}{2}(1-t_1-t_2-t_3-t_4)\right),\\
&(t_1,t_2,t_3,t_4,t_5) \text{ cannot be partitioned into } (m,n)\in II,
\end{aligned}$$

$$\sigma-2\varpi \leqslant t_6<\min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5)\right),$$
$$(t_1,t_2,t_3,t_4,t_5,t_6) \text{ cannot be partitioned into } (m,n)\in II,$$
$$(t_1,t_2,t_3,t_4,t_5,t_6,t_6) \text{ can be partitioned into } (m,n)\in I,$$
$$\sigma-2\varpi \leqslant t_7<\min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6)\right),$$
$$(t_1,t_2,t_3,t_4,t_5,t_6,t_7) \text{ cannot be partitioned into } (m,n)\in II,$$
$$\sigma-2\varpi \leqslant t_8<\min\left(t_5, \frac{1}{2}(1-t_1-t_2-t_3-t_4-t_5-t_6-t_7)\right),$$
$$(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) \text{ cannot be partitioned into } (m,n)\in II,$$
$$\left.\sigma-2\varpi \leqslant t_1<\frac{1}{2},\ \sigma-2\varpi \leqslant t_2<\min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}.$$

For $S_B$ we use Buchstab's identity to get

$$S_B=\sum_{(\alpha_1,\alpha_2)\in B} S\left(\mathcal{A}_{p_1p_2}^d, X^{\sigma-2\varpi}\right)-\sum_{\substack{(\alpha_1,\alpha_2)\in B\\ \sigma-2\varpi\leqslant\alpha_3<\min\left(\alpha_2,\frac{1-\alpha_1-\alpha_2}{2}\right)}} S\left(\mathcal{A}_{p_1p_2p_3}^d, p_3\right). \tag{5}$$

Here we cannot decompose it directly using Buchstab's identity once more, since we cannot give an asymptotic formula for part of the negative sum

$$\sum_{\substack{(\alpha_1,\alpha_2)\in B\\ \sigma-2\varpi\leqslant\alpha_3<\min\left(\alpha_2,\frac{1-\alpha_1-\alpha_2}{2}\right)\\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II}} S\left(\mathcal{A}_{p_1p_2p_3}^d, X^{\sigma-2\varpi}\right).$$

However, we can use a role–reversal to transfer the last sum in (5) into a form that have an asymptotic formula. Note that in [10] role–reversals were not used. We refer the readers to [6] and [8] for more applications of role–reversals. By a standard process, we have (where $\beta\sim X^{1-\alpha_1-\alpha_2-\alpha_3}$ and $(\beta, P(p_3))=1$)

$$\begin{aligned}
S_B=&\sum_{(\alpha_1,\alpha_2)\in B} S\left(\mathcal{A}_{p_1p_2}^d, p_2\right)\\
=&\sum_{(\alpha_1,\alpha_2)\in B} S\left(\mathcal{A}_{p_1p_2}^d, X^{\sigma-2\varpi}\right)-\sum_{\substack{(\alpha_1,\alpha_2)\in B\\ \sigma-2\varpi\leqslant\alpha_3<\min\left(\alpha_2,\frac{1-\alpha_1-\alpha_2}{2}\right)\\ (\alpha_1,\alpha_2,\alpha_3) \text{ can be partitioned into } (m,n)\in II}} S\left(\mathcal{A}_{p_1p_2p_3}^d, p_3\right)\\
&-\sum_{\substack{(\alpha_1,\alpha_2)\in B\\ \sigma-2\varpi\leqslant\alpha_3<\min\left(\alpha_2,\frac{1-\alpha_1-\alpha_2}{2}\right)\\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II}} S\left(\mathcal{A}_{\beta p_2p_3}^d, X^{\sigma-2\varpi}\right)\\
&+\sum_{\substack{(\alpha_1,\alpha_2)\in B\\ \sigma-2\varpi\leqslant\alpha_3<\min\left(\alpha_2,\frac{1-\alpha_1-\alpha_2}{2}\right)\\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II\\ \sigma-2\varpi\leqslant\alpha_4<\frac{1}{2}\alpha_1\\ (1-\alpha_1-\alpha_2-\alpha_3,\alpha_2,\alpha_3,\alpha_4) \text{ can be partitioned into } (m,n)\in II}} S\left(\mathcal{A}_{\beta p_2p_3p_4}^d, p_4\right)\\
&+\sum_{\substack{(\alpha_1,\alpha_2)\in B\\ \sigma-2\varpi\leqslant\alpha_3<\min\left(\alpha_2,\frac{1-\alpha_1-\alpha_2}{2}\right)\\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II\\ \sigma-2\varpi\leqslant\alpha_4<\frac{1}{2}\alpha_1\\ (1-\alpha_1-\alpha_2-\alpha_3,\alpha_2,\alpha_3,\alpha_4) \text{ cannot be partitioned into } (m,n)\in II}} S\left(\mathcal{A}_{\beta p_2p_3p_4}^d, p_4\right)\\
=&S_{B1}-S_{B2}-S_{B3}+S_{B4}+S_{B5}.
\end{aligned} \tag{6}$$

We can give asymptotic formulas for $S_{B1}$–$S_{B4}$ by Lemmas 2.1–2.3. We can also decompose part of $S_{B5}$ if the variables (with $\alpha_1$ replaced by $1-\alpha_1-\alpha_2-\alpha_3$) satisfy the same conditions as in the decomposable part of $S_{B5}$. Again, the total loss from $S_B$ can be bounded by

$$\begin{aligned}
&\Bigg(\int_{\sigma-2\varpi}^{\frac{1}{2}} \int_{\sigma-2\varpi}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\sigma-2\varpi}^{\frac{1}{2}t_1} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4) \in S_{B51}] \frac{\omega_1\left(\frac{t_1-t_4}{t_4}\right)\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4^2} dt_4 dt_3 dt_2 dt_1\Bigg) \\
&- \Bigg(\int_{\sigma-2\varpi}^{\frac{1}{2}} \int_{\sigma-2\varpi}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\sigma-2\varpi}^{\frac{1}{2}t_1} \int_{t_4}^{\frac{t_1-t_4}{2}} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5) \in S_{B52}] \frac{\omega_0\left(\frac{t_1-t_4-t_5}{t_5}\right)\omega_0\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4 t_5^2} dt_5 dt_4 dt_3 dt_2 dt_1\Bigg) \\
&+ \Bigg(\int_{\sigma-2\varpi}^{\frac{1}{2}} \int_{\sigma-2\varpi}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_2, \frac{1-t_1-t_2}{2}\right)} \int_{\sigma-2\varpi}^{\frac{1}{2}t_1} \int_{\sigma-2\varpi}^{\min\left(t_4, \frac{t_1-t_4}{2}\right)} \int_{\sigma-2\varpi}^{\min\left(t_5, \frac{t_1-t_4-t_5}{2}\right)} \\
&\qquad \texttt{Boole}[(t_1,t_2,t_3,t_4,t_5,t_6) \in S_{B53}] \frac{\omega_1\left(\frac{t_1-t_4-t_5-t_6}{t_6}\right)\omega_1\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_2 t_3^2 t_4 t_5 t_6^2} dt_6 dt_5 dt_4 dt_3 dt_2 dt_1\Bigg) \\
&< 0.006249, \qquad (7)
\end{aligned}$$

where

$$\begin{aligned}
S_{B51}(t_1,t_2,t_3,t_4) := \ \{&(t_1,t_2) \in S_B, \\
&\sigma-2\varpi \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
&(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n) \in II, \\
&\sigma-2\varpi \leqslant t_4 < \frac{1}{2}t_1, \\
&(1-t_1-t_2-t_3,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in II, \\
&(1-t_1-t_2-t_3,t_2,t_3,t_4,t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
&\sigma-2\varpi \leqslant t_1 < \frac{1}{2},\ \sigma-2\varpi \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Big\}, \\
S_{B52}(t_1,t_2,t_3,t_4,t_5) := \ \{&(t_1,t_2) \in S_B, \\
&\sigma-2\varpi \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right), \\
&(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n) \in II, \\
&\sigma-2\varpi \leqslant t_4 < \frac{1}{2}t_1, \\
&(1-t_1-t_2-t_3,t_2,t_3,t_4) \text{ cannot be partitioned into } (m,n) \in II, \\
&(1-t_1-t_2-t_3,t_2,t_3,t_4,t_4) \text{ cannot be partitioned into } (m,n) \in I, \\
&t_4 < t_5 < \frac{1}{2}(t_1-t_4), \\
&(1-t_1-t_2-t_3,t_2,t_3,t_4,t_5) \text{ can be partitioned into } (m,n) \in II, \\
&\sigma-2\varpi \leqslant t_1 < \frac{1}{2},\ \sigma-2\varpi \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\Big\}, \\
S_{B53}(t_1,t_2,t_3,t_4,t_5,t_6) := \ \{&(t_1,t_2) \in S_B,
\end{aligned}$$

$$\begin{aligned}
&\sigma - 2\varpi \leqslant t_3 < \min\left(t_2, \frac{1}{2}(1-t_1-t_2)\right),\\
&(t_1,t_2,t_3) \text{ cannot be partitioned into } (m,n) \in II,\\
&\sigma - 2\varpi \leqslant t_4 < \frac{1}{2}t_1,\\
&(1-t_1-t_2-t_3, t_2, t_3, t_4) \text{ cannot be partitioned into } (m,n) \in II,\\
&(1-t_1-t_2-t_3, t_2, t_3, t_4, t_4) \text{ can be partitioned into } (m,n) \in I,\\
&\sigma - 2\varpi \leqslant t_5 < \min\left(t_4, \frac{1}{2}(t_1-t_4)\right),\\
&(1-t_1-t_2-t_3, t_2, t_3, t_4, t_5) \text{ cannot be partitioned into } (m,n) \in II,\\
&\sigma - 2\varpi \leqslant t_6 < \min\left(t_5, \frac{1}{2}(t_1-t_4-t_5)\right),\\
&(1-t_1-t_2-t_3, t_2, t_3, t_4, t_5, t_6) \text{ cannot be partitioned into } (m,n) \in II,\\
&\left.\sigma - 2\varpi \leqslant t_1 < \frac{1}{2},\ \sigma - 2\varpi \leqslant t_2 < \min\left(t_1, \frac{1}{2}(1-t_1)\right)\right\}.
\end{aligned}$$

For $S_C$ we cannot use Buchstab's identity in a straightforward manner, but we can use Buchstab's identity in reverse to make some almost–primes visible. The details of using Buchstab's identity in reverse are similar to those in [6] and [8]. By using Buchstab's identity in reverse twice, we have

$$\begin{aligned}
S_C =& \sum_{(\alpha_1,\alpha_2)\in C} S\left(\mathcal{A}^d_{p_1p_2}, p_2\right)\\
=& \sum_{(\alpha_1,\alpha_2)\in C} S\left(\mathcal{A}^d_{p_1p_2}, 2\left(\frac{X}{p_1p_2}\right)^{\frac{1}{2}}\right)\\
&+ \sum_{\substack{(\alpha_1,\alpha_2)\in C\\ \alpha_2<\alpha_3<\frac{1-\alpha_1-\alpha_2}{2}\\ (\alpha_1,\alpha_2,\alpha_3) \text{ can be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3}, p_3\right)\\
&+ \sum_{\substack{(\alpha_1,\alpha_2)\in C\\ \alpha_2<\alpha_3<\frac{1-\alpha_1-\alpha_2}{2}\\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3}, 2\left(\frac{X}{p_1p_2p_3}\right)^{\frac{1}{2}}\right)\\
&+ \sum_{\substack{(\alpha_1,\alpha_2)\in C\\ \alpha_2<\alpha_3<\frac{1-\alpha_1-\alpha_2}{2}\\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II\\ \alpha_3<\alpha_4<\frac{1-\alpha_1-\alpha_2-\alpha_3}{2}\\ (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \text{ can be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3p_4}, p_4\right)\\
&+ \sum_{\substack{(\alpha_1,\alpha_2)\in C\\ \alpha_2<\alpha_3<\frac{1-\alpha_1-\alpha_2}{2}\\ (\alpha_1,\alpha_2,\alpha_3) \text{ cannot be partitioned into } (m,n)\in II\\ \alpha_3<\alpha_4<\frac{1-\alpha_1-\alpha_2-\alpha_3}{2}\\ (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \text{ cannot be partitioned into } (m,n)\in II}} S\left(\mathcal{A}^d_{p_1p_2p_3p_4}, p_4\right)\\
=& S_{C1} + S_{C2} + S_{C3} + S_{C4} + S_{C5}. \qquad (8)
\end{aligned}$$

We can give asymptotic formulas for $S_{C2}$ and $S_{C4}$ by Lemma 2.3, hence we can subtract them from the loss. In this way we obtain a loss from $S_C$ of

$$\left(\int_{\sigma-2\varpi}^{\frac{1}{2}} \int_{\sigma-2\varpi}^{\min\left(t_1, \frac{1-t_1}{2}\right)} \texttt{Boole}[(t_1,t_2)\in S_C] \frac{\omega\left(\frac{1-t_1-t_2}{t_2}\right)}{t_1 t_2^2} dt_2 dt_1\right.$$

$$
\begin{aligned}
&-\left(\int_{\sigma-2\varpi}^{\frac{1}{2}}\int_{\sigma-2\varpi}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{t_2}^{\frac{1-t_1-t_2}{2}}\texttt{Boole}[(t_1,t_2,t_3)\in S_{C2}]\frac{\omega\left(\frac{1-t_1-t_2-t_3}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\right)\\
&-\left(\int_{\sigma-2\varpi}^{\frac{1}{2}}\int_{\sigma-2\varpi}^{\min\left(t_1,\frac{1-t_1}{2}\right)}\int_{t_2}^{\frac{1-t_1-t_2}{2}}\int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}}\texttt{Boole}[(t_1,t_2,t_3,t_4)\in S_{C4}]\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_4}\right)}{t_1t_2t_3t_4^2}dt_4dt_3dt_2dt_1\right)\\
&<0.990258, \qquad (9)
\end{aligned}
$$

where

$$
\begin{aligned}
S_{C2}(t_1,t_2,t_3):=\Bigg\{&(t_1,t_2)\in S_C,\ t_2<t_3<\frac{1}{2}(1-t_1-t_2),\\
&(t_1,t_2,t_3)\text{ can be partitioned into }(m,n)\in II,\\
&\sigma-2\varpi\leqslant t_1<\frac{1}{2},\ \sigma-2\varpi\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\Bigg\},\\
S_{C4}(t_1,t_2,t_3,t_4):=\Bigg\{&(t_1,t_2)\in S_C,\ t_2<t_3<\frac{1}{2}(1-t_1-t_2),\\
&(t_1,t_2,t_3)\text{ cannot be partitioned into }(m,n)\in II,\\
&t_3<t_4<\frac{1}{2}(1-t_1-t_2-t_3),\\
&(t_1,t_2,t_3,t_4)\text{ can be partitioned into }(m,n)\in II,\\
&\sigma-2\varpi\leqslant t_1<\frac{1}{2},\ \sigma-2\varpi\leqslant t_2<\min\left(t_1,\frac{1}{2}(1-t_1)\right)\Bigg\}.
\end{aligned}
$$

Finally, by (2)–(9), the total loss is less than

$$0.990258+0.002515+0.006249<0.9991<1$$

and the proof of Theorem 1.1 is completed.

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