A REMARK ON LARGE EVEN INTEGERS OF THE FORM $p + P_3$

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Abstract. Let N denotes a sufficiently large even integer, p denotes a prime and P_r denotes an integer with at most r prime factors. In this paper, we study the solutions of the equation $N - p = P_3$ and consider two special cases where p is small, and p, P_3 are within short intervals.

Contents

1. Introduction

Let N denotes a sufficiently large even integer, p denotes a prime, and let P_r denotes an integer with at most r prime factors counted with multiplicity. For each $N \geqslant 4$ and $r \geqslant 2$, we define

$$D_{1,r}(N) := |\{p : p \leqslant N, N - p = P_r\}| . \tag{1}$$

In 1966 Jingrun Chen [7] proved his remarkable Chen's theorem: let N denotes a sufficiently large even integer, then

$$D_{1,2}(N) \geqslant 0.67 \frac{C(N)N}{(\log N)^2} \tag{2}$$

where

$$C(N) := \prod_{\substack{p \mid N \\ p>2}} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) . \tag{3}$$

and the detail was published in [8]. The original proof of Jingrun Chen was simplified by Pan, Ding and Wang [15], Halberstam and Richert [12], Halberstam [11], Ross [17]. As Halberstam and Richert indicated in [12], it would be interesting to know whether a more elaborate weighting procedure could be adapted to the purpose of (2). This might lead to numerical improvements and could be important. Chen's constant 0.67 was improved successively to

$$0.689, 0.7544, 0.81, 0.8285, 0.836, 0.867, 0.899$$

by Halberstam and Richert [12] [11], Chen [10] [9], Cai and Lu [6], Wu [22], Cai [2] and Wu [23] respectively.

Chen's theorem with small primes was first studied by Cai [1]. For $0 < \theta \leqslant 1$, we define

$$D_{1,r}^{\theta}(N) := \left|\left\{p : p \leqslant N^{\theta}, N - p = P_r\right\}\right| . \tag{4}$$

Then it is proved in [1] that for $0.95 \leqslant \theta \leqslant 1$, we have

$$D_{1,2}^{\theta}(N) \gg \frac{C(N)N^{\theta}}{(\log N)^2} . \tag{5}$$

Cai's range $0.95 \leqslant \theta \leqslant 1$ was extended successively to $0.945 \leqslant \theta \leqslant 1$ in [4] and to $0.941 \leqslant \theta \leqslant 1$ in [3].

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Chen's theorem in short intervals was first studied by Ross [18]. For $0<\kappa\leqslant 1$, we define

$$D_{1,r}(N,\kappa):=\left|\{p: N/2-N^{\kappa}\leqslant p, P_r\leqslant N/2+N^{\kappa}, N=p+P_r\}\right|. \tag{6}$$

Then it is proved in [18] that for $0.98\leqslant\kappa\leqslant 1$, we have

$$D_{1,2}(N,\kappa)\gg\frac{C(N)N^{\kappa}}{(\log N)^2}. \tag{7}$$

The constant 0.98 was improved successively to

$$0.974, 0.973, 0.9729, 0.972, 0.971, 0.97$$

by Wu [20] [21], Salerno and Vitolo [19], Cai and Lu [5], Wu [22] and Cai [2] respectively.

In this paper, we aim to relax the number of prime factors of $N-p$, and at the same time extend the range of θ. Our improvement partially relies on the cancellation of the use of Wu's mean value theorem. Our main result is the following theorem.

Theorem 1.1. *for $0.838\leqslant\theta\leqslant 1$ and $0.919\leqslant\kappa\leqslant 1$, we have*

$$D_{1,3}^{\theta}(N)\gg\frac{C(N)N^{\theta}}{(\log N)^2}\quad\text{and}\quad D_{1,3}(N,\kappa)\gg\frac{C(N)N^{\kappa}}{(\log N)^2}.$$

We also generalize our results to integers of the form $ap+bP_3$. For two relatively prime square-free positive integers a and b, let M denotes a sufficiently large integer that is relatively prime to both a and b, $a,b<M^{\varepsilon}$ and let M be even if a and b are both odd. Let $R_{a,b}^{\theta}(M)$, $R_{a,b}(M,\kappa)$, $R_{a,b}^{\theta}(M,c,d)$ and $R_{a,b}(M,c,d,\kappa)$ denote the number of primes similar to those of [14] but satisfy $\frac{M-ap}{b}=P_3$ instead of P_2. By using similar arguments as in [14], we prove that

Theorem 1.2. *For $0.838\leqslant\theta\leqslant 1$, $0.919\leqslant\kappa\leqslant 1$ and $c\leqslant(\log N)^C$ where C is a positive constant, we have*

$$R_{a,b}^{\theta}(M)\gg\frac{M^{\theta}}{ab(\log M)^2},\quad R_{a,b}(M,\kappa)\gg\frac{M^{\kappa}}{ab(\log M)^2},$$

$$R_{a,b}^{\theta}(M,c,d)\gg\prod_{\substack{p\mid c\\ p\nmid M\\ p>2}}\left(\frac{p-1}{p-2}\right)\frac{M^{\theta}}{\varphi(c)ab(\log M)^2}$$

and

$$R_{a,b}(M,c,d,\kappa)\gg\prod_{\substack{p\mid c\\ p\nmid M\\ p>2}}\left(\frac{p-1}{p-2}\right)\frac{M^{\kappa}}{\varphi(c)ab(\log M)^2}.$$

Since the detail of the proof of Theorem 1.2 are similar to those of [14] and Theorem 1.1 so we omit it in this paper.

2. Preliminary lemmas

Let $\mathcal{A}$ denote a finite set of positive integers, $\mathcal{P}$ denote an infinite set of primes and $z\geqslant 2$. Suppose that $|\mathcal{A}|\sim X_{\mathcal{A}}$ and for square-free d, put

$$\mathcal{P}=\{p:(p,N)=1\},\quad \mathcal{P}(r)=\{p:p\in\mathcal{P},(p,r)=1\},$$

$$P(z)=\prod_{\substack{p\in\mathcal{P}\\ p<z}}p,\quad \mathcal{A}_d=\{a:a\in\mathcal{A}, a\equiv 0(\operatorname{mod} d)\},\quad S(\mathcal{A};\mathcal{P},z)=\sum_{\substack{a\in\mathcal{A}\\ (a,P(z))=1}}1.$$

Lemma 2.1. *([13], Lemma 1]). If*

$$\sum_{z_1\leqslant p<z_2}\frac{\omega(p)}{p}=\log\frac{\log z_2}{\log z_1}+O\left(\frac{1}{\log z_1}\right),\quad z_2>z_1\geqslant 2,$$

where $\omega(d)$ is a multiplicative function, $0 \leqslant \omega(p) < p, X > 1$ is independent of d. Then

$$S(\mathcal{A};\mathcal{P},z) \geqslant X_{\mathcal{A}} W(z) \left\{ f\left(\frac{\log D}{\log z}\right) + O\left(\frac{1}{\log^{\frac{1}{3}} D}\right) \right\} - \sum_{\substack{n \leqslant D \\ n \mid P(z)}} \eta(X_{\mathcal{A}}, n)$$

$$S(\mathcal{A};\mathcal{P},z) \leqslant X_{\mathcal{A}} W(z) \left\{ F\left(\frac{\log D}{\log z}\right) + O\left(\frac{1}{\log^{\frac{1}{3}} D}\right) \right\} + \sum_{\substack{n \leqslant D \\ n \mid P(z)}} \eta(X_{\mathcal{A}}, n)$$

where

$$W(z) = \prod_{\substack{p<z \\ (p,N)=1}} \left(1 - \frac{\omega(p)}{p}\right), \quad \eta(X_{\mathcal{A}}, n) = \left| |\mathcal{A}_n| - \frac{\omega(n)}{n} X_{\mathcal{A}} \right| = \left| \sum_{\substack{a \in \mathcal{A} \\ a \equiv 0 (\bmod n)}} 1 - \frac{\omega(n)}{n} X_{\mathcal{A}} \right|,$$

γ denotes the Euler's constant, $f(s)$ and $F(s)$ are determined by the following differential-difference equation

$$\begin{cases} F(s) = \frac{2e^{\gamma}}{s}, \quad f(s) = 0, & 0 < s \leqslant 2, \\ (sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), & s \geqslant 2. \end{cases}$$

Lemma 2.2. *([2], Lemma 2], deduced from [12]).*

$$\begin{aligned}
F(s) =& \frac{2e^{\gamma}}{s}, \quad 0 < s \leqslant 3; \\
F(s) =& \frac{2e^{\gamma}}{s} \left(1 + \int_2^{s-1} \frac{\log(t-1)}{t} dt\right), \quad 3 \leqslant s \leqslant 5; \\
F(s) =& \frac{2e^{\gamma}}{s} \left(1 + \int_2^{s-1} \frac{\log(t-1)}{t} dt + \int_2^{s-3} \frac{\log(t-1)}{t} dt \int_{t+2}^{s-1} \frac{1}{u} \log \frac{u-1}{t+1} du\right), \quad 5 \leqslant s \leqslant 7; \\
f(s) =& \frac{2e^{\gamma} \log(s-1)}{s}, \quad 2 \leqslant s \leqslant 4; \\
f(s) =& \frac{2e^{\gamma}}{s} \left(\log(s-1) + \int_3^{s-1} \frac{dt}{t} \int_2^{t-1} \frac{\log(u-1)}{u} du\right), \quad 4 \leqslant s \leqslant 6; \\
f(s) =& \frac{2e^{\gamma}}{s} \left(\log(s-1) + \int_3^{s-1} \frac{dt}{t} \int_2^{t-1} \frac{\log(u-1)}{u} du\right. \\
&\left. + \int_2^{s-4} \frac{\log(t-1)}{t} dt \int_{t+2}^{s-2} \frac{1}{u} \log \frac{u-1}{t+1} \log \frac{s}{u+2} du\right), \quad 6 \leqslant s \leqslant 8.
\end{aligned}$$

Lemma 2.3. *([16], Theorem]). For any given constant $A > 0$, there exists a constant $B = B(A) > 0$ such that*

$$\sum_{d \leqslant x^{t-1/2} (\log x)^{-B}} \max_{x/2 \leqslant y \leqslant x} \max_{(l,d)=1} \max_{h \leqslant x^t} \left| \pi(y+h;d,l) - \pi(y;d,l) - \frac{h}{\varphi(d)} \right| \ll \frac{x^t}{\log^A x},$$

where

$$\frac{3}{5} < t \leqslant 1.$$

Lemma 2.4. *If we define the function ω as $\omega(p) = 0$ for primes $p \mid N$ and $\omega(p) = \frac{p}{p-1}$ for other primes and $N^{\frac{1}{\alpha}-\varepsilon} < z \leqslant N^{\frac{1}{\alpha}}$, then we have*

$$W(z) = \frac{2\alpha e^{-\gamma} C(N)(1+o(1))}{\log N}.$$

Proof. By similar arguments as in [1], we have

$$W(z) = \prod_{p \mid N} \frac{p}{p-1} \prod_{(p,N)=1} \left(1 - \frac{\omega(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-1} \frac{\alpha e^{-\gamma}(1+o(1))}{\log N}$$

$$= \frac{2\alpha e^{-\gamma} C(N)(1+o(1))}{\log N}.$$

$\square$

3. Proof of Theorem 1.1

Let $\theta = 0.838$ and $\kappa = 0.919$ in this section. Put

$$\mathcal{A} = \{N - p : p \leqslant N^{\theta}\} \quad \text{and} \quad \mathcal{B} = \{N - p : N/2 - N^{\kappa} \leqslant p \leqslant N/2 + N^{\kappa}\}.$$

Clearly we have

$$D_{1,3}^{\theta}(N) \geqslant S\left(\mathcal{A}; \mathcal{P}, N^{\frac{1}{11.99}}\right) - \frac{1}{2} \sum_{\substack{N^{\frac{1}{11.99}} \leqslant p < N^{\frac{1}{3}} \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}, N^{\frac{1}{11.99}}\right) = S_1 - \frac{1}{2} S_2, \tag{8}$$

$$D_{1,3}(N,\kappa) \geqslant S\left(\mathcal{B}; \mathcal{P}, N^{\frac{1}{11.99}}\right) - \frac{1}{2} \sum_{\substack{N^{\frac{1}{11.99}} \leqslant p < N^{\frac{1}{3}} \\ (p,N)=1}} S\left(\mathcal{B}_p; \mathcal{P}, N^{\frac{1}{11.99}}\right) = S_1' - \frac{1}{2} S_2'. \tag{9}$$

Now we define the function ω as $\omega(p) = 0$ for primes $p \mid N$ and $\omega(p) = \frac{p}{p-1}$ for other primes. We can take

$$X_{\mathcal{A}} \sim \frac{N^{\theta}}{\theta \log N} \quad \text{and} \quad X_{\mathcal{B}} \sim \frac{2N^{\kappa}}{\log N}.$$

By Lemmas 2.1–2.4, Bombieri's theorem and some routine arguments, we have

$$S_1 \geqslant (1+o(1)) \frac{8\Delta_1 C(N) N^{\theta}}{\theta^2 (\log N)^2}, \quad S_2 \leqslant (1+o(1)) \frac{8\Delta_2 C(N) N^{\theta}}{\theta^2 (\log N)^2}, \tag{10}$$

$$S_1' \geqslant (1+o(1)) \frac{16\Delta_3 C(N) N^{\kappa}}{(2\kappa - 1)(\log N)^2}, \quad S_2' \leqslant (1+o(1)) \frac{16\Delta_4 C(N) N^{\kappa}}{(2\kappa - 1)(\log N)^2}, \tag{11}$$

where

$$\Delta_1 = \log(5.995\theta - 1) + \int_2^{5.995\theta - 2} \frac{\log(s-1)}{s} \log \frac{5.995\theta - 1}{s+1} ds, \tag{12}$$

$$\Delta_2 = \log\left(\frac{11.99\theta - 2}{3\theta - 2}\right) + \int_2^{5.995\theta - 2} \frac{\log(s-1)}{s} \log \frac{(5.995\theta - 1)(5.995\theta - 1 - s)}{s+1} ds, \tag{13}$$

$$\Delta_3 = \log(11.99\kappa - 6.995) + \int_2^{11.99\kappa - 7.995} \frac{\log(s-1)}{s} \log \frac{11.99\kappa - 6.995}{s+1} ds, \tag{14}$$

$$\Delta_4 = \log\left(\frac{23.98\kappa - 13.99}{6\kappa - 5}\right) + \int_2^{11.99\kappa - 7.995} \frac{\log(s-1)}{s} \log \frac{(11.99\kappa - 6.995)(11.99\kappa - 6.995 - s)}{s+1} ds. \tag{15}$$

By numerical calculations we get that

$$\Delta_1 - \frac{1}{2}\Delta_2 \geqslant 0.0009 \quad \text{and} \quad \Delta_3 - \frac{1}{2}\Delta_4 \geqslant 0.0009. \tag{16}$$

Then by (8)–(16) we have

$$D_{1,3}^{\theta}(N) \gg \frac{C(N) N^{\theta}}{(\log N)^2} \quad \text{and} \quad D_{1,3}(N,\kappa) \gg \frac{C(N) N^{\kappa}}{(\log N)^2}.$$

Theorem 1.1 is proved.

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