On the Goldbach's conjecture

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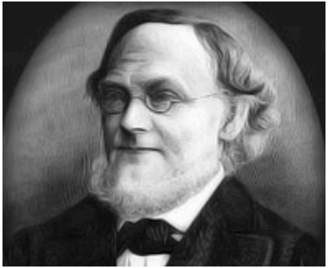
Goldbach's conjecture 哥德巴赫猜想




Goldbach (1690-1764) and Euler (1707-1783)

Goldbach's conjecture

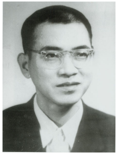
Every even integer greater than 2 can be written as the sum of two primes.

Goldbach's conjecture for large integers

Every sufficiently large even integer can be written as the sum of two primes.

Historical records

- Goldbach's conjecture: 素数 $+$ 素数 $(1+1)$



- Chen's Theorem: 素数 $+$ 素数 or 素数 $+$ 素数 $\times$ 素数 $(1+2)$, 陈景润, 1973.

Chen's Theorem 陈景润定理

Theorem (Chen, 1973)

Every sufficiently large even integer can be written as the sum of a prime and a P_2.
Moreover, let N denote a sufficiently large even integer and define

$$D_{1,2}(N) := \left|\{p : p \leqslant N, N - p = P_2\}\right|,$$

we have

$$D_{1,2}(N) \geqslant 0.67 \frac{C(N)N}{(\log N)^2}.$$

Two results claimed by Chen

In 1980, Chen announced two unpublished results of himself:

1. 0.9;
2. $9 \Rightarrow$ Goldbach's conjecture.

- 0.67, 陈景润, 1973;
- 0.899, 吴杰, 2008.

$$C_x = \prod_{\substack{p \mid x \\ p>2}} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) .$$

Let $P_x(1,2)$ be the number of primes p satisfying the following conditions: either $x-p = p_1$ or $x-p = p_2p_3$, where p_1, p_2, p_3 are primes. In 1966, I proved that

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$$P_x(1,2) \ge \frac{BC_x x}{(\log x)^2} \qquad (1)$$

where $B \ge 0.098$. This was successively improved by myself in 1973 to $B \ge 0.67$, by Halberstam and Richert in 1975 to $B \ge 0.689$, and by myself in 1978 to $B \ge 0.81$. Recently, in an unpublished manuscript, I have shown that $B \ge 0.9$. The improvement of the constant B in the inequality (1) is important because it implies an increase in the number of solutions of our problem. Moreover, if B is larger than 9, then we can solve $(1,1)$.

Main result

Theorem 1 (L. 2024)

We have

$$D_{1,2}(N) \geq {\color{red}1.733}\frac{C(N)N}{(\log N)^2}.$$

Theorem 2 (L. 2025)

We have

$$D_{1,2}(N) \geq {\color{red}1.9728}\frac{C(N)N}{(\log N)^2}.$$

One important significance of our Theorems is to make us truly achieve and exceed the constant 0.9 claimed by Chen.
Our constant 1.9728 gives a 119% refinement of Wu's prior record 0.899. This is the greatest refinement on this problem since Chen from 1973.

Main tools

In order to prove our Theorem 2, we mainly utilize the following tools:

1. **Weighted sieve inequalities**
   加权筛法不等式;
2. **Lichtman's new distribution levels**
   素数在等差数列上的分布水平;
3. **Chen's double sieve**
   陈景润的双筛法;
4. **Harman's sieve**
   Harman 筛法;
5. **Optimization of various bounds**.

Proposition 6.6. *Let $(D_1, \ldots, D_r) \in \mathbf{D}_r^{\mathrm{well}}(D)$ and write $D = x^{\boldsymbol{\vartheta}}$, $D_i = x^{t_i}$ for $i \leqslant r$. If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}(t_1) - \epsilon$ as in (6.2), then*

$$\sum_{\substack{b=p_1\cdots p_r \\ D_i<p_i \leqslant D_i^{1+\epsilon^9}}} \sum_{\substack{d=bc \leqslant x^{\boldsymbol{\vartheta}} \\ c|P(p_r) \\ (d,a)=1}} \widetilde{\lambda}^{\pm}(d) \left(\pi(x;d,a) - \frac{\pi(x)}{\varphi(d)}\right) \ll_{a,A,\epsilon} \frac{x}{(\log x)^A}. \tag{6.12}$$

And if $t_1 \leqslant \min(\frac{1-\theta}{4-3\theta}, \frac{1-\alpha\theta}{4})$ and $r \geqslant 3$, then (6.12) holds if $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}(t_1, t_2, t_3) - \epsilon$ as in (6.4).

be difficult. Chen improved on the sieve (3.3) by introducing two new functions $H(s)$ and $h(s)$ such that (3.3) holds with $f(s) + h(s)$ and $F(s) - H(s)$ in place of $f(s)$ and $F(s)$ respectively [54].

$$S(\mathcal{A}; \mathcal{P}, z) \leq XV(z) \left\{ (F(s) - H(s)) \left(\frac{\log Q}{\log z} \right) + E \right\} + \text{error}, \tag{3.7}$$

Chen proved that $h(s) > 0$ and $H(s) > 0$ (which is obviously a required property, as otherwise these functions would make the bound on $S(\mathcal{A}; \mathcal{P}, z)$ worse) using three set of complicated inequalities (the largest had 43 terms!).

$$\begin{aligned} S_{14} \leqslant (1+o(1)) \Bigg(\int_{\frac{1}{11.49}}^{\frac{1}{6.18}} \int_{\frac{1}{11.49}}^{t_1} \int_{\frac{1}{11.49}}^{t_2} \int_{\frac{1}{11.49}}^{t_3} & \Big(\mathtt{Boole}[(D_1, \ldots, D_4) \in \mathbf{D}_4^{well}(D)] \times \\ & \min \left(\frac{2}{e^\gamma} \frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3) - t_1 - t_2 - t_3 - t_4)}{t_4}\right)}{t_1 t_2 t_3 t_4^2}, \frac{2G_1}{e^\gamma} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_3}\right)}{t_1 t_2 t_3^2 t_4} \right) \\ & + \mathtt{Boole}[(D_1, \ldots, D_4) \notin \mathbf{D}_4^{well}(D)] \frac{2G_1}{e^\gamma} \frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_3}\right)}{t_1 t_2 t_3^2 t_4} \Big) dt_4 dt_3 dt_2 dt_1 \Bigg) \frac{C(N)N}{(\log N)^2} \end{aligned}$$

$$\begin{aligned} S_3 \leqslant (1+o(1)) \frac{2}{e^\gamma} \Bigg(\int_{\frac{1}{11.49}}^{\frac{25}{128}} \min \Bigg(& 11.49 \frac{F(11.49(\boldsymbol{\vartheta}_1(t_1, \frac{1}{11.49}, \frac{1}{11.49}) - t_1))}{t_1} \\ & - \frac{22.98 e^\gamma H(11.49(\frac{1}{2} - t_1))}{(11.49(\frac{1}{2} - t_1))t_1}, \min_{11.49 \leqslant k \leqslant 500} \Bigg(k \frac{F(k(\boldsymbol{\vartheta}_1(t_1, \frac{1}{k}, \frac{1}{k}) - t_1))}{t_1} \\ & - \frac{2ke^\gamma H(k(\frac{1}{2} - t_1))}{(k(\frac{1}{2} - t_1))t_1} - k \int_{\frac{1}{k}}^{\frac{1}{11.49}} \frac{f(k(\boldsymbol{\vartheta}_1(t_1, t_2, \frac{1}{k}) - t_1 - t_2))}{t_1 t_2} dt_2 \\ & - 2ke^\gamma \int_{\frac{1}{k}}^{\frac{1}{11.49}} \frac{h(k(\frac{1}{2} - t_1 - t_2))}{(k(\frac{1}{2} - t_1 - t_2))t_1 t_2} dt_2 \\ & + \int_{\frac{1}{k}}^{\frac{1}{11.49}} \int_{\frac{1}{k}}^{t_2} \frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3) - t_1 - t_2 - t_3)}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 \Bigg) \Bigg) dt_1 \Bigg) \frac{C(N)N}{(\log N)^2} \end{aligned}$$

Thank you!