A NOTE ON VARIANTS OF BUCHSTAB'S IDENTITY

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Abstract. The author proves variants of Buchstab's identity on sieve functions, refining the previous work on new iteration rules of Brady. The main tool used in the proof is a special form of combinatorial identities related to the binomial coefficients. As a by–product, the author obtains better inequalities of $F_\kappa(s)$ and $f_\kappa(s)$ for dimensions $\kappa > 1$.

Contents

1. Introduction

Let $\mathcal{A}$ be a set of numbers, $\mathcal{A}_d = \{a : ad \in \mathcal{A}\}$ and $S(\mathcal{A}, z) = \sum_{\substack{a\in\mathcal{A} \\ (a,\prod_{p<z} p)=1}} 1$. Suppose that κ, z, y are such that for every squarefree integer d, all of whose prime factors are less than z, we have

$$\left| |\mathcal{A}_d| - \kappa^{\omega(d)} \frac{y}{d} \right| \leqslant 1. \tag{1}$$

Suppose that $y = z^s$ and define $F_\kappa(s)$ and $f_\kappa(s)$ by

$$(1 + o(1))f_\kappa(s)y \prod_{p<z} \left(1 - \frac{\kappa}{p}\right) \leqslant S(\mathcal{A}, z) \leqslant (1 + o(1))F_\kappa(s)y \prod_{p<z} \left(1 - \frac{\kappa}{p}\right) \tag{2}$$

with $f_\kappa(s)$ as large as possible and $F_\kappa(s)$ as small as possible, given that (2) holds for all choices of $\mathcal{A}$ satisfying (1). Selberg [3] has shown that $F_\kappa(s)$ and $f_\kappa(s)$ are continuous, monotone, and computable for $s > 1$, and that they tend to 1 exponentially as s goes to infinity.

When $\kappa \leqslant 1$, the optimal estimates for $F_\kappa(s)$ and $f_\kappa(s)$ arise from Buchstab's identity

$$S(\mathcal{A}, z) = S(\mathcal{A}, w) - \sum_{w \leqslant p < z} S(\mathcal{A}_p, p) \tag{3}$$

for $w \leqslant z$. Simply let $w = 2$, this becomes

$$S(\mathcal{A}, z) = |\mathcal{A}| - \sum_{p<z} S(\mathcal{A}_p, p). \tag{4}$$

This leads to the inequalities

$$s^\kappa F_\kappa(s) \leqslant s^\kappa - \kappa \int_{t>s} t^{\kappa-1} (f_\kappa(t - 1) - 1)\, dt, \tag{5}$$

$$s^\kappa f_\kappa(s) \geqslant s^\kappa - \kappa \int_{t>s} t^{\kappa-1} (F_\kappa(t - 1) - 1)\, dt. \tag{6}$$

Infinite iteration of these inequalities leads to the β–sieve.

2020 Mathematics Subject Classification. 11N35.
Key words and phrases. Buchstab's identity, sieve methods.

However, there are better estimates for $F_\kappa(s)$ and $f_\kappa(s)$ when $\kappa > 1$. Taking Selberg's upper bound sieve as a starting point and using similar iteration rules, Diamond, Halberstam and Richert [2] developed their DHR–sieve.

In 2017, Brady mentioned and proved lots of new sieve iteration rules in his PhD thesis. One of his simplest upper bound sieve is

$$S(\mathcal{A}, z) \leqslant S(\mathcal{A}, w) - \frac{2}{3} \sum_{w \leqslant p_1 < z} S(\mathcal{A}_{p_1}, w) + \frac{1}{3} \sum_{w \leqslant p_2 < p_1 < z} S(\mathcal{A}_{p_1 p_2}, w).$$

He proved this inequality using a combinatorial identity

$$1 - \frac{2}{3} n + \frac{1}{3} \binom{n}{2} = \left(1 - \frac{n}{2}\right)\left(1 - \frac{n}{3}\right). \tag{7}$$

Clearly, this leads to an inequality of $F_\kappa(s)$:

$$s^\kappa F_\kappa(s) \leqslant t^\kappa F_\kappa(t) - \frac{2}{3} \int_{\frac{1}{t}}^{\frac{1}{s}} \frac{t^\kappa f_\kappa(t(1-x_1))}{x_1} dx_1 + \frac{1}{3} \int_{\frac{1}{t}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{x_1} \frac{t^\kappa F_\kappa(t(1-x_1-x_2))}{x_1 x_2} dx_2 dx_1. \tag{8}$$

In this note, we further develop his method and prove a series of generalized iteration rules.

2. Upper bound iteration

We first prove a simple upper bound iteration, which is a direct generalization of [[1], Theorem 34].

Theorem 2.1. *For any odd positive integer k and $w \leqslant z$, we have*

$$\begin{aligned} S(\mathcal{A}, z) \leqslant\ & S(\mathcal{A}, w) - \frac{k-1}{k} \sum_{w \leqslant p_1 < z} S(\mathcal{A}_{p_1}, w) + \frac{k-2}{k} \sum_{w \leqslant p_2 < p_1 < z} S(\mathcal{A}_{p_1 p_2}, w) \\ & - \frac{k-3}{k} \sum_{w \leqslant p_3 < p_2 < p_1 < z} S(\mathcal{A}_{p_1 p_2 p_3}, w) + \cdots \\ & - \frac{2}{k} \sum_{w \leqslant p_{k-2} < \cdots < p_2 < p_1 < z} S\left(\mathcal{A}_{p_1 p_2 \cdots p_{k-2}}, w\right) \\ & + \frac{1}{k} \sum_{w \leqslant p_{k-1} < \cdots < p_2 < p_1 < z} S\left(\mathcal{A}_{p_1 p_2 \cdots p_{k-1}}, w\right). \end{aligned}$$

Proof. We follow the essential steps in the proof of [[1], Theorem 34]. Let $a \in \mathcal{A}$. If a has any prime factor below w, then both quantities are clearly zero. Assume that a has no prime factors below w and has exactly n prime factors between w and z. If $n = 0$ then both sides count a once. Thus we only need to show that for any integer $n \geqslant 1$ we have

$$0 \leqslant 1 - \frac{k-1}{k} n + \frac{k-2}{k} \binom{n}{2} - \frac{k-3}{k} \binom{n}{3} + \cdots + \frac{1}{k} \binom{n}{k-1}. \tag{9}$$

Note that we have the following identity

$$1 - \frac{k-1}{k} n + \frac{k-2}{k} \binom{n}{2} - \frac{k-3}{k} \binom{n}{3} + \cdots + \frac{1}{k} \binom{n}{k-1} = \left(1 - \frac{n}{2}\right)\left(1 - \frac{n}{3}\right) \cdots \left(1 - \frac{n}{k}\right) \tag{10}$$

and the right hand side of (), which has even number of terms, is clearly $\geqslant 0$, Theorem 2.1 is proved. Note that [[1], Theorem 34] is just Theorem 2.1 with $k = 3$. ☐

Corollary 2.2. *For any odd positive integer k and real $2 \leqslant s \leqslant t$, we have*

$$\begin{aligned} s^\kappa F_\kappa(s) \leqslant\ & t^\kappa F_\kappa(t) - \frac{k-1}{k} \int_{\frac{1}{t}}^{\frac{1}{s}} \frac{t^\kappa f_\kappa(t(1-x_1))}{x_1} dx_1 + \frac{k-2}{k} \int_{\frac{1}{t}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{x_1} \frac{t^\kappa F_\kappa(t(1-x_1-x_2))}{x_1 x_2} dx_2 dx_1 \\ & - \frac{k-3}{k} \int_{\frac{1}{t}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{x_1} \int_{\frac{1}{t}}^{x_2} \frac{t^\kappa f_\kappa(t(1-x_1-x_2-x_3))}{x_1 x_2 x_3} dx_3 dx_2 dx_1 + \cdots \end{aligned}$$

$$- \frac{2}{k} \int_{\frac{1}{t}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{x_1} \cdots \int_{\frac{1}{t}}^{x_{k-3}} \frac{t^\kappa f_\kappa(t(1 - x_1 - x_2 - \cdots - x_{k-2}))}{x_1 x_2 \cdots x_{k-2}} dx_{k-2} \cdots dx_2 dx_1$$
$$+ \frac{1}{k} \int_{\frac{1}{t}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{x_1} \cdots \int_{\frac{1}{t}}^{x_{k-2}} \frac{t^\kappa F_\kappa(t(1 - x_1 - x_2 - \cdots - x_{k-1}))}{x_1 x_2 \cdots x_{k-1}} dx_{k-1} \cdots dx_2 dx_1.$$

However, we can use more flexible parameters to get more variants of this iteration. Before stating the next result, we first define

$$\mathcal{U} = \{(x_1, x_2) : x_1, x_2 \in (0, 1] \cup [2, 3] \cup \cdots \cup [k-1, k] \text{ with all odd } k,\ |x_1 - x_2| \leqslant 1\}. \tag{11}$$

Theorem 2.3. *For any m_1, m_2 such that $(m_1, m_2) \in \mathcal{U}$ and $w \leqslant z$, we have*

$$S(\mathcal{A}, z) \leqslant S(\mathcal{A}, w) - \frac{m_1 + m_2 - 1}{m_1 m_2} \sum_{w \leqslant p_1 < z} S(\mathcal{A}_{p_1}, w) + \frac{2}{m_1 m_2} \sum_{w \leqslant p_2 < p_1 < z} S(\mathcal{A}_{p_1 p_2}, w).$$

Proof. Again, we use the essentially same arguments as the proof of Theorem 2.1. Let $a \in \mathcal{A}$. If a has any prime factor below w, then both quantities are clearly zero. Assume that a has no prime factors below w and has exactly n prime factors between w and z. If $n = 0$ then both sides count a once. Thus we only need to show that for any integer $n \geqslant 1$ we have

$$0 \leqslant 1 - \frac{m_1 + m_2 - 1}{m_1 m_2} n + \frac{2}{m_1 m_2} \binom{n}{2}. \tag{12}$$

By the following identity

$$1 - \frac{m_1 + m_2 - 1}{m_1 m_2} n + \frac{2}{m_1 m_2} \binom{n}{2} = \left(1 - \frac{n}{m_1}\right)\left(1 - \frac{n}{m_2}\right) \tag{13}$$

and $(x_1, x_2) \in \mathcal{U}$, which means that , we know that the right–hand side of (4) is clearly $\geqslant 0$, Theorem 2.3 is proved. Note that [[1], Theorem 34] is just Theorem 2.3 with $m_1 = 2$ and $m_2 = 3$. □

Corollary 2.4. *For any m_1, m_2 such that $(m_1, m_2) \in \mathcal{U}$ and real $2 \leqslant s \leqslant t$, we have*

$$s^\kappa F_\kappa(s) \leqslant t^\kappa F_\kappa(t) - \frac{m_1 + m_2 - 1}{m_1 m_2} \int_{\frac{1}{t}}^{\frac{1}{s}} \frac{t^\kappa f_\kappa(t(1 - x_1))}{x_1} dx_1$$
$$+ \frac{2}{m_1 m_2} \int_{\frac{1}{t}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{x_1} \frac{t^\kappa F_\kappa(t(1 - x_1 - x_2))}{x_1 x_2} dx_2 dx_1.$$

Using the same method but with more parameters, we can get lots of upper bound iterations of this type. For the sake of simplicity, we write

$$M_k^r = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant k} m_{i_1} m_{i_2} \cdots m_{i_r}.$$

Theorem 2.5. *For any m_1, m_2, m_3, m_4 such that $(m_1, m_2) \in \mathcal{U}$, $(m_3, m_4) \in \mathcal{U}$ and $w \leqslant z$, we have*

$$S(\mathcal{A}, z) \leqslant S(\mathcal{A}, w) - \frac{M_4^3 - M_4^2 + M_4^1 - 1}{m_1 m_2 m_3 m_4} \sum_{w \leqslant p_1 < z} S(\mathcal{A}_{p_1}, w) + \frac{2\left(M_4^2 - 3M_4^1 + 7\right)}{m_1 m_2 m_3 m_4} \sum_{w \leqslant p_2 < p_1 < z} S(\mathcal{A}_{p_1 p_2}, w)$$
$$- \frac{6\left(M_4^1 - 6\right)}{m_1 m_2 m_3 m_4} \sum_{w \leqslant p_3 < p_2 < p_1 < z} S(\mathcal{A}_{p_1 p_2 p_3}, w) + \frac{24}{m_1 m_2 m_3 m_4} \sum_{w \leqslant p_4 < p_3 < p_2 < p_1 < z} S(\mathcal{A}_{p_1 p_2 p_3 p_4}, w).$$

Corollary 2.6. *For any m_1, m_2, m_3, m_4 such that $(m_1, m_2) \in \mathcal{U}$, $(m_3, m_4) \in \mathcal{U}$ and real $2 \leqslant s \leqslant t$, we have*

$$s^\kappa F_\kappa(s) \leqslant t^\kappa F_\kappa(t) - \frac{M_4^3 - M_4^2 + M_4^1 - 1}{m_1 m_2 m_3 m_4} \int_{\frac{1}{t}}^{\frac{1}{s}} \frac{t^\kappa f_\kappa(t(1 - x_1))}{x_1} dx_1$$
$$+ \frac{2\left(M_4^2 - 3M_4^1 + 7\right)}{m_1 m_2 m_3 m_4} \int_{\frac{1}{t}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{x_1} \frac{t^\kappa F_\kappa(t(1 - x_1 - x_2))}{x_1 x_2} dx_2 dx_1$$

$$
\begin{aligned}
&- \frac{6\left(M_4^1-6\right)}{m_1m_2m_3m_4}\int_{\frac{1}{t}}^{\frac{1}{s}}\int_{\frac{1}{t}}^{x_1}\int_{\frac{1}{t}}^{x_2}\frac{t^\kappa f_\kappa(t(1-x_1-x_2-x_3))}{x_1x_2x_3}dx_3dx_2dx_1\\
&+ \frac{24}{m_1m_2m_3m_4}\int_{\frac{1}{t}}^{\frac{1}{s}}\int_{\frac{1}{t}}^{x_1}\int_{\frac{1}{t}}^{x_2}\int_{\frac{1}{t}}^{x_3}\frac{t^\kappa F_\kappa(t(1-x_1-x_2-x_3-x_4))}{x_1x_2x_3x_4}dx_4dx_3dx_2dx_1.
\end{aligned}
$$

Theorem 2.7. *For any $m_1, m_2, m_3, m_4, m_5, m_6$ such that $(m_1, m_2) \in \mathcal{U}$, $(m_3, m_4) \in \mathcal{U}$, $(m_5, m_6) \in \mathcal{U}$ and $w \leqslant z$, we have*

$$
\begin{aligned}
S(\mathcal{A},z) \leqslant S(\mathcal{A},w) &- \frac{M_6^5-M_6^4+M_6^3-M_6^2+M_6^1-1}{m_1m_2m_3m_4m_5m_6}\sum_{w\leqslant p_1<z} S(\mathcal{A}_{p_1},w)\\
&+ \frac{2\left(M_6^4-3M_6^3+7M_6^2-15M_6^1+31\right)}{m_1m_2m_3m_4m_5m_6}\sum_{w\leqslant p_2<p_1<z} S(\mathcal{A}_{p_1p_2},w)\\
&- \frac{6\left(M_6^3-6M_6^2+25M_6^1-90\right)}{m_1m_2m_3m_4m_5m_6}\sum_{w\leqslant p_3<p_2<p_1<z} S(\mathcal{A}_{p_1p_2p_3},w)\\
&+ \frac{24\left(M_6^2-10M_6^1+65\right)}{m_1m_2m_3m_4m_5m_6}\sum_{w\leqslant p_4<p_3<p_2<p_1<z} S(\mathcal{A}_{p_1p_2p_3p_4},w)\\
&- \frac{120\left(M_6^1-15\right)}{m_1m_2m_3m_4m_5m_6}\sum_{w\leqslant p_5<p_4<p_3<p_2<p_1<z} S(\mathcal{A}_{p_1p_2p_3p_4p_5},w)\\
&+ \frac{720}{m_1m_2m_3m_4m_5m_6}\sum_{w\leqslant p_6<p_5<p_4<p_3<p_2<p_1<z} S(\mathcal{A}_{p_1p_2p_3p_4p_5p_6},w).
\end{aligned}
$$

Corollary 2.8. *For any $m_1, m_2, m_3, m_4, m_5, m_6$ such that $(m_1, m_2) \in \mathcal{U}$, $(m_3, m_4) \in \mathcal{U}$, $(m_5, m_6) \in \mathcal{U}$ and real $2 \leqslant s \leqslant t$, we have*

$$
\begin{aligned}
s^\kappa F_\kappa(s) \leqslant t^\kappa F_\kappa(t) &- \frac{M_6^5-M_6^4+M_6^3-M_6^2+M_6^1-1}{m_1m_2m_3m_4m_5m_6}\int_{\frac{1}{t}}^{\frac{1}{s}}\frac{t^\kappa f_\kappa(t(1-x_1))}{x_1}dx_1\\
&+ \frac{2\left(M_6^4-3M_6^3+7M_6^2-15M_6^1+31\right)}{m_1m_2m_3m_4m_5m_6}\int_{\frac{1}{t}}^{\frac{1}{s}}\int_{\frac{1}{t}}^{x_1}\frac{t^\kappa F_\kappa(t(1-x_1-x_2))}{x_1x_2}dx_2dx_1\\
&- \frac{6\left(M_6^3-6M_6^2+25M_6^1-90\right)}{m_1m_2m_3m_4m_5m_6}\int_{\frac{1}{t}}^{\frac{1}{s}}\int_{\frac{1}{t}}^{x_1}\int_{\frac{1}{t}}^{x_2}\frac{t^\kappa f_\kappa(t(1-x_1-x_2-x_3))}{x_1x_2x_3}dx_3dx_2dx_1\\
&+ \frac{24\left(M_6^2-10M_6^1+65\right)}{m_1m_2m_3m_4m_5m_6}\int_{\frac{1}{t}}^{\frac{1}{s}}\int_{\frac{1}{t}}^{x_1}\int_{\frac{1}{t}}^{x_2}\int_{\frac{1}{t}}^{x_3}\frac{t^\kappa F_\kappa(t(1-x_1-x_2-x_3-x_4))}{x_1x_2x_3x_4}dx_4dx_3dx_2dx_1\\
&- \frac{120\left(M_6^1-15\right)}{m_1m_2m_3m_4m_5m_6}\times\\
&\int_{\frac{1}{t}}^{\frac{1}{s}}\int_{\frac{1}{t}}^{x_1}\int_{\frac{1}{t}}^{x_2}\int_{\frac{1}{t}}^{x_3}\int_{\frac{1}{t}}^{x_4}\frac{t^\kappa F_\kappa(t(1-x_1-x_2-x_3-x_4-x_5))}{x_1x_2x_3x_4x_5}dx_5dx_4dx_3dx_2dx_1\\
&+ \frac{720}{m_1m_2m_3m_4m_5m_6}\times\\
&\int_{\frac{1}{t}}^{\frac{1}{s}}\int_{\frac{1}{t}}^{x_1}\int_{\frac{1}{t}}^{x_2}\int_{\frac{1}{t}}^{x_3}\int_{\frac{1}{t}}^{x_4}\int_{\frac{1}{t}}^{x_5}\frac{t^\kappa F_\kappa(t(1-x_1-x_2-x_3-x_4-x_5-x_6))}{x_1x_2x_3x_4x_5x_6}dx_6dx_5dx_4dx_3dx_2dx_1.
\end{aligned}
$$

3. Lower bound iteration

In this section we shall use a similar method to prove corresponding lower bound iterations.

Theorem 3.1. *For any $0 < m_0 \leqslant 1$ and m_1, m_2 such that $(m_1, m_2) \in \mathcal{U}$ and $w \leqslant z$, we have*

$$\begin{aligned} S(\mathcal{A}, z) \geqslant S(\mathcal{A}, w) &- \frac{m_0m_1 + m_0m_2 + m_1m_2 - m_0 - m_1 - m_2 + 1}{m_0m_1m_2} \sum_{w \leqslant p_1 < z} S(\mathcal{A}_{p_1}, w) \\ &+ \frac{2(m_0 + m_1 + m_2 - 3)}{m_0m_1m_2} \sum_{w \leqslant p_2 < p_1 < z} S(\mathcal{A}_{p_1p_2}, w) \\ &- \frac{6}{m_0m_1m_2} \sum_{w \leqslant p_3 < p_2 < p_1 < z} S(\mathcal{A}_{p_1p_2p_3}, w). \end{aligned}$$

Proof. By the same arguments as in the proof of Theorem , we only need to show that for any integer $n \geqslant 1$ we have

$$0 \geqslant 1 - \frac{m_0m_1 + m_0m_2 + m_1m_2 - m_0 - m_1 - m_2 + 1}{m_0m_1m_2} n + \frac{2(m_0 + m_1 + m_2 - 3)}{m_0m_1m_2} \binom{n}{2} - \frac{6}{m_0m_1m_2} \binom{n}{3}. \quad (14)$$

Here, we have the identity

$$\begin{aligned} &1 - \frac{m_0m_1 + m_0m_2 + m_1m_2 - m_0 - m_1 - m_2 + 1}{m_0m_1m_2} n + \frac{2(m_0 + m_1 + m_2 - 3)}{m_0m_1m_2} \binom{n}{2} - \frac{6}{m_0m_1m_2} \binom{n}{3} \\ &= \left(1 - \frac{n}{m_0}\right) \left(1 - \frac{n}{m_1}\right) \left(1 - \frac{n}{m_2}\right). \end{aligned} \quad (15)$$

One can easily check that for $0 < m_0 \leqslant 1$ and m_1, m_2 such that $(m_1, m_2) \in \mathcal{U}$, the right–hand side is zero or negative for any positive integer n. Hence Theorem 3.1 is proved. □

Corollary 3.2. *For any $0 < m_0 \leqslant 1$ and m_1, m_2 such that $(m_1, m_2) \in \mathcal{U}$ and $3 \leqslant s \leqslant t$, we have*

$$\begin{aligned} s^\kappa f_\kappa(s) \geqslant t^\kappa f_\kappa(t) &- \frac{m_0m_1 + m_0m_2 + m_1m_2 - m_0 - m_1 - m_2 + 1}{m_0m_1m_2} \int_{\frac{1}{t}}^{\frac{1}{s}} \frac{t^\kappa F_\kappa(t(1 - x_1))}{x_1} dx_1 \\ &+ \frac{2(m_0 + m_1 + m_2 - 3)}{m_0m_1m_2} \int_{\frac{1}{t}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{x_1} \frac{t^\kappa f_\kappa(t(1 - x_1 - x_2))}{x_1x_2} dx_2dx_1 \\ &- \frac{6}{m_0m_1m_2} \int_{\frac{1}{t}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{x_1} \int_{\frac{1}{t}}^{x_2} \frac{t^\kappa F_\kappa(t(1 - x_1 - x_2 - x_3))}{x_1x_2x_3} dx_3dx_2dx_1. \end{aligned}$$

Again, for the sake of simplicity, we write

$$N_k^r = \sum_{0 \leqslant i_1 < i_2 < \cdots < i_r \leqslant k-1} m_{i_1} m_{i_2} \cdots m_{i_r}.$$

Theorem 3.3. *For any $0 < m_0 \leqslant 1$, m_1, m_2, m_3, m_4 such that $(m_1, m_2) \in \mathcal{U}$, $(m_3, m_4) \in \mathcal{U}$ and $w \leqslant z$, we have*

$$\begin{aligned} S(\mathcal{A}, z) \geqslant S(\mathcal{A}, w) &- \frac{N_5^4 - N_5^3 + N_5^2 - N_5^1 + 1}{m_0m_1m_2m_3m_4} \sum_{w \leqslant p_1 < z} S(\mathcal{A}_{p_1}, w) \\ &+ \frac{2\left(N_5^3 - 3N_5^2 + 7N_5^1 - 15\right)}{m_0m_1m_2m_3m_4} \sum_{w \leqslant p_2 < p_1 < z} S(\mathcal{A}_{p_1p_2}, w) \\ &- \frac{6\left(N_5^2 - 6N_5^1 + 25\right)}{m_0m_1m_2m_3m_4} \sum_{w \leqslant p_3 < p_2 < p_1 < z} S(\mathcal{A}_{p_1p_2p_3}, w) \\ &+ \frac{24\left(N_5^1 - 10\right)}{m_0m_1m_2m_3m_4} \sum_{w \leqslant p_4 < p_3 < p_2 < p_1 < z} S(\mathcal{A}_{p_1p_2p_3p_4}, w) \\ &- \frac{120}{m_0m_1m_2m_3m_4} \sum_{w \leqslant p_5 < p_4 < p_3 < p_2 < p_1 < z} S(\mathcal{A}_{p_1p_2p_3p_4p_5}, w). \end{aligned}$$

Corollary 3.4. *For any $0 < m_0 \leqslant 1$, m_1, m_2, m_3, m_4 such that $(m_1, m_2) \in \mathcal{U}$, $(m_3, m_4) \in \mathcal{U}$ and $3 \leqslant s \leqslant t$, we have*

$$\begin{aligned} s^\kappa f_\kappa(s) \geqslant t^\kappa f_\kappa(t) &- \frac{N_5^4 - N_5^3 + N_5^2 - N_5^1 + 1}{m_0 m_1 m_2 m_3 m_4} \int_{\frac{1}{t}}^{\frac{1}{s}} \frac{t^\kappa F_\kappa(t(1-x_1))}{x_1} dx_1 \\ &+ \frac{2\left(N_5^3 - 3N_5^2 + 7N_5^1 - 15\right)}{m_0 m_1 m_2 m_3 m_4} \int_{\frac{1}{t}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{x_1} \frac{t^\kappa f_\kappa(t(1-x_1-x_2))}{x_1 x_2} dx_2 dx_1 \\ &- \frac{6\left(N_5^2 - 6N_5^1 + 25\right)}{m_0 m_1 m_2 m_3 m_4} \int_{\frac{1}{t}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{x_1} \int_{\frac{1}{t}}^{x_2} \frac{t^\kappa F_\kappa(t(1-x_1-x_2-x_3))}{x_1 x_2 x_3} dx_3 dx_2 dx_1 \\ &+ \frac{24\left(N_5^1 - 10\right)}{m_0 m_1 m_2 m_3 m_4} \int_{\frac{1}{t}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{x_1} \int_{\frac{1}{t}}^{x_2} \int_{\frac{1}{t}}^{x_3} \frac{t^\kappa f_\kappa(t(1-x_1-x_2-x_3-x_4))}{x_1 x_2 x_3 x_4} dx_4 dx_3 dx_2 dx_1 \\ &- \frac{120}{m_0 m_1 m_2 m_3 m_4} \times \\ &\int_{\frac{1}{t}}^{\frac{1}{s}} \int_{\frac{1}{t}}^{x_1} \int_{\frac{1}{t}}^{x_2} \int_{\frac{1}{t}}^{x_3} \int_{\frac{1}{t}}^{x_4} \frac{t^\kappa F_\kappa(t(1-x_1-x_2-x_3-x_4-x_5))}{x_1 x_2 x_3 x_4 x_5} dx_5 dx_4 dx_3 dx_2 dx_1. \end{aligned}$$

4. Further prospect

In this note, we only give some sieve inequalities and do not mention any possible application of these inequalities. In fact, these may be helpful in bounding the "sifting limits" β_κ for $\kappa > 1$. The bounds for β_κ are quite important in many high–dimensional sieve problems. We hope someone can accomplish this work.

There are many other iteration rules proved in Brady's thesis [1]. We state two of them in the rest of this note, and we hope someone can generalize them.

Theorem 4.1. *([1], Theorem 35]). For any $w \leqslant z^2$, we have*

$$\begin{aligned} S(\mathcal{A}, z) \geqslant S\left(\mathcal{A}, w^{\frac{1}{2}}\right) &- \sum_{w^{\frac{1}{2}} \leqslant p_1 < z} S\left(\mathcal{A}_{p_1}, \frac{w}{p_1}\right) + \frac{5}{6} \sum_{\frac{w}{p_1} \leqslant p_2 < p_1 < z} S\left(\mathcal{A}_{p_1 p_2}, \frac{w}{p_1}\right) \\ &- \frac{2}{3} \sum_{\substack{\frac{w}{p_1} \leqslant p_3 < p_2 < p_1 < z \\ p_2 p_3 < w}} S\left(\mathcal{A}_{p_1 p_2 p_3}, \frac{w}{p_1}\right) - \frac{1}{2} \sum_{\frac{w}{p_2} \leqslant p_3 < p_2 < p_1 < z} S\left(\mathcal{A}_{p_1 p_2 p_3}, \frac{w}{p_1}\right). \end{aligned}$$

Theorem 4.2. *([1], Theorem 42]). If every element of $\mathcal{A}$ has size at most $y^{\frac{13}{12}}$ and $z^{\frac{12}{5}} < y < z^{\frac{5}{2}}$, we have*

$$\begin{aligned} S(\mathcal{A}, z) \leqslant S\left(\mathcal{A}, \frac{y}{z^2}\right) &- \frac{4}{5} \sum_{\frac{y}{z^2} \leqslant p_1 < \frac{z^3}{y}} S\left(\mathcal{A}_{p_1}, \frac{y}{z^2}\right) - \frac{2}{3} \sum_{\frac{z^3}{y} \leqslant p_1 < \frac{y^2}{z^4}} S\left(\mathcal{A}_{p_1}, \frac{y}{z^2}\right) - \frac{8}{15} \sum_{\frac{y^2}{z^4} \leqslant p_1 < z} S\left(\mathcal{A}_{p_1}, \frac{y}{z^2}\right) \\ &+ \frac{3}{5} \sum_{\frac{y}{z^2} \leqslant p_2 < p_1 < \frac{z^3}{y}} S\left(\mathcal{A}_{p_1 p_2}, \frac{y}{z^2}\right) + \frac{7}{15} \sum_{\frac{y}{z^2} \leqslant p_2 < \frac{z^3}{y} \leqslant p_1 < \frac{y^2}{z^4}} S\left(\mathcal{A}_{p_1 p_2}, \frac{y}{z^2}\right) \\ &+ \frac{1}{3} \sum_{\frac{y}{z^2} \leqslant p_2 < \frac{z^3}{y} < \frac{y^2}{z^4} \leqslant p_1 < z} S\left(\mathcal{A}_{p_1 p_2}, \frac{y}{z^2}\right) + \frac{1}{3} \sum_{\frac{z^3}{y} \leqslant p_2 < p_1 < \frac{y^2}{z^4}} S\left(\mathcal{A}_{p_1 p_2}, \frac{y}{z^2}\right) \\ &+ \frac{4}{15} \sum_{\frac{z^3}{y} \leqslant p_2 < \frac{y^2}{z^4} \leqslant p_1 < z} S\left(\mathcal{A}_{p_1 p_2}, \frac{y}{z^2}\right) + \frac{1}{5} \sum_{\frac{y^2}{z^4} \leqslant p_2 < p_1 < z} S\left(\mathcal{A}_{p_1 p_2}, \frac{y}{z^2}\right) \\ &- \frac{2}{5} \sum_{\substack{\frac{y}{z^2} \leqslant p_3 < p_2 < p_1 < \frac{z^3}{y} \\ p_1 p_2 p_3^2 < z^2}} S\left(\mathcal{A}_{p_1 p_2 p_3}, \frac{y}{z^2}\right) \\ &- \frac{4}{15} \sum_{\frac{y}{z^2} \leqslant p_3 < p_2 < \frac{z^3}{y} \leqslant p_1 < \frac{y^2}{z^4}} \left(1 - \frac{3\log(p_2 p_3)}{8\log(y/p_1)}\right) S\left(\mathcal{A}_{p_1 p_2 p_3}, \frac{y}{z^2}\right) \end{aligned}$$

$$
+\frac{1}{5}\sum_{\substack{\frac{y}{z^2}\leqslant p_4<p_3<p_2<p_1<\frac{z^3}{y}\\ p_1p_2p_3^2<z^2}} S\left(\mathcal{A}_{p_1p_2p_3},\frac{y}{z^2}\right)
$$

$$
+\frac{1}{10}\sum_{\frac{y}{z^2}\leqslant p_4<p_3<p_2<\frac{z^3}{y}\leqslant p_1<\frac{y^2}{z^4}} \left(1-\frac{\log(p_2p_3p_4)}{\log(y/p_1)}\right) S\left(\mathcal{A}_{p_1p_2p_3},\frac{y}{z^2}\right).
$$

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